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MOMENT PROBLEM FOR SOME OPERATORS

Miloslav Duchoň — Ján Haluška

ABSTRACT. We present a generalized moment problem in the context of majorized operators on the space of continuous functions. A generalization of the Hausdorff moment theorem is formulated and proved, and there are given some corollaries of it.

1. Introduction

If g is a real-valued or complex-valued function of bounded variation on the unit interval I of the real line, the numbers

$$a_k = \int_0^1 t^k dg(t), \qquad k \in \mathbb{N}.$$

are called the (power) *moments* of g.

For every sequence of real or complex numbers a_k, $k \in \mathbb{N}$ and every pair of non-negative integers n, k, set

$$\nabla^n a_k := \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{k+j}.$$

It is known [W, H] that the sequence $(c_k,\ k \in \mathbb{N})$ of complex numbers is a sequence of moments of some complex function of bounded variation on I or, equivalently, of some complex measure μ on borelian sets in I, if and only if there exists a positive finite constant M such that

$$\sum_{k=0}^{n} \binom{n}{k} \left|\nabla^{n-k} c_k\right| \le M, \qquad n = 0, 1, \ldots$$

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Recall that $\nabla^m c_k$ can be written as the mth differences for the sequence (c_k) therefore defined inductively by equalities

$$\nabla^{m+1} c_k = \nabla^m c_k - \nabla^m c_{k+1}\,, \quad \nabla^0 c_k = c_k\,, \qquad m = 0, 1, \dots;\ k = 0, 1, \dots \tag{1}$$

In this paper we consider a question of moments of majorized mappings on the space of continuous functions into a Banach space.

2. Moment problem

Recall [D] that the function $g(t)\colon [a,b] \to Y$, Y being a Banach space, is said to have bounded variation $|g|$ if $\sup \sum\limits_j \|g(s_j) - g(s_{j-1})\| < \infty$, where all possible partitions of $[a,b]$ are considered. We can formulate a task in the considered case as follows: Decide under which conditions there exists such a function of bounded variation $g(t)\colon [a,b] \to Y$ that

$$\int\limits_a^b t^n \, dg(t) = y_n\,, \qquad n = 0, 1, \dots \tag{2}$$

Now we require to recall the definition of a majorized mapping. For each subset A of $[a,b]$, let $C([a,b], A)$ denote the space of continuous functions on $[a,b]$ vanishing outside A. If $F\colon C([a,b]) \to X$ is a linear mapping, define for each A,

$$|||F_A||| = \sup \sum \|F(\psi_i)\|\,,$$

where supremum is over all finite families ψ_i in $C\big([a,b], A\big)$ with $\sum |\psi_i| \le \chi_A(t)$ for all t in $[a,b]$.

If $F\colon C\big([a,b]\big) \to X$ is a bounded linear mapping, then it is called majorized (also dominated, [D]) if

$$|||F_A||| < \infty$$

for all A in $B\big([a,b]\big)$.

An equivalent definition is as follows. If $F\colon C\big([a,b]\big) \to X$ is a linear mapping, it is majorized (dominated) if and only if there exists a nonnegative Borel measure μ on $B([0,1])$ such that

$$\|F(\psi)\| \le \int\limits_a^b |\psi(t)| \, d\mu(t)\,, \qquad \psi \in C\big([a,b]\big)\,.$$

Some authors use the term absolutely summing mapping for majorized mapping. It is known [T, D] that if $F\colon C\big([a,b]\big) \to X$ is a majorized linear mapping, then it can be extended to all bounded Borel real functions on $[a,b]$, in particular to characteristic functions of Borel sets.

We shall derive a concrete result relating to a power moment problem in the interval $[0,1]$.

THEOREM. *In order that there exists a function of bounded variation* $g(t)$: $[0,1] \to Y$ *such that*

$$\int_0^1 t^n \, dg(t) = y_n \,, \qquad y_n \in Y,\ n = 0, 1, \dots \,, \tag{3}$$

it is necessary and sufficient that there exists a positive finite constant M *such that*

$$\sum_{k=0}^{n} \binom{n}{k} \left\| \nabla^{n-k} y_k \right\| \le M \,, \qquad n = 0, 1, \dots \,, \tag{4}$$

P r o o f . Let the moment problem (3) be solved. Denote by L a bounded linear operator on the space $C[0,1]$ generated by a function of bounded variation, $g(t)$: $[0,1] \to Y$,

$$L(f) = \int_0^1 f(t) \, dg(t) \,, \qquad f \in C[0,1] \,.$$

Put

$$x_k^{(m)}(t) = t^k (1-t)^m, \qquad m, k = 0, 1, \dots \tag{5}$$

Since

$$x_k^{(m+1)}(t) = t^k (1-t)^{m+1} = t^k (1-t)^m - t^{k+1}(1-t)^m = x_k^{(m)}(t) - x_{k+1}^{(m)}(t) \,,$$

we have

$$L\big(x_k^{(m+1)}\big) = L\big(x_k^{(m)}\big) - L\big(x_{k+1}^{(m)}\big) \,, \qquad m, k = 0, 1, \dots$$

Further

$$L\big(x_k^{(0)}\big) = y_k \,.$$

If we take into the consideration (5), we can easily see (by induction) that

$$L\big(x_k^{(m)}\big) = \nabla^m y_k \,, \qquad m, k = 0, 1, \dots$$

From this we deduce that (4) is satisfied. For we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\|\nabla^{n-k}y_k\| &= \sum_{k=0}^{n}\binom{n}{k}\left\|\int_0^1 x_k^{(n-k)}(t)\,dg(t)\right\| \\ &\le \sum_{k=0}^{n}\binom{n}{k}\int_0^1 \left|x_k^{(n-k)}(t)\right| d|g|(t) \\ &= \int_0^1\left(\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\right)d|g|(t) \\ &= \int_0^1 \big[t+(1-t)\big]^n d|g|(t) \le |g|\big([0,1]\big)\,, \qquad n=1,2,\dots\,,\end{aligned}$$

which proves the necessity of the condition putting $M = |g|\big([0,1]\big)$.

The sufficiency. Let L_0 denote the operator defined on the set of functions (x_n), $x_n(t) = t^n$, $n = 0,1,\dots$ into Y by formula $L_0(x_n) = y_n$, $n = 0,1,\dots$ Extend L_0 to the linear hull of the set (x_n), i.e., to the set of all polynomials. Namely if $x(t) = c_0 + c_1t + \dots + c_nt^n$, we put

$$L(x) = c_0y_0 + c_1y_1 + \dots + c_ny_n\,.$$

Since the functions x_n are linearly independent, the definition of L is unambiguous.

The operator L defined above is clearly additive and homogeneous. We shall see that condition (4) implies that L is bounded.

Let us note that (even without this condition) the operator L is bounded on the set P_m of polynomials the degree of which is $\le m$, because P_m is finite-dimensional space (as coordinates we take coefficients of polynomial), hence the convergence in P_m is coordinatewise.

We have

$$L(x_k^{(s)}) = \nabla y_k^s\,, \qquad s,k = 0,1,\dots$$

Take any polynomial $p(t)$. Let the degree of $p(t)$ be m. Form the sequence of corresponding Bernstein polynomials (of $p(t)$)

$$p_n(t) = B_n(p;t) = \sum_{k=0}^{n}\binom{n}{k}p\left(\frac{k}{n}\right)t^k(1-t)^{n-k}.$$

It is well-known that the degree of the polynomial $p_n(t)$ for any $n = 1,2,\dots$ is not greater than m, and since $p_n(t)$ uniformly converges to $p(t)$ for $n\to\infty$, we have (according to remarks above) $L(p_n)\to L(p)$.

To obtain our required function, we shall prove that the mapping L is majorized (see [D], § 19, p. 380, 383).

To prove that

$$|||L_A||| = \sup \sum \|L(\psi_i)\|$$

is finite, we take for every ψ_i corresponding Bernstein polynomial

$$p_n^i(t) = B_n(\psi_i; t) = \sum_{k=0}^{n} \binom{n}{k} \psi_i\left(\frac{k}{n}\right) t^k (1-t)^{n-k}.$$

Then we have

$$\sum_i \|L(p_n^i)\| \le \sum_{k=0}^{n} \binom{n}{k} \left(\sum \left|\psi_i\left(\frac{k}{n}\right)\right|\right) \|\nabla^{n-k} y_k\| \le M\,, \quad n = 0, 1, \dots$$

It follows

$$\sum_i \|L(\psi_i)\| \le M\,.$$

Since L can be extended to the bounded Borel functions, we may define a function $f(t) = L(c_{[0,t]})$ with values in the Banach space Y. It can be verified that f has bounded variation. Now similarly as in [DDD] we can prove that

$$L(t^n) = \int_0^1 t^n df(t)\,, \qquad n \in \mathbb{N}\,.$$

□

If we take into consideration correspondence between functions and vector measures of finite variation [D], we obtain an equivalent form of the moment theorem.

Theorem. *In order that there exists a vector measure of bounded variation* $\boldsymbol{m}\colon B([0,1]) \to Y$ *such that*

$$\int_0^1 t^n d\boldsymbol{m}(t) = y_n\,, \qquad y_n \in Y,\ n = 0, 1, \dots\,, \tag{3'}$$

it is necessary and sufficient that there exists a constant M *such that*

$$\sum_{k=0}^{n} \binom{n}{k} \|\nabla^{n-k} y_k\| \le M\,, \qquad n = 0, 1, \dots \tag{4}$$

3. Representation theorem

We can obtain a representation theorem for majorized operators from the [D].

THEOREM M. *Every majorized linear operator L on $C(I)$ into Y is presentable in the form*

$$L(f) = \int_0^1 f(t)\, dq(t)\,, \tag{6}$$

where q is a function of bounded variation on I into Y. Conversely, every mapping of the form (6) *is majorized.*

Thus (6) represents majorized linear operators on $C(I)$ with values in Y.

P r o o f. Put

$$L(t^n) = y_n\,, \qquad n = 0, 1, \dots$$

Take

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} \|\nabla^{n-k} y_k\| &= \sum_{k=0}^{n} \binom{n}{k} \|L(t_k(1-t)^{n-k})\| \\ &= \sum_{k=0}^{n} \left\| L\left(\binom{n}{k} t_k(1-t)^{n-k}\right)\right\| \le M < \infty\end{aligned}$$

since L is majorized. Hence by preceding theorem

$$L(t^n) = \int_0^1 t^n dq(t)$$

for some function q of bounded variation on I with values in Y. By Weierstrass's theorem we can extend the last equality for every continuous function on I. □

We can also give an equivalent version.

THEOREM M'. *Every majorized linear operator L on $C(I)$ into Y is presentable in the form*

$$L(f) = \int_0^1 f(t)\, d\boldsymbol{m}(t) \tag{6'}$$

where $\boldsymbol{m}$ is a vector measure of bounded variation on $B(I)$ into Y. Conversely, every mapping of the form (6') *is majorized.*

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Miloslav Duchoň
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA
E-mail: duchon@mat.savba.sk

Ján Haluška
Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK– 040 01 Košice
SLOVAKIA
E-mail: jhaluska@saske.sk