Tatra Mt. Math. Publ. **24** (2002), 117–123

TATRA MOUNTAINS
Mathematical Publications

SUMS, DIFFERENCES, PRODUCTS AND QUOTIENTS OF CLOSED GRAPH FUNCTIONS

JÁN BORSÍK

ABSTRACT. The family of functions which can be written as the sum, the difference, the product, and the quotient of functions with closed graphs is characterized.

Let X be a topological space. A function $f\colon X \to \mathbb{R}$ has the closed graph if the set $\{(x, f(x)) : x \in X\}$ is a closed set in $X \times \mathbb{R}$. Functions with closed graphs play an important rule in functional analysis and they have been studied extensively. Denote by $\mathcal{U}$ the family of all closed graph functions (in $\mathbb{R}^X$).

A function $f\colon X \to \mathbb{R}$ is piecewise continuous if there are closed sets $X_n \subset X$ such that $X = \bigcup\limits_{n=1}^{\infty} X_n$ and the restriction $f \restriction X_n$ is continuous for each $n \in \mathbb{N}$. A function $f\colon X \to \mathbb{R}$ is a function of the class $\mathcal{B}_1^*$ (Baire-one-star function) if for each nonempty open set $F \subset X$ there exists an open set G such that $F \cap G \neq \emptyset$ and $f \restriction (F \cap G)$ is continuous ([4]). Denote by $\mathcal{C}$ the family of all continuous functions, by $\mathcal{P}$ the family of all piecewise continuous functions and by $\mathcal{B}_1^*$ the family of all Baire-one-star functions.

If $\mathcal{F} \subset \mathbb{R}^X$ is a family of functions, denote by

$\mathcal{F} + \mathcal{F} = \{f\colon X \to \mathbb{R}\,;\ f = g + h \text{ for some } g, h \in \mathcal{F}\}$,

$\mathcal{F} - \mathcal{F} = \{f\colon X \to \mathbb{R}\,;\ f = g - h \text{ for some } g, h \in \mathcal{F}\}$,

$\mathcal{F} \cdot \mathcal{F} = \{f\colon X \to \mathbb{R}\,;\ f = g \cdot h \text{ for some } g, h \in \mathcal{F}\}$,

$\mathcal{F}/\mathcal{F} = \{f\colon X \to \mathbb{R}\,;\ f = g/h \text{ for some } g, h \in \mathcal{F} \text{ with } h(x) \neq 0 \text{ for each } x \in X\}$.

Further denote by $\mathcal{F}^+$ the family of all positive functions from $\mathcal{F}$. The purpose of this paper is to characterize families $\mathcal{F} + \mathcal{F}$, $\mathcal{F} - \mathcal{F}$, $\mathcal{F} \cdot \mathcal{F}$ and $\mathcal{F}/\mathcal{F}$ for the families $\mathcal{U}$ and $\mathcal{U}^+$.

2000 Mathematics Subject Classification: Primary 54C08; Secondary 54C30.

Keywords: functions with closed graph, sum, difference, product, quotient, piecewise continuous functions.

Supported by the Grant VEGA 2/7192/20

The sum and the product of two closed graph functions need not be such function. However, for arbitrary topological space X we have $\mathcal{U}^+ + \mathcal{U}^+ = \mathcal{U}^+$ ([2]). In [3] it is shown that $\mathcal{U} + \mathcal{U} = \mathcal{B}_1^*$ if $X = \mathbb{R}$. For arbitrary (Baire metric) space this is not true because the set $\mathcal{U} \setminus \mathcal{B}_1^*$ can be nonempty. We shall show that $\mathcal{U} + \mathcal{U} = \mathcal{P}$ (for a perfectly normal space X). Let us remark that for complete metric spaces we have $\mathcal{P} = \mathcal{B}_1^*$ ([4]).

To show that f has the closed graph, a local characterization of the closedness of the graph is useful. We say that a function $f\colon X \to \mathbb{R}$ has the closed graph at $x \in X$ if $C(f,x) = \{f(x)\}$, where $C(f,x) = \bigcap\{\mathrm{Cl}\, f(U)\colon U$ is a neighbourhood of $x\}$ ([1]). If we denote by $H(f)$ the set of all closedness graph points of f, then f has the closed graph iff $H(f) = X$. Therefore to show that $f \in \mathcal{U}$ it is enough to prove that for each $x \in X$ and for each $m \in \mathbb{N}$ we can find a neighbourhood V of x such that $f(y) \in (-\infty, -m) \cup \big(f(x) - 1/m, f(x) + 1/m\big) \cup (m, \infty)$ for each $y \in V$.

Theorem 1. *Let X be a perfectly normal space. Then $\mathcal{U} + \mathcal{U} = \mathcal{P}$.*

Proof. Let $f \in \mathcal{P}$ and let $\{F_i\}_{i=1}^{\infty}$ be an increasing sequence of closed subsets of X such that $\bigcup\limits_{i=1}^{\infty} F_i = X$ and $f \restriction F_i$ is continuous. Put $F_0 = \emptyset$ and $E_i = F_i \setminus F_{i-1}$. For each $i \in \mathbb{N}$ let $t_i\colon X \to \mathbb{R}$ be a nonnegative continuous function such that $t_i^{-1}(0) = F_{i-1}$ and $h_i = \min\{t_1, t_2, \dots, t_i\}$. Then h_i are nonnegative continuous functions, $h_i^{-1}(0) = F_{i-1}$ and $h_{i+1} \le h_i$. Define a function $g\colon X \to \mathbb{R}$ as $g(x) = \frac{1}{h_i(x)}$ for $x \in E_i$. Further, let $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$. Finally, define functions $f_1, f_2 : X \to \mathbb{R}$ as

$$f_1 = f^+ + g \qquad \text{and} \qquad f_2 = f^- - g\,.$$

Evidently $f_1 + f_2 = f$. We shall show that f_1 and f_2 have the closed graphs. Fix $x \in X$ and $m \in \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that $x \in E_k$. Now, there exists a neighbourhood V of x such that:

(1) $0 \le h_{k+1}(y) < \frac{1}{m}$ for each $y \in V$,

(2) $\left|\frac{1}{h_k(y)} - \frac{1}{h_k(x)}\right| < \frac{1}{2m}$ for each $y \in V$ and

(3) $\left|f^+(x) - f^+(y)\right| < \frac{1}{2m}$ for each $y \in V \cap F_k$.

Moreover, we can assume that $V \cap F_{k-1} = \emptyset$.

Let $y \in V$.

If $y \in V \cap F_k$ then $y \in F_k \setminus F_{k-1}$ and

$$\left|f_1(x) - f_1(y)\right| \le \left|f^+(x) - f^+(y)\right| + \left|\frac{1}{h_k(x)} - \frac{1}{h_k(y)}\right| < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}\,.$$

If $y \in V \setminus F_k$ then $y \in E_i$ for some $i > k$ and

$$f_1(y) = f^+(y) + 1/h_i(y) \geq 1/h_{k+1}(y) > m\,.$$

Therefore $y \in \big(f(x) - 1/m, f(x) + 1/m\big) \cup (m, \infty)$ and $y \in H(f_1)$. Similarly we can prove that f_2 has the closed graph.

On the other hand, $\mathcal{U} \subset \mathcal{P}$ (for $f \in \mathcal{U}$, $F_n = f^{-1}\big([-n, n]\big)$ are closed sets, $\bigcup\limits_{n=1}^{\infty} F_n = X$ and $f{\restriction}F_n$ are continuous) and therefore $\mathcal{U} + \mathcal{U} \subset \mathcal{P} + \mathcal{P} = \mathcal{P}$. $\square$

It is easy to see that the following lemma holds.

Lemma 1. *Let X be a topological space and $f\colon X \to \mathbb{R}$. If $x \in H(f)$ and $\alpha \in \mathbb{R}$ then $x \in H(\alpha f)$ and $x \in H\big(|f|\big)$.*

Since the function f_1 from the proof of Theorem 1 is positive and the function f_2 is negative, by Lemma 1 we have the following theorem.

Theorem 2. *Let X be a perfectly normal space. Then $\mathcal{U} - \mathcal{U} = \mathcal{U}^+ - \mathcal{U}^+ = \mathcal{P}$.*

Lemma 2. *Let X be a topological space and $f, g\colon X \to \mathbb{R}$. If $x \in H(f) \cap H(g)$ and $f(x) \neq 0 \neq g(x)$ then $x \in H(f \cdot g)$.*

Proof. Fix $m \in \mathbb{N}$.

a) At first, we shall assume that $|f(x)| > 1$ and $|g(x)| > 1$. Let $k > m$ be such that $\frac{1}{k} < \frac{1}{m(m+|f(x)|)}$. Then there exists a neighbourhood V of x such that

(1) $f(y) \in (-\infty, -k) \cup \big(f(x) - 1/k, f(x) + 1/k\big) \cup (k, \infty)$ for each $y \in V$ and

(2) $g(y) \in (-\infty, -k) \cup \big(g(x) - 1/k, g(x) + 1/k\big) \cup (k, \infty)$ for each $y \in V$.

Let $y \in V$. If $f(y) > m$ then $|(f \cdot g)(y)| = |f(y)| \cdot |g(y)| > m$; similarly $|g(y)| > m$ implies $|f(y) \cdot g(y)| > m$. Now let $|f(y)| \leq m$ and $|g(y)| \leq m$. Then $|f(x) - f(y)| < 1/k$ and $|g(x) - g(y)| < 1/k$ and hence $|(f \cdot g)(y) - (f \cdot g)(x)| \leq |f(y) - f(x)| \cdot |g(y)| + |g(y) - g(x)| \cdot |f(x)| < m \cdot 1/k + |f(x)| \cdot 1/k < 1/m$ and $x \in H(f \cdot g)$.

b) Now let $f(x) \neq 0$, $g(x) \neq 0$ be arbitrary. Then there is $\alpha > 0$ such that $|f(x)| > \alpha$ and $|g(x)| > \alpha$. If we put $f_1 = (1/\alpha)f$ and $g_1 = (1/\alpha)g$ then $|f_1(x)| > 1$, $|g_1(x)| > 1$ and by Lemma 1 and a) we obtain $x \in H(f_1) \cap H(g_1)$ and hence $x \in H(f_1 \cdot g_1)$. Again by Lemma 1 we have $x \in H(\alpha^2 \cdot f_1 \cdot g_1) = H(f \cdot g)$. $\square$

Corollary 1. *If f, g have closed graphs and the inverse images of 0 under f and g are open sets then, $f \cdot g$ has the closed graph.*

Corollary 2. *Let X be a topological space. Then $\mathcal{U}^+ \cdot \mathcal{U}^+ = \mathcal{U}^+$.*

Lemma 3. *Let X be a topological space and let f and g have the closed graphs. Then the set $(f \cdot g)^{-1}(0)$ is closed.*

Proof. Since f and g have the closed graphs, the sets $f^{-1}(0)$ and $g^{-1}(0)$ are closed. Therefore the set $(f \cdot g)^{-1}(0) = f^{-1}(0) \cup g^{-1}(0)$ is closed. □

Lemma 4. *Let X be a topological space and let f, g have the closed graphs. Then $X \setminus (f \cdot g)^{-1}(0) \subset H(f \cdot g)$.*

Proof. Let $x \in X \setminus (f \cdot g)^{-1}(0)$. Then $f(x) \neq 0 \neq g(x)$ and $x \in H(f \cdot g)$. □

Lemma 5. *Let (X, d) be a metric space. Let $f \colon X \to \mathbb{R}$ be a positive function with the closed graph. Then there exists a positive continuous function $g \colon X \to \mathbb{R}$ such that $g \leq f$.*

Proof. Define a function $g \colon X \to \mathbb{R}$ as

$$g(x) = \inf\bigl\{f(y) + d(x, y) \colon y \in X\bigr\}.$$

We shall show that g has all required properties.

Let $x_1, x_2 \in X$ and let $\varepsilon > 0$. Then there is $y \in X$ such that $g(x_2) + \varepsilon > f(y) + d(x_2, y)$ and evidently $g(x_1) \leq f(y) + d(x_1, y)$. Then $g(x_1) - g(x_2) < f(y) + d(x_1, y) - f(y) - d(x_2, y) + \varepsilon \leq d(x_2, y) + d(x_1, x_2) - d(x_2, y) + \varepsilon = d(x_1, x_2) + \varepsilon$.

Since ε is arbitrary we obtain $g(x_1) - g(x_2) \leq d(x_1, x_2)$ and similarly $g(x_2) - g(x_1) \leq d(x_1, x_2)$; therefore g is continuous. Since $f(x) \in \bigl\{f(y) + d(x, y) \colon y \in X\bigr\}$ we have $g(x) \leq f(x)$.

Let $x \in X$. Let $m \in \mathbb{N}$ be such that $m > f(x)/2$ and $1/m < f(x)/2$. Then there exists a δ, $0 < \delta < f(x)/2$ such that $f(y) \in \bigl(f(x) - 1/m, f(x) + 1/m\bigr) \cup (m, \infty)$ for each $y \in S(x, \delta) = \bigl\{z \in X \colon d(x, z) < \delta\bigr\}$. If $y \in S(x, \delta)$ then $f(y) + d(x, y) \geq f(y) > f(x) - 1/m > f(x) - f(x)/2 = f(x)/2 > \delta$ or $f(y) + d(x, y) \geq f(y) > m > \delta$. If $y \in X \setminus S(x, \delta)$ then $f(y) + d(x, y) > d(x, y) \geq \delta$. Therefore for each $y \in X$ we have $f(y) + d(x, y) > \delta$ and this yields $g(x) \geq \delta > 0$. □

Now we shall characterize products of closed graphs functions. Denote by

$$\mathcal{H} = \bigl\{h \colon X \to \mathbb{R};\ h^{-1}(0) \text{ is closed and } X \setminus h^{-1}(0) \subset H(h)\bigr\}.$$

Lemma 6. *Let X be a topological space. Then $\mathcal{H} \cdot \mathcal{H} = \mathcal{H}$.*

Proof. Let $f, g \in \mathcal{H}$. Then $(f \cdot g)^{-1}(0) = f^{-1}(0) \cup g^{-1}(0)$ is a closed sets. If $(f \cdot g)(x) \neq 0$ then $f(x) \neq 0 \neq g(x)$ and hence $x \in H(f) \cap H(g)$. According to Lemma 2 we obtain $x \in H(f \cdot g)$. □

THEOREM 3. *Let (X, d) be a metric space. Then $\mathcal{U} \cdot \mathcal{U} = \mathcal{H} = \mathcal{U} \cdot \mathcal{C}$.*

Proof. Evidently $\mathcal{C} \subset \mathcal{U} \subset \mathcal{H}$ and hence by Lemma 6 we have $\mathcal{U} \cdot \mathcal{C} \subset \mathcal{U} \cdot \mathcal{U} \subset \mathcal{H}$.

Let $f \in \mathcal{H}$. Put $A = h^{-1}(0)$. The function $t \colon X \to \mathbb{R}$, $t(x) = \operatorname{dist}(x, A) = \inf\{d(x, y) \colon y \in A\}$, is continuous. The function $s = \left|h\restriction(X \setminus A)\right| \colon X \setminus A \to \mathbb{R}$ has the closed graph by Lemma 1 and $s(x) > 0$ for each $x \in X \setminus A$. According to Lemma 5 there is a continuous positive function $p \colon X \setminus A \to \mathbb{R}$ such that $0 < p \leq s$. We can assume that $p \leq 1$ (otherwise we can take $\max\{p, 1\}$). Define functions $f, g \colon X \to \mathbb{R}$ as

$$f(x) = \begin{cases} t(x) \cdot p(x) & \text{for} \quad x \in X \setminus A, \\ 0 & \text{for} \quad x \in A, \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{t(x)} \cdot \frac{1}{p(x)} \cdot h(x) & \text{for} \quad x \in X \setminus A, \\ 0 & \text{for} \quad x \in A. \end{cases}$$

Then evidently $f \cdot g = h$. We shall show that f is continuous and g has the closed graph.

Let $x \in A$ and $\varepsilon > 0$. Then there is a neighbourhood U of x such that $0 \leq t(y) < \varepsilon$ for each $y \in U$. We have $f(y) = 0$ for $y \in U \cap A$ and $0 \leq f(y) = t(y) \cdot p(y) \leq t(y) < \varepsilon$ for $y \in U \setminus A$, thus f is continuous at $x \in A$. Since A is closed and f is continuous on $X \setminus A$, f is continuous on X.

If $x \in X \setminus A$ then $1/p$ and $1/t$ are continuous at x and $x \in H(h)$. Since $1/p(x) \neq 0$, $1/t(x) \neq 0$ and $h(x) \neq 0$, according to Lemma 1 we have $x \in H\left(\frac{1}{t} \cdot \frac{1}{p} \cdot h\right)$, therefore $x \in H(g)$.

Now let $x \in A$ and let $m \in \mathbb{N}$. Then there is a neighbourhood U of x such that $0 \leq t(y) < 1/m$ for each $y \in U$. Let $y \in U$. If $y \in U \cap A$ then $g(y) = 0 \in \left(g(x) - 1/m,\, g(x) + 1/m\right)$. If $y \in U \setminus A$ then $\frac{1}{t}(y) > m$, $\left|\frac{h(y)}{p(y)}\right| = \frac{s(y)}{p(y)} \geq 1$ and hence $\left|g(y)\right| > m$. Therefore $x \in H(g)$. $\square$

Remark 1. Theorem 2 is not true for an arbitrary topological space. Let X be an uncountable set with the topology $\mathcal{T}$, where a set A is closed iff A is countable or $A = X$. Then every function with closed graph is constant. (If $f \colon X \to \mathbb{R}$ has closed graph then there exists $n \in \mathbb{N}$ such that the set $f^{-1}\left([-n, n]\right)$ is uncountable. If there is $x \in X$ with $f(x) \notin [-n, n]$ then $\left|f(x)\right| - 1/m > n$ for some $m > n$. Now for $V = (-\infty, -m) \cup \left(f(x) - 1/m, f(x) + 1/m\right) \cup (m, \infty)$ and for each neighbourhood U of x we have $f(U) \not\subset V$, therefore $x \notin H(f)$. Thus $f(X) \subset [-n, n]$ and this implies that f is continuous. Hence f must be constant.) On the other hand, $f \colon X \to \mathbb{R}$, $f(a) = 0$ for some $a \in X$ and $f(x) = 1$ otherwise, belongs to $\mathcal{H} \setminus \mathcal{U}$ and hence $\mathcal{U} \cdot \mathcal{U} = \mathcal{U} \neq \mathcal{H}$.

Denote by $\mathcal{S} = \left\{h \colon X \to \mathbb{R};\ h^{-1}(0) \text{ is closed and } h \in \mathcal{P}\right\}$.

THEOREM 4. *Let X be a perfectly normal space. Then $\mathcal{U}/\mathcal{U} = \mathcal{S}$ and $\mathcal{U}^+/\mathcal{U}^+ = \mathcal{S}^+$.*

Proof. Let $f \in \mathcal{S}$. Therefore $f \in \mathcal{P}$ and the set $A = f^{-1}(0)$ is a closed set. Let $\{X_n\}_{n=1}^{\infty}$ be a increasing sequence of closed sets such that $\bigcup_{n=1}^{\infty} X_n = X$ and $f \restriction X_n$ are continuous for each $n \in \mathbb{N}$. Put $F_0 = A$ and $F_n = X_n \cup A$ for $n \in \mathbb{N}$. Then F_i are closed sets, $X = \bigcup_{i=0}^{\infty} F_i$ and $f \restriction F_i$ are continuous for each $i \geq 0$. Further, for each $i \in \mathbb{N}$ put $E_i = F_i \setminus F_{i-1}$ and let $t_i \colon X \to \mathbb{R}$ be a nonnegative continuous function such that $t_i^{-1}(0) = F_{i-1}$. If $h_i = \min\{t_1, t_2, \dots, t_i\}$, then h_i are continuous nonnegative functions, $h_{i+1} \leq h_i$ and $h_i^{-1}(0) = F_{i-1}$. Define a function $g \colon X \to \mathbb{R}$ as $g(x) = 1/h_i(x)$ for $x \in E_i$ and a function $p \colon X \setminus A \to \mathbb{R}$ as $p(x) = \max\{1, 1/|f(x)|\}$. Then p and g are positive and piecewise continuous on $X \setminus A$.

Finally, define functions $f_1, f_2 \colon X \to \mathbb{R}$ as

$$f_1(x) = \begin{cases} g(x) \cdot p(x) \cdot f(x) & \text{for } x \in X \setminus A, \\ 0 & \text{for } x \in A. \end{cases}$$

$$f_2(x) = \begin{cases} g(x) \cdot p(x) & \text{for } x \in X \setminus A, \\ 1 & \text{for } x \in A. \end{cases}$$

Then evidently $f = f_1/f_2$. We shall show that f_1 and f_2 have the closed graph.

Let $x \in X$ and $m \in \mathbb{N}$.

If $x \in A$ then there is a neighbourhood V of x such that $0 \leq h_1(x) < 1/m$ for each $y \in V$.

Let $y \in V$.

If $y \in V \setminus A$ then $y \in E_i$ for some $i \in \mathbb{N}$. We have

$$|f_1(x)| \geq \left| g(y) \cdot \frac{1}{|f(y)|} \cdot f(y) \right| = \frac{1}{h_i(y)} \geq \frac{1}{h_1(y)} > m$$

and

$$f_2(y) \geq g(y) \geq \frac{1}{h_i(y)} > m\,.$$

If $x \in X \setminus A$ then $x \in E_k$ for some $k \in \mathbb{N}$. The function $1/h_k$ is continuous on $X \setminus F_{k-1}$ and the function f is continuous on F_k. Therefore $1/h_k$ and $1/|f|$ are continuous on $F_k \setminus F_{k-1}$ and $p \cdot g$ and $p \cdot g \cdot f$ are continuous on E_k. Now there is a neighbourhood V of x such that

(1) $0 \leq h_{k+1}(y) < 1/m$ for each $y \in V$,

(2) $|p(x) \cdot g(x) - p(y) \cdot g(y)| < 1/m$ for each $y \in V \cap E_k$ and

(3) $|p(x) \cdot g(x) \cdot f(x) - g(y) \cdot g(y) \cdot f(y)| < 1/m$ for each $y \in V \cap E_k$.

Moreover we can assume that $V \cap F_{k-1} = \emptyset$. Let $y \in V$.

If $y \in V \setminus F_k$ then $y \in E_i$ for some $i > k$. We have

$$\left| f_1(y) \right| \geq \left| g(y) \cdot \frac{1}{|f(y)|} \cdot f(y) \right| \geq \frac{1}{h_i(y)} \geq \frac{1}{h_{k+1}(y)} > m$$

and

$$f_2(y) \geq g(y) > m \,.$$

If $y \in V \cap F_k$ then $y \in F_k \setminus F_{k-1} = E_k$. By (5) and (6) we have

$$\left| f_1(x) - f_1(y) \right| < 1/m \qquad \text{and} \qquad \left| f_2(x) - f_2(y) \right| < 1/m \,.$$

Therefore $x \in H(f_1) \cap H(f_2)$.

The function f_2 is positive, and if f is positive then also f_1 is positive.

On the other hand $\mathcal{U}/\mathcal{U} \subset \mathcal{P}/\mathcal{P} = \mathcal{P}$ and $\mathcal{U}^+/\mathcal{U}^+ \subset \mathcal{P}^+/\mathcal{P}^+ = \mathcal{P}^+$. Moreover, if $f = f_1/f_2$, where $f_1, f_2 \in \mathcal{U}$ and $f_2(x) \neq 0$ for $x \in X$, then $f^{-1}(0) = f_1^{-1}(0)$ is a closed set. $\square$

Remark 2. For metric spaces we have $\mathcal{H} \subset \mathcal{S} \subset \mathcal{P}$ (all inclusions are proper). This is not true for an arbitrary topological space (the function f from Remark 1 belongs to $\mathcal{H} \setminus \mathcal{P}$).

REFERENCES

[1] BORSÍK, J.: *Local characterization of functions with closed graphs*, Demonstratio Math. **29** (1996), 643–650.

[2] DOBOŠ, J.: *On the set of points of discontinuity for functions with closed graphs*, Časopis Pěst. Mat. **110** (1985), 60–68.

[3] DOBOŠ, J.: *Sums of closed graphs functions*, Tatra Mountains Math. Publ. **14** (1998), 9–11.

[4] KIRCHHEIM, B.: *Baire one star functions*, Real Anal. Exchange **18** (1992–1993), 385–399.

Received September 29, 2000

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

E-mail: borsik@saske.sk