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SAC PROPERTY AND SEMICONTINUITY

FRANCISZEK PRUS-WIŚNIOWSKI

ABSTRACT. Semicontinuity of two auxiliary functions related to Z. Grande's SAC problem is studied. An example of unbounded approximately continuous function without SAC property is given.

The notion of the property SAC has been introduced by Z. Grande in [2] in connection with his investigation of equiderivatives and approximate equicontinuity. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have the property SAC if for every $\eta > 0$ there is an approximately continuous positive function $r\colon \mathbb{R} \to \mathbb{R}$ such that for every x and every h with $0 < |h| < r(x)$ we have

$$\left| \frac{1}{h} \int\limits_{x}^{x+h} f(t)\,dt - f(x) \right| < \eta\,.$$

Grande has proven that every function with the property SAC must be approximately continuous. He also asked whether the converse holds, that is, whether every approximately continuous function has the property SAC.

In an incomplete discussion of the above problem contained in [4] a function $p(x)$ appeared in a natural way. Below we redefine it in a slightly altered manner in order to avoid infinite values. Given an $\varepsilon > 0$, let

$$p_\varepsilon(x) = \sup \left\{ \gamma \in (0,1]\colon \forall h \neq 0 \ |h| < \gamma \Rightarrow \left| \frac{1}{h} \int\limits_{x}^{x+h} f(t)\,dt - f(x) \right| < \varepsilon \right\}.$$

PROPOSITION 1. *Let a function f be bounded and measurable. If f is continuous at a point x_0, then for any $\varepsilon > 0$ the function p_ε is lower semicontinuous at x_0.*

Proof. Denote the value $\sup\limits_{t \neq s} |f(t) - f(s)|$ by M. Given an $\varepsilon > 0$, we are going to prove that

$$\liminf_{x \to x_0} p_\varepsilon(x) \geq p_\varepsilon(x_0)\,.$$

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The continuity of f at x_0 implies that p_ε is positive there. Take any $\theta \in \big(0,\, p_\varepsilon(x_0)\big)$. Let $\delta \in \big(0, p_\varepsilon(x_0) - \theta\big)$ be such that $|x - x_0| \le \delta$ implies $\big|f(x_0) - f(x)\big| < \frac{\varepsilon}{2}$. Now take any x such that $|x - x_0| \le \frac{\delta}{2}$. Then for any h satisfying the condition $0 < |h| \le \frac{\delta}{2}$ we get

$$\left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)\right| \le \frac{1}{|h|}\int_{[x,x+h]} \Big(\big|f(t) - f(x_0)\big| + \big|f(x_0) - f(x)\big|\Big)\,dt$$

$$< \frac{1}{|h|}\int_{[x,x+h]} \Big(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\Big)\,dt = \varepsilon\,.$$

Therefore $p_\varepsilon(x) \ge \frac{\delta}{2}$ for $x \in \big(x_0 - \frac{\delta}{2},\, x_0 + \frac{\delta}{2}\big)$.

Since the value $\left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - f(x_0)\right|$ is a continuous function in h, it attains its maximum on the set $\big\{h\colon \frac{\delta}{2} \le |h| \le p_\varepsilon(x_0) - \theta\big\}$ and hence

$$\inf\left\{\varepsilon - \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - f(x_0)\right| : \frac{\delta}{2} \le |h| \le p_\varepsilon(x_0) - \theta\right\} > 0\,. \tag{1}$$

Denote half of the infimum by η. Take x such that

$$\big|f(x) - f(x_0)\big| < \frac{\eta}{2} \qquad \text{and} \qquad \frac{4|x - x_0|M}{\delta} < \frac{\eta}{2}\,. \tag{2}$$

Then for $|h| \in \big[\frac{\delta}{2},\, p_\varepsilon(x_0) - \theta\big]$ we get

$$\left|\left|f(x) - \frac{1}{h}\int_x^{x+h} f(t)\,dt\right| - \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - f(x_0)\right|\right|$$

$$\le \big|f(x) - f(x_0)\big| + \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt\right|$$

$$< \frac{\eta}{2} + \frac{2|x - x_0|M}{|h|} \le \frac{\eta}{2} + \frac{2|x - x_0|M}{\frac{\delta}{2}} < \eta\,,$$

and hence by (1)

$$\left|f(x) - \frac{1}{h}\int_x^{x+h} f(t)\,dt\right| \le \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - f(x_0)\right| + \eta < \varepsilon\,.$$

Thus for x satisfying (2) and $|x - x_0| < \frac{\delta}{2}$ and for any h with $|h| < p_\varepsilon(x_0) - \theta$ we get

$$\left| \frac{1}{h} \int\limits_{x}^{x+h} f(t)\,dt - f(x) \right| < \varepsilon\,.$$

Therefore $p_\varepsilon(x) \geq p_\varepsilon(x_0) - \theta$ for those x. Hence

$$\liminf_{x \to x_0} p_\varepsilon(x) \geq p_\varepsilon(x_0) - \theta\,.$$

Since θ was arbitrary, the proof is complete. $\square$

Observe that the function p_ε does not have to be upper semicontinuous under the assumptions of Proposition 1. Indeed, if we define $f(x)$ to be twice the distance from a point x to the set $(-\infty, 0] \cup [1, +\infty)$ and if $\varepsilon = \frac{1}{2}$, then we get

$$p_{\frac{1}{2}}(x) = \begin{cases} \frac{1}{2} & \text{for} \quad x \in \left\{0, \frac{1}{2}, 1\right\}, \\ 1 & \text{otherwise}\,. \end{cases}$$

However, by changing the second inequality in the definition of $p_\varepsilon(x)$ from a sharp one to a weak one, we obtain in the above example a function that is upper semicontinuous.

Proposition 2. *Let f be bounded and measurable. If f is continuous at a point x_0, then for all $\varepsilon > 0$ the function*

$$\bar{p}_\varepsilon(x) = \sup \left\{ \gamma \in (0,1] \colon \forall h \neq 0 \ |h| < \gamma \Rightarrow \left| \frac{1}{h} \int\limits_{x}^{x+h} f(t)\,dt - f(x) \right| \leq \varepsilon \right\}$$

is upper semicontinuous at x_0.

P r o o f. If $\limsup\limits_{x \to x_0} \bar{p}_\varepsilon(x) = 0$, then $\bar{p}_\varepsilon$ is upper semicontinuous at x_0 for it takes positive values everywhere.

Assume now that $\limsup\limits_{x \to x_0} \bar{p}_\varepsilon(x) = \delta > 0$ and take any $\gamma \in (0, \delta)$. Let a sequence $x_n \to x_0$, $x_n \neq x_0$, be such that $\bar{p}_\varepsilon(x_n) > \gamma$ for all n. Then

$$\left| \frac{1}{h} \int\limits_{x_n}^{x_n+h} f(t)\,dt - f(x_n) \right| \leq \varepsilon$$

for all $0 < |h| \leq \gamma$ and for all positive integers n. Thus for all $0 < |h| < \gamma$, passing to a limit with $n \to \infty$, we get

$$\left| \frac{1}{h} \int\limits_{x_0}^{x_0+h} f(t)\,dt - f(x_0) \right| \leq \varepsilon\,.$$

Hence $\bar{p}_\varepsilon(x) \geq \gamma$ which implies $\bar{p}_\varepsilon(x) \geq \delta$. The proof is complete. $\square$

We conclude the note with an example of a function showing that weakening both assumptions on f by requiring only that f is approximately continuous at x_0, does not guarantee lower semicontinuity of $p_\varepsilon(x)$. It does not have to be approximately lower semicontinuous, either. However, it is still an open question whether p_ε must be approximately lower semicontinuous at x_0 if f is approximately continuous there and bounded.

A lemma will be useful in constructing our example. Given two positive numbers b_1 and b_2 and two nonnegative numbers a_1 and a_2, we get

$$\frac{a_1 + a_2}{b_1 + b_2} = \frac{b_1}{b_1 + b_2} \cdot \frac{a_1}{b_1} + \left(1 - \frac{b_1}{b_1 + b_2}\right) \frac{a_2}{b_2}.$$

Thus $\frac{a_1+a_2}{b_1+b_2}$ is a point from the interval having $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ as endpoints (possibly a degenerate interval), and hence

$$\min_{i=1,2} \frac{a_i}{b_i} \leq \frac{a_1 + a_2}{b_1 + b_2} \leq \max_{i=1,2} \frac{a_i}{b_i}.$$

These inequalities can be easily generalized to the following form.

Lemma 1. *Let $\sum_i a_i$ and $\sum_i b_i$ be convergent series, the first with nonnegative terms, the second one with positive terms. Then for any $k = 2, 3, \ldots, \infty$ the following inequalities hold*

$$\inf_{1 \leq i \leq k} \frac{a_i}{b_i} \leq \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \leq \sup_{1 \leq i \leq k} \frac{a_i}{b_i}.$$

We now start the construction of our example. Denote the interval $\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ by P_n and divide it into $2n$ subintervals $P_n^1, \ldots, P_n^{2n}$ of equal length and denote their centers by c_n^i, $i = 1, \ldots, 2n$, respectively. Let I_n^i, $i = 1, \ldots, 2n$, be an interval of length $\frac{1}{n^2 2^n}$ centered at c_n^i. For every odd i, $1 \leq i \leq 2n-1$, define $f(x)$ on P_n^i to be $n^3 2^{n+1}$ times the distance from x to the set $P_n^i \setminus I_n^i$. For every even i, $2 \leq i \leq 2n$ define $f(x)$ on P_n^i to be minus $n^3 2^{n+1}$ times the distance from x to the set $P_n^i \setminus I_n^i$. Finally, set $f(x) = 0$ for all x outside $(0, 1]$. In particular, we got $f(c_n^i) = n$ for every odd $i \in \{1, \ldots, 2n\}$ and $f(c_n^i) = -n$ for every even $i \in \{1, \ldots, 2n\}$.

It is easy to see that f is continuous except for right-hand discontinuity at 0. We are now going to show that f is approximately continuous at 0. Denoting the set $\{x \colon f(x) = 0\}$ by A and its complement by CA, we get

$$\frac{\mu(CA \cap P_n)}{|P_n|} = \frac{2n \cdot \frac{1}{n^2 2^n}}{\frac{1}{2^n}} = \frac{2}{n}.$$

Given an $\eta > 0$, let $n = n(\eta)$ be such that $\eta \in P_n$. Then

$$\frac{\mu(CA \cap [0,\eta])}{\eta} \le \frac{\mu(CA \cap [0,\ \frac{1}{2^{n-1}}])}{\frac{1}{2^n}} = \frac{\sum\limits_{k \ge n} \mu(CA \cap P_k)}{\sum\limits_{k \ge n} \frac{1}{2^{k+1}}}.$$

Hence by our lemma

$$\frac{\mu(CA \cap [0,\ \eta])}{\eta} \le \sup_{k \ge n} \frac{\mu(CA \cap P_k)}{\frac{1}{2^{k+1}}} = \sup_{k \ge n} 2 \cdot \frac{2}{k} = \frac{4}{n}$$

which tends to 0 as $\eta \to 0+$ for then $n = n(\eta) \to +\infty$. Therefore 0 is a density point of the set A which implies approximate continuity of f at 0.

Given an $x \in P_n^i \setminus I_n^i$, we get

$$\left| \frac{\int\limits_{x_n}^{c_n^i} f(t)\,dt}{c_n^i - x_n} \right| \ge \frac{\int\limits_{\inf I_n^i}^{c_n^i} f(t)\,dt}{\frac{1}{2 \cdot 2^n \cdot 2n}} = 1\,.$$

Hence for all positive $\varepsilon \le 1$

$$p_\varepsilon(x) \le |x - c_n^i| \le \frac{1}{n^2 2^{n+1}}\,.$$

Further, since

$$A = \bigcup_n \bigcup_{i=1}^{2n} P_n^i \setminus I_n^i$$

and since 0 is a density point of the set A, we get

$$\operatorname{app} \liminf_{x \to 0} p_\varepsilon(x) = \lim_{A \ni x \to 0} p_\varepsilon(x) = 0$$

for any $\varepsilon \in (0,1]$.

On the other hand, since $\int\limits_{P_n^i \cup P_n^{i+1}} f(t)\,dt = 0$ for all positive integers n and all $i = 1, \dots, 2n-1$, we get for $\eta \in P_n$

$$\frac{\int\limits_0^\eta f(t)\,dt}{\eta} = \frac{\int\limits_{\inf P_n}^\eta f(t)\,dt}{\eta} \le \frac{\int\limits_{I_n^1} f(t)\,dt}{\frac{1}{2^n}} = \frac{1}{2n}\,.$$

Hence $p_\varepsilon(0) \geq \frac{1}{2^{k-1}}$ for $\varepsilon \in \left(\frac{1}{2k}, \frac{1}{2(k-1)}\right]$. Thus

$$p_\varepsilon(0) > \operatorname{app}\liminf_{x\to 0} p_\varepsilon(x) = 0$$

showing that for any $\varepsilon \in (0,1]$ the function p_ε is not approximately lower semicontinuous at 0.

Note that the function f, although unbounded at 0, is in a sense as close to being a bounded function as it gets. Namely, f is p-integrable for any $p \in (0,+\infty)$. Indeed

$$\int |f(t)|^p \, dt = \sum_n \sum_{i=1}^{2n} \frac{n^p}{2n^2 2^n} = \sum_n \frac{n^{p-1}}{2^n} < +\infty$$

for any $p \in (0,+\infty)$.

Finally, observe that the above function f is also an example of an unbounded approximately continuous function not having the SAC property. In the light of Proposition 2 from [4] it is of some interest to know if weakening only the assumption about continuity at x_0 by assuming that f is approximately continuous there, and retaining the assumptions about boundedness of f allows to conclude that p_ε is approximately lower semicontinuous at x_0.

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Institute of Mathematics
Szczecin University
ul. Wielkopolska 15
PL–70-451 Szczecin
POLAND
E-mail: wisniows@univ.szczecin.pl