Tatra Mt. Math. Publ. **26** (2003), 49–57

TATRA MOUNTAINS
Mathematical Publications

OPTIMUM CHEMICAL BALANCE WEIGHING DESIGNS

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ABSTRACT. The paper deals with the problem of estimating individual weights of objects using a chemical balance weighing design with diagonal variance-covariance matrix of errors under the restriction on the number times in which each object is weighted. A lower bound for the variance of each of the estimated weights from this chemical balance weighing design is obtained and a necessary and sufficient condition for this lower bound to be attained is given. The incidence matrices of ternary balanced block designs are used to construct the matrix of optimum chemical balance weighing designs.

I. Introduction

The results of n weighing operations to determine the individual weights of p objects with a balance that is corrected for bias will fit into the linear model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}\,, \tag{1.1}$$

where $\mathbf{y}$ is an $n \times 1$ random column vector of the observed weights, $\mathbf{X} = (x_{ij})$, $i = 1, 2 \dots, n$, $j = 1, 2 \dots, p$, is an $n \times p$ matrix of known elements with $x_{ij} = -1, 1$ or 0 if the jth object is kept on the right pan, left pan, or is not included in the particular weighing operation, respectively, $\mathbf{w}$ is the $p \times 1$ column vector representing the unknown weights of objects and $\mathbf{e}$ is an $n \times 1$ random column vector of errors such that $E(\mathbf{e}) = \mathbf{0}_n$ and $E(\mathbf{e}\mathbf{e}') = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is the $n \times 1$ column vector with zero elements everywhere, $\mathbf{G}$ is the $n \times n$ positive definite diagonal matrix of known elements, $E(\cdot)$ stands for the expectation and $\mathbf{e}'$ is used for transpose of $\mathbf{e}$.

The normal equations estimating $\mathbf{w}$ are of the form

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}\,, \tag{1.2}$$

where $\hat{\mathbf{w}}$ is the vector of the weights estimated by the least squares method.

2000 Mathematics Subject Classification: 62K15.
Keywords: chemical balanced weighing design, ternary balanced block design.

A chemical balance weighing design is said to be singular or nonsingular depending on whether the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular, respectively. It is obvious that the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if the matrix $\mathbf{X}$ is of full column rank ($= p$).

Now, if $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, then the least squares estimator of $\mathbf{w}$ is given by

$$\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y} \tag{1.3}$$

and the variance-covariance matrix of $\hat{\mathbf{w}}$ is

$$\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}. \tag{1.4}$$

Various aspects of chemical balance weighing designs were studied by Raghavarao (1971) and Banerjee (1975). Katulska (1989) showed that the minimum attainable variance for each of the estimated weights for a chemical balance weighing design with positive definite diagonal variance-covariance matrix of errors is $\sigma^2/\mathrm{tr}(\mathbf{G}^{-1})$ and proved the theorem that each of the variance of the estimated weights attains the minimum if and only if $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mathrm{tr}(\mathbf{G}^{-1})\mathbf{I}_p$. This design is said to be optimum chemical balance weighing design. In this case several methods of constructing optimum chemical balance weighing design are available in the literature.

In the present paper we investigate the necessary and sufficient condition under which the minimum variance is attained for estimated weights. We give some methods of constructing optimum chemical balance weighing design with positive definite diagonal variance-covariance matrix of errors under the restriction on the number of objects placed on the either pan based on the incidence matrices of ternary balanced block designs.

II. Variance limit of estimated weights

Let $\mathbf{X}$ be an $n \times p$ matrix of rank p and $\mathbf{c}$ denote any $p \times 1$ column vector. Then, from Section 1e. 1 (ii) (b) given by Rao (1973), we have

Lemma 2.1. *For a positive definite $n \times n$ diagonal matrix $\mathbf{G}$, an $n \times p$ matrix $\mathbf{X}$ of rank p and a $p \times 1$ column vector $\mathbf{c}$, the inequality*

$$\mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{c} \geq \frac{(\mathbf{c}'\mathbf{c})^2}{\mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})\mathbf{c}} \tag{2.1}$$

holds, with equality attained if and only if $\mathbf{c}$ is an eigenvector of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$.

We assume that matrix $\mathbf{G}$ is given in the form

$$\mathbf{G} = \begin{bmatrix} \frac{1}{a_1}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} & \dots & \mathbf{0}_{n_1}\mathbf{0}'_{n_t} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & \frac{1}{a_2}\mathbf{I}_{n_2} & \dots & \mathbf{0}_{n_2}\mathbf{0}'_{n_t} \\ \dots & \dots & \dots & \dots \\ \mathbf{0}_{n_t}\mathbf{0}'_{n_1} & \mathbf{0}_{n_t}\mathbf{0}'_{n_2} & \dots & \frac{1}{a_t}\mathbf{I}_{n_t} \end{bmatrix}, \tag{2.2}$$

where $\sum_{h=1}^{t} n_h = n$, $a_h > 0$, *and* $\mathbf{I}_h$ *is the* $n_h \times n_h$ *identity matrix,* $h = 1, 2, \ldots, t$. *Suppose further that matrix* $\mathbf{X}$ *is likewise partitioned as the matrix,* $\mathbf{G}$, *i.e.,*

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1^* \\ \mathbf{X}_2^* \\ \vdots \\ \mathbf{X}_t^* \end{bmatrix} \tag{2.3}$$

and let m_{hj}^* *be number of times in which* j*th object is weighted for a particular* $n_h \times p$ *matrix* $\mathbf{X}_h^* = (x_{qj}^h)$, $q = 1, 2, \ldots, n_h$, $j = 1, 2, \ldots, p$, $h = 1, 2, \ldots, t$.

Theorem 2.1. *For any nonsingular chemical balance weighing design with design matrix* $\mathbf{X}$ *given in the form* (2.3) *and with variance-covariance matrix* $\sigma^2\mathbf{G}$, *where matrix* $\mathbf{G}$ *is given in the form* (2.2), *the variance of* $\hat{w}_j$ *for a particular* j, *such that* $1 \leq j \leq p$, *cannot be less than* σ^2/m, *where* $m = \sum_{h=1}^{t} a_h m_h^*$ *and* $m_h^* = \max\{m_{h1}^*, m_{h2}^*, \ldots, m_{hp}^*\}$.

P r o o f. Let $\mathbf{c}_j$, $j = 1, 2, \ldots, p$, be the vector equal to the jth column of $\mathbf{I}_p$. Then it follows that

$$\hat{w}_j = \mathbf{c}_j'\hat{\mathbf{w}}$$

and

$$\mathrm{Var}(\hat{w}_j) = \sigma^2 \mathbf{c}_j'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{c}_j .$$

Since the matrix $\mathbf{X}$ is of full column rank, then from Lemma 2.1 we have

$$\begin{aligned} \mathrm{Var}(\hat{w}_j) \geq \sigma^2 \frac{(\mathbf{c}_j'\mathbf{c}_j)^2}{\mathbf{c}_j'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})\mathbf{c}_j} &= \frac{\sigma^2}{\sum_{h=1}^{t} a_h \sum_{q=1}^{n_h} (x_{qj}^h)^2} \\ = \frac{\sigma^2}{\sum_{h=1}^{t} a_h m_{hj}^*} &\geq \frac{\sigma^2}{\sum_{h=1}^{t} a_h m_h^*} = \frac{\sigma^2}{m}, \end{aligned} \tag{2.4}$$

because elements $x_{qj}^h = -1, 1$ or 0 only, and number of elements equal to -1 or 1 in each column of $\mathbf{X}_h^*$ is equal to m_{hj}^*, where $q = 1, 2, \ldots, n_h$, $j = 1, 2, \ldots, p$, $h = 1, 2, \ldots, t$. Hence the Theorem is proved. □

Since $\mathrm{tr}(\mathbf{G}^{-1}) = \sum_{h=1}^{t} a_h n_h$, then if $m_h^* = n_h$ for each h, then $\mathrm{Var}(\hat{w}_j) \geq \frac{\sigma^2}{\mathrm{tr}(\mathbf{G}^{-1})}$. This Theorem was originally proved by K a t u l s k a (1989) for the case $m_h^* = n_h$, $h = 1, 2, \ldots, t$.

Definition 2.1. Any nonsingular chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where matrix $\mathbf{G}$ is given in the form (2.2), is said be optimal for the estimated weights of objects if the variances of estimators attain the lower bound given by Theorem 2.1, i.e., if

$$\mathrm{Var}(\hat{w}_j) = \frac{\sigma^2}{m}, \qquad j = 1, 2, \ldots, p,$$

where $m = \sum_{h=1}^{t} a_h m_h^*$.

Now, we investigate the necessary and sufficient condition under which the minimum variance for the estimated weights is attained.

Theorem 2.2. *For any positive definite diagonal matrix $\mathbf{G}$ given in the form (2.2) and any $n \times p$ matrix $\mathbf{X}$ given in the form (2.3) of a nonsingular chemical balance weighing design, each of the variances of the estimated weights attains the minimum if and only if*

$$\mathbf{X}'G^{-1}\mathbf{X} = m\,\mathbf{I}_p. \tag{2.5}$$

P r o o f. To prove the necessity part we observe that the equality in (2.4) holds for any $j = 1, 2, \ldots, p$ if and only if

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{c}_j = \mu_j \mathbf{c}_j, \qquad \mu_j > 0,$$

and

$$\mathbf{c}_j'\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{c}_j = \sum_{h=1}^{t} a_h m_h^*, \qquad j = 1, 2, \ldots, p.$$

These conditions imply that $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_p\}$ and $\mu_1 = \mu_2 = \cdots = \mu_p = \sum_{h=1}^{t} a_h m_h^* = m$. Then we have condition (2.5). The proof for the sufficiency part is obvious. □

Theorem 2.3. *If the $n_s \times p$ design matrices $\mathbf{X}_s$ are the matrices of optimum chemical balance weighing designs with the variance-covariance matrices of errors $\sigma^2\mathbf{G}_s$, where $\mathbf{G}_s$ are $n_s \times n_s$ known positive definite diagonal matrices, $s = 1, 2, \ldots, u$, respectively, then $\mathbf{X} = \left[\mathbf{X}_1' \vdots \mathbf{X}_2' \vdots \cdots \vdots \mathbf{X}_u'\right]'$ is the $\left(\sum_{s=1}^{u} n_s\right) \times p$ design matrix of the optimum chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where matrix $\mathbf{G} = \mathrm{diag}\{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_u\}$.*

P r o o f. It is obvious that

$$\mathbf{X}'G^{-1}\mathbf{X} = \left(\sum_{s=1}^{u} m_s\right)\mathbf{I}_p = m\mathbf{I}_p.$$

□

Theorem 2.4. *If $\mathbf{X}_s$ are the $n_s \times p$ matrices of the optimum chemical balance weighing designs with the variance-covariance matrices of errors $\sigma^2 \mathbf{G}_s$, where $\mathbf{G}_s$ are $n_s \times n_s$ known positive definite diagonal matrices, $s = 1, 2$, and $\mathbf{X}_1' \mathbf{G}_1^{-1} \mathbf{1}_{n_1} = \mathbf{0}_p$, then*

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{1}_{n_1} \\ \mathbf{X}_2 & \mathbf{0}_{n_2} \end{bmatrix} \tag{2.6}$$

is the $(n_1 + n_2) \times (p+1)$ design matrix of the optimum chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2 \mathbf{G}$, where matrix $\mathbf{G} = \operatorname{diag}\{\mathbf{G}_1, \mathbf{G}_2\}$ if and only if $m = m_1 + m_2 = \operatorname{tr}(\mathbf{G}_1^{-1})$, where $\mathbf{1}_{n_1}$ denotes $n_1 \times 1$ column vector of ones.

P r o o f . For the design matrix $\mathbf{X}$ given by (2.6) we have

$$\begin{aligned} \mathbf{X}'\mathbf{G}^{-1}\mathbf{X} &= \begin{bmatrix} \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{G}_2^{-1}\mathbf{X}_2 & \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{1}_{n_1} \\ \mathbf{1}_{n_1}'\mathbf{G}_1^{-1}\mathbf{X}_1 & \mathbf{1}_{n_1}'\mathbf{G}_1^{-1}\mathbf{1}_{n_1} \end{bmatrix} \\ &= \begin{bmatrix} (m_1 + m_2)\mathbf{I}_p & \mathbf{0}_p \\ \mathbf{0}_p' & \operatorname{tr}(\mathbf{G}_1^{-1}) \end{bmatrix} = m\mathbf{I}_{p+1} \,. \end{aligned} \tag{2.7}$$

Hence the thesis is proved. □

Theorem 2.5. *If $\mathbf{X}_s$ are the $n_s \times p$ matrices of the optimum chemical balance weighing designs with the variance-covariance matrices of errors $\sigma^2 \mathbf{G}_s$, where $\mathbf{G}_s$ are $n_s \times n_s$ known positive definite diagonal matrices, $s = 1, 2$, and $\mathbf{X}_1' \mathbf{G}_1^{-1} \mathbf{X}_1 = \mathbf{X}_2' \mathbf{G}_2^{-1} \mathbf{X}_2$, $\mathbf{X}_1' \mathbf{G}_1^{-1} \mathbf{1}_{n_1} = \mathbf{0}_p$, then*

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_1 & \mathbf{1}_{n_1} \\ \mathbf{X}_2 & -\mathbf{X}_2 & \mathbf{0}_{n_2} \end{bmatrix} \tag{2.8}$$

is the $(n_1 + n_2) \times (2p+1)$ design matrix of the optimum chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2 \mathbf{G}$, where matrix $\mathbf{G} = \operatorname{diag}\{\mathbf{G}_1, \mathbf{G}_2\}$ if and only if $m_1 = m_2$ and $m = \operatorname{tr}(\mathbf{G}_1^{-1})$, where $m = m_1 + m_2$.

P r o o f . For the design matrix $\mathbf{X}$ given by (2.8) we have

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} =$$

$$= \begin{bmatrix} \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{G}_2^{-1}\mathbf{X}_2 & \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{X}_1 - \mathbf{X}_2'\mathbf{G}_2^{-1}\mathbf{X}_2 & \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{1}_{n_1} \\ \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{X}_1 - \mathbf{X}_2'\mathbf{G}_2^{-1}\mathbf{X}_2 & \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{G}_2^{-1}\mathbf{X}_2 & \mathbf{X}_1'\mathbf{G}_1^{-1}\mathbf{1}_{n_1} \\ \mathbf{1}_{n_1}'\mathbf{G}_1^{-1}\mathbf{X}_1 & \mathbf{1}_{n_1}'\mathbf{G}_1^{-1}\mathbf{X}_1 & \mathbf{1}_{n_1}'\mathbf{G}_1^{-1}\mathbf{1}_{n_1} \end{bmatrix} = m\mathbf{I}_{2p+1} \,.$$

Hence the thesis is proved. □

It is obvious that we have many interesting possibilities for matrix $\mathbf{G}$. The constructions of optimum chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2 \mathbf{G}$ for each form of matrix $\mathbf{G}$ must be investigated separately. Now, we will consider the simplest form of $\mathbf{G} \neq \mathbf{I}_n$, i.e.,

$$\mathbf{G} = \begin{bmatrix} \frac{1}{a}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & \mathbf{I}_{n_2} \end{bmatrix}, \qquad a > 0. \tag{2.9}$$

Suppose further that the matrix $\mathbf{X}$ is likewise partitioned as the matrix $\mathbf{G}$, i.e.,

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1^* \\ \mathbf{X}_2^* \end{bmatrix}. \tag{2.10}$$

Then from Theorem 2.2 the matrix $\mathbf{X}$ given by (2.10) is the matrix of the optimum chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where matrix $\mathbf{G}$ is given by (2.9) if and only if

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = (am_1^* + m_2^*)\mathbf{I}_p. \tag{2.11}$$

In the next section we will present a construction of the matrix $\mathbf{X}$ given in the form (2.10) of the optimum chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where matrix $\mathbf{G}$ is given in the form (2.9), based on the incidence matrices of ternary balanced block designs.

III. Construction of the design matrix using ternary balanced block designs

For the construction of the design matrix $\mathbf{X}$ we will use the incidence matrices of the ternary balanced block designs. The definition of that design was given by Tocher (1952) and it was used, later, by Billington and Robinson (1983).

A ternary balanced block design with the parameters $v, b, r, k, \lambda, \rho_1, \rho_2$ is a design consisting of b blocks, each of size k, chosen from a set of size v in such a way that each of the v elements occurs r times altogether and 0, 1 or 2 times in each block, 2 occurs at least once and each of the distinct pairs of elements occurs λ times. Any ternary balanced block design is regular, that is, each element occurs singly in ρ_1 blocks and is repeated in ρ_2 blocks. Let $\mathbf{N}$ be the incidence matrix of ternary balanced block design. Elements of this matrix are equal to 0, 1 or 2. It follows from the paper of Tocher (1952) that

$$\mathbf{N}\mathbf{N}' = (r + 2\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}'_v.$$

Not all parameters of ternary balanced block design are independent. It is straightforward to verify that

$$\begin{aligned} vr &= bk, \\ r &= \rho_1 + 2\rho_2, \\ \lambda(v-1) &= \rho_1(k-1) + 2\rho_2(k-2) = r(k-1) - 2\rho_2. \end{aligned}$$

Let $\mathbf{N}_i$, $i = 1, 2$, denote the incidence matrices of ternary balanced block designs with parameters v, b_i, r_i, k_i, λ_i, ρ_{1i}, ρ_{2i}. Matrix $\mathbf{X}$ of a chemical balance weighing design we define in the following form

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' - \mathbf{1}_{b_1}\mathbf{1}_v' \\ \mathbf{N}_2' - \mathbf{1}_{b_2}\mathbf{1}_v' \end{bmatrix}. \tag{3.1}$$

In this design we have $n = b_1 + b_2$ and $p = v$. Thus, each column of $\mathbf{X}$ will contain $\rho_{21} + \rho_{22}$ elements equal to 1, $b_1 + b_2 - \rho_{11} - \rho_{12} - \rho_{21} - \rho_{22}$ elements equal to -1, and $\rho_{11} + \rho_{12}$ elements equal to 0. Clearly, such a design implies that each object is weighted $m = b_1 + b_2 - \rho_{11} - \rho_{12}$ times in the $n = b_1 + b_2$ weighing operations.

Lemma 3.1. *A design given by* $\mathbf{X}$ *of the form* (3.1) *is nonsingular if and only if* $k_1 \neq k_2$ *or* $k_1 = k_2 \neq v$.

P r o o f . If the matrix $\mathbf{G}$ is positive defined diagonal matrix, then $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if $\mathbf{X}'\mathbf{X}$ is nonsingular. Thus for the design matrix $\mathbf{X}$ given by (3.1) we have

$$\begin{aligned} \mathbf{X}'\mathbf{X} = {} & \left[r_1 + 2\rho_{21} - \lambda_1 + r_2 + 2\rho_{22} - \lambda_2\right]\mathbf{I}_v \\ & + \left[b_1 - 2r_1 + \lambda_1 + b_2 - 2r_2 + \lambda_2\right]\mathbf{1}_v\mathbf{1}_v' . \end{aligned} \tag{3.2}$$

It is easy to see, that

$$\det(\mathbf{X}'\mathbf{X}) = \left[r_1 + 2\rho_{21} - \lambda_1 + r_2 + 2\rho_{22} - \lambda_2\right]^{v-1} \left[\frac{r_1(v-k_1)^2}{k_1} + \frac{r_2(v-k_2)^2}{k_2}\right].$$

Evidently $r_1 + 2\rho_{21} - \lambda_1 + r_2 + 2\rho_{22} - \lambda_2$ is positive, then $\det(\mathbf{X}'\mathbf{X}) \neq 0$ if and only if $k_1 \neq k_2$ or $k_1 = k_2 \neq v$. So, the Lemma is proved. □

Theorem 3.1. *A chemical balance weighing design with the matrix* $\mathbf{X}$ *given by* (3.1) *and with the variance-covariance matrix of errors* $\sigma^2\mathbf{G}$, *where matrix* $\mathbf{G}$ *is given by* (2.9), *is optimal if and only if*

$$a(b_1 - 2r_1 + \lambda_1) + (b_2 - 2r_2 + \lambda_2) = 0 . \tag{3.3}$$

P r o o f . From the conditions (2.11) and (3.2) it follows, that a chemical balance weighing design is optimal if and only if the condition (3.3) holds. Hence the theorem is proved. □

If the chemical balance weighing design given by matrix $\mathbf{X}$ of the form (3.1) and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where matrix $\mathbf{G}$ is given by (2.9), is optimal, then

$$\operatorname{Var}(\hat{w}_j) = \frac{\sigma^2}{a(b_1 - \rho_{11}) + (b_2 - \rho_{12})}, \qquad j = 1, 2, \ldots, p .$$

Corollary 3.1. *If* $b_i - 2r_i + \lambda_i = 0$, $i = 1,2$, *then a chemical balance weighing design given by matrix* $\mathbf{X}$ *of the form* (3.1) *and with the variance-covariance matrix of errors* $\sigma^2\mathbf{G}$, *where matrix* $\mathbf{G}$ *is given by* (2.9), *is optimal for any* a, $a > 0$.

Series of chemical balance weighing designs satisfying the conditions of Corollary 3.1 are given by Ceranka and Graczyk (2001).

We can notice that if parameters of ternary balanced block designs satisfy the condition (3.3) for a given a, then the chemical balance weighing design with the matrix $\mathbf{X}$ given in the form (3.1) and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where matrix $\mathbf{G}$ is given by (2.9), is optimal.

In the Table 1, $s = 1,2,\ldots,$ $q = 0,1,2,$ $l = 1,2$.

Table 1

a	v	1st design						2nd design					
		b_1	r_1	k_1	λ_1	ρ_{11}	ρ_{21}	b_2	r_2	k_2	λ_2	ρ_{12}	ρ_{22}
$\frac{2}{7}$	7	28	12	3	3	6	3	$s+13$	$s+13$	7	$s+11$	$s+1$	6
$\frac{1}{3}$	9	27	15	5	6	3	6	$s+8$	$s+8$	9	$s+7$	s	4
$\frac{1}{3}$	11	22	12	6	5	3	6	$s+10$	$s+10$	s	$s+8$	s	5
$\frac{1}{3}$	12	14	7	6	3	5	1	18	15	10	11	1	7
$\frac{1}{3}$	15	25	15	9	8	7	4	$s+14$	$s+14$	s	$s+13$	s	7
$\frac{2}{5}$	7	21	9	3	2	3	3	$s+13$	$s+13$	7	$s+11$	$s+1$	6
$\frac{2}{5}$	8	26	13	4	5	9	2	15	15	8	13	1	7
$\frac{1}{2}$	5	$s+9$	$s+9$	5	$s+7$	$s+1$	4	$5(s+2)$	$3(s+2)$	3	$s+3$	$s+6$	s
$\frac{1}{2}$	5	$5(s+4)$	$3(s+4)$	3	$s+6$	$s+12$	s	$5(s+1)$	$4(s+1)$	4	$3s+2$	$4s$	2
$\frac{1}{2}$	9	$s+17$	$s+17$	9	$s+15$	$s+1$	8	$3(s+4)$	$2(s+4)$	6	$s+5$	8	s
$\frac{1}{2}$	11	$s+21$	$s+21$	11	$s+19$	$s+1$	10	11	7	7	4	5	1
$\frac{1}{2}$	12	$s+22$	$s+22$	12	$s+20$	s	11	$3(2q+5)$	$2(2q+5)$	8	$2(q+3)$	$6-2q$	$3q+2$
$\frac{1}{2}$	15	$s+29$	$s+29$	15	$s+27$	$s+1$	14	$3(l+4)$	$2(l+4)$	10	$l+5$	$6-2l$	$2l+1$
$\frac{2}{3}$	5	15	15	5	12	3	6	25	15	3	7	13	1

On the base of parameters of ternary balanced block designs given by Billington (1984) in the Table 1, series of parameters of that two designs are given. For a given a and for a given number of objects v we can choose the parameters of two ternary balanced block designs, and on the basis of that parameters we can construct two incidence matrices of that designs. As the next step from that incidence matrices we construct the design matrix $\mathbf{X}$ of the form (3.1) and with variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where matrix $\mathbf{G}$ is given by (2.9). This matrix is the matrix of optimum chemical balance weighing design.

We can notice that if parameters $v, b_1, r_1, k_1, \lambda_1, \rho_{11}, \rho_{21}$ and $v, b_2, r_2, k_2, \lambda_2, \rho_{12}, \rho_{22}$ of ternary balanced block designs satisfy the condition (3.3) for a given a, then parameters $v^* = v$, $b_1^* = b_2$, $r_1^* = r_2$, $k_1^* = k_2$, $\lambda_1^* = \lambda_2$, $\rho_{11}^* = \rho_{12}$, $\rho_{21}^* = \rho_{22}$ and $v^* = v$, $b_2^* = b_1$, $r_2^* = r_1$, $k_2^* = k_1$, $\lambda_2^* = \lambda_1$, $\rho_{12}^* = \rho_{11}$, $\rho_{22}^* = \rho_{12}$ of two ternary balanced block designs satisfy the condition

$$a^*\left(b_1^* - 2r_1^* + \lambda_1^*\right) + \left(b_2^* - 2r_2^* + \lambda_2^*\right) = 0$$

for $a^* = \frac{1}{a}$, and the chemical balance weighing design with matrix $\mathbf{X}$ given by (3.1) and with variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2.9), is optimal for estimating individual unknown weights of objects.

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Received February 15, 2002

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