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CRITERIA FOR DETERMINING COMPACTNESS OF THE INTEGRATION OPERATOR ASSOCIATED WITH A VECTOR MEASURE

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ABSTRACT. Criteria are presented for determining whether or not the integration operator $f \mapsto \int f dm$ is compact for a given particular vector measure m. We also present a method for constructing both compact and non-compact integration operators associated with ℓ^1-valued vector measures.

1. Introduction

Associated with each Banach space valued vector measure m is the corresponding Banach space $L^1(m)$ of all (scalar-valued) m-integrable functions. The continuous linear map which sends a function $f \in L^1(m)$ to its integral $\int f dm$ (an element of the underlying Banach space in which m takes its values) is called the *integration operator* associated with m. The importance of such maps stems from the fact that many classical kernel operators (e.g., the Fourier transform on $L^1([-\pi, \pi])$, Volterra operators, the Riemann–Liouville fractional integral, etc.) are of this form. The same is true for representations of cyclic Banach spaces induced by spectral measures, and so on. We refer to [2, 5, 9, 14, 15, 18], for example, and the references therein.

In the recent paper [16] a detailed study is made of the *compactness* of the integration operator I_m. In particular, such (compact) operators are characterized in terms of certain properties involving the variation of m. The aim of this note is to continue the investigation begun in [16]. Particular emphasis is given to formulating *sufficient conditions* which can effectively be used to determine whether or not I_m is compact for a given *particular* vector measure m; see Sections 3 and 4. The final Section 5 is devoted to outlining a procedure for the

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construction of certain ℓ^1-valued vector measures m for which it is possible to determine precisely when (or when not) I_m is compact.

2. Preliminaries

Let X be a (complex) Banach space with norm $\|\cdot\|$ and dual space X'. The unit sphere of X is denoted by $S[X] := \{x \in X : \|x\| = 1\}$. A subset W of X is said to be *w-seminorm dominated* if there exists a finite set $F \subseteq X'$ such that

$$\|x\| \le \sum_{x' \in F} \left|\langle x, x'\rangle\right| , \qquad x \in W .$$

This concept was introduced in [16] where it played an important role.

Let Σ be a σ-algebra of subsets of a non-empty set Ω and $m\colon \Sigma \to X$ be a vector measure, that is, m is a σ-additive set function. The variation of m is denoted by $|m|$. It is a (possibly infinite) non-negative measure on Σ; the definition of variation can be found, for example, in [7, Definition I.1.4]. For each $x' \in X'$, let $\langle m, x'\rangle$ denote the complex measure $E \mapsto \langle m(E), x'\rangle$, for $E \in \Sigma$; its variation $|\langle m, x'\rangle|$ is then a finite measure. A Σ-measurable function $f\colon \Omega \to \mathbb{C}$ is called *m-integrable* if it is $\langle m, x'\rangle$-integrable, for all $x' \in X'$, and if there is a set function $fm\colon \Sigma \to X$, necessarily unique, satisfying $\big\langle (fm)(E), x'\big\rangle = \int_E f d\langle m, x'\rangle$ for all $x' \in X'$ and $E \in \Sigma$. It follows from the Orlicz–Pettis theorem that fm is also a vector measure. The classical notation $\int_E f dm := (fm)(E)$, for $E \in \Sigma$, will also be used. The vector space of all m-integrable functions is denoted by $L^1(m)$. Define a seminorm on $L^1(m)$ by

$$\|f\|_m := \sup \left\{ \int_\Omega |f|\, d\big|\langle m,\, x'\rangle\big| : x' \in B[X'] \right\} , \qquad f \in L^1(m) ,$$

where for any Banach space Y we define $B[Y] := \{y \in Y : \|y\| \le 1\}$. The space $L^1(m)$ is then a complete seminormed space in which the set of all $\mathbb{C}$-valued, Σ-simple functions is dense; see [11, Chapter IV] or [12, Theorem 2.4]. Associated with m is its integration operator $I_m\colon L^1(m) \to X$ defined by

$$I_m f := \int_\Omega f\, dm , \qquad f \in L^1(m) .$$

Clearly, I_m is linear. Furthermore, the operator I_m is also continuous because of the inequality $\|I_m f\| \le \|f\|_m$ for every $f \in L^1(m)$.

For each set $E \in \Sigma$, denote the characteristic function of E by χ_E, and let $\|m\|(E) := \|\chi_E\|_m$. The so defined set function $\|m\|\colon E \mapsto \|\chi_E\|_m$, for $E \in \Sigma$, is called the *semivariation* of m. In particular, $\|m\|$ is always finite (unlike the variation $|m|$, in general). An element $f \in L^1(m)$ is called *m-null* if fm is the zero vector measure. This is equivalent to $\|f\|_m = 0$. The space of all m-null functions is denoted by $N(m)$; it is a closed subspace of $L^1(m)$. The quotient space $L^1(m)/N(m)$ equipped with the quotient norm induced by $\|\cdot\|_m$ is then a Banach space. Unless otherwise specified, the seminormed space $L^1(m)$ is identified with this quotient Banach space (in the usual manner). In fact, such a distinction is only needed in Section 3.

Sets $E \in \Sigma$ satisfying $\|m\|(E) = 0$ are called m-null. The m-null and $|m|$-null sets coincide, as is easily seen from the inequalities $\|m(E)\| \le \|m\|(E) \le |m|(E)$, for $E \in \Sigma$. A property which holds outside an m-null set is said to hold *m-almost everywhere* (briefly, m-a.e.). The space of all Σ-measurable and m-essentially bounded (i.e., bounded m-a.e.) functions is denoted by $L^\infty(m)$ and is equipped with the essential sup-norm $\|\cdot\|_\infty$. It is known that $L^\infty(m) \subseteq L^1(m)$ and that

$$\left\| \int_E f\, dm \right\| \le \|f\|_\infty \cdot \|m\|(E), \qquad f \in L^\infty(m),\ E \in \Sigma, \tag{2.1}$$

see [11, Theorem II.3.1] and [7, p.6]. The quotient Banach space of $L^\infty(m)$ with respect to its closed subspace $N(m)$ is also denoted by $L^\infty(m)$. In particular, $L^\infty\big(|m|\big) = L^\infty(m)$.

For the definition of *Bochner integrals* we refer to [7, Chapter II]. Let $\lambda\colon \Sigma \to [0,\infty)$ be a finite measure and let $\mathcal{B}(\lambda, X)$ denote the space of all X-valued, Bochner λ-integrable functions on Ω. Given $G \in \mathcal{B}(\lambda, X)$, the Bochner integral of G over a set $E \in \Sigma$ (with respect to λ) is denoted by (B)-$\int_E G\, d\lambda$ and is an element of X. The *indefinite Bochner λ-integral* of G is the vector measure $G \cdot \lambda$ on Σ given by $E \mapsto$ (B)-$\int_E G\, d\lambda$. Given a vector measure $m\colon \Sigma \to X$ with finite variation, if there exists $G \in \mathcal{B}(\lambda, X)$, necessarily unique, such that $m = G\cdot\lambda$, then G is called the *Radon–Nikodým derivative* of m with respect to λ and we write $G = dm/d\lambda$. A function $H\colon \Omega \to X$ is said to have *λ-essentially relatively compact range* if $H(\Omega \setminus E)$ is relatively compact in X for some λ-null set $E \in \Sigma$.

Given a vector measure m, the following result characterizes compactness of the integration operator I_m, [16, Theorems 1 and 4].

Theorem A. *Let X be a Banach space and $m\colon \Sigma \to X$ be a vector measure. Then the integration operator $I_m\colon L^1(m) \to X$ is compact if and only if both of the following conditions hold:*

(i) *the measure m has finite variation and admits a Radon–Nikodým derivative $G = dm/d|m| \in \mathcal{B}\big(|m|, X\big)$, and*

(ii) *the function G has $|m|$-essentially relatively compact range in X.*

In this case, the identity $L^1(m) = L^1(|m|)$ *necessarily holds, and* $I_m f =$ (B)-$\int_\Omega f \cdot G \, d|m|$ *for every* $f \in L^1(m)$.

3. Factorization of vector measures

Throughout this section let X be a Banach space and Σ be a σ-algebra of subsets of a non-empty set Ω. Let $m \colon \Sigma \mapsto X$ be a vector measure. The seminormed space $L^1(m)$ and its quotient space $L^1(m)/N(m)$ with respect to the closed subspace $N(m)$ of all m-null functions are *not* identified in the usual way, but will be considered as separate entities.

We say that the vector measure m *factors* through a Banach space Y if there exist a vector measure $\mu \colon \Sigma \to Y$ and a continuous linear map $T \colon Y \to X$ such that

(F1) $L^1(m) = L^1(\mu)$;

(F2) $N(m) = N(\mu)$; and

(F3) $I_m = T \circ I_\mu$.

In this case m is said to factor through Y *via* μ and T. The Banach spaces $L^1(m)/N(m)$ and $L^1(\mu)/N(\mu)$ are (topologically) isomorphic, [15, p. 178]. Moreover, m has finite variation whenever μ does and

$$|m|(E) \le \|T\| \cdot |\mu|(E), \qquad E \in \Sigma.$$

Let $\Pi_m \colon L^1(m) \to L^1(m)/N(m)$ denote the quotient map. Associated with m is the set function $[m] \colon \Sigma \to L^1(m)/N(m)$ defined by

$$[m](E) := \Pi_m(\chi_E), \qquad E \in \Sigma.$$

Thanks to the Lebesgue dominated convergence theorem for a vector measure, [11, Theorem II.4.2], the set function $[m]$ is a vector measure. Moreover,

$$L^1(m) = L^1([m]) \qquad \text{and} \qquad N(m) = N([m]). \tag{3.1}$$

The Beppo–Levi theorem, [11, Theorem II.4.1], is needed to prove the first equality in (3.1), and the second equality is obvious. There is a unique continuous linear operator $[I_m] \colon L^1(m)/N(m) \to X$ such that $I_m = [I_m] \circ \Pi_m$. Since

$$\big\|[m](E)\big\| \le \|\Pi_m\| \cdot |m|(E), \qquad E \in \Sigma,$$

whenever m has finite variation, it follows in this case that $[m]$ also has finite variation.

The following lemma is clear from the definition of integrability with respect to a vector measure.

Lemma 3.1. *Let Y be a Banach space and $\nu\colon \Sigma \to Y$ be a vector measure. Suppose that $T\colon Y \to X$ be a continuous linear operator. Then the set function $T \circ \nu\colon \Sigma \to X$ is a vector measure such that*

(i) *$L^1(\nu) \subseteq L^1(T \circ \nu)$ and $N(\nu) \subseteq N(T \circ \nu)$; and*

(ii) *the integration operators I_ν and $I_{T\circ\nu}$ coincide on $L^1(\nu)$.*

The above lemma is needed to establish the following result.

Lemma 3.2. *Let $m\colon \Sigma \to X$ be a vector measure. Suppose that there exist a Banach space Y, and continuous linear operators $U\colon L^1(m)/N(m) \to Y$ and $T\colon Y \to X$ such that $[I_m] = T \circ U$. Then the vector measure m factors through Y via the vector measure $\mu := U \circ [m]\colon \Sigma \to Y$ and the map T.*

Proof. From Lemma 3.1 (i) applied to the measure $[m]$ and the operator U, it follows that

$$L^1\big([m]\big) \subseteq L^1(\mu) \qquad \text{and} \qquad N\big([m]\big) \subseteq N(\mu)\,. \tag{3.2}$$

For each $E \in \Sigma$, we have

$$m(E) = \big([I_m] \circ \Pi_m\big)(\chi_E) = \big(T \circ U \circ [m]\big)(E) = (T \circ \mu)(E)\,,$$

that is, $m = T \circ \mu$. Again by Lemma 3.1 (i)

$$L^1(\mu) \subseteq L^1(m) \qquad \text{and} \qquad N(\mu) \subseteq N(m)\,. \tag{3.3}$$

So, (F1) and (F2) hold by (3.1), (3.2) and (3.3). Finally (F3) follows from Lemma 3.1 (ii). □

When $[I_m]$ is compact, we have the following factorization.

Proposition 3.3. *Let $m\colon \Sigma \to X$ be a vector measure. The associated integration operator $[I_m]\colon L^1(m)/N(m) \to X$ is compact if and only if m factors through a closed linear subspace Y of the Banach space c_0, via some Y-valued measure of finite variation on Σ and a compact linear operator from Y into X.*

Proof. One direction is clear. So, assume that $[I_m]$ is compact. Apply [16, Theorem 4] (in which I_m and $[I_m]$ are identified) to deduce that m has finite variation. Then, as noted above, $[m]$ also has finite variation. Moreover, Theorem 16.5 of [1] implies that $[I_m]$ factors through a closed subspace Y of c_0 with $[I_m] = T \circ U$ for some compact linear operators $U\colon L^1(m)/N(m) \to Y$ and $T\colon Y \to X$. Lemma 3.2 guarantees that the measure m factors through Y via the measure $\mu := U \circ [m]\colon \Sigma \to Y$ and the map T. The inequalities

$$\big\|\mu(E)\big\| \le \|U\| \cdot \big\|[m](E)\big\| \le \|U\| \cdot \big|[m]\big|(E)\,, \qquad E \in \Sigma\,,$$

show that μ has finite variation. □

Problem 3.4. Given that $[I_m]$ is compact, does m actually factor through c_0?

4. Testing for compactness of I_m

In this section we exhibit criteria which can be used to decide, for a given *particular* vector measure m (of finite variation), whether or not I_m is compact. Unless specified, m is a vector measure on a σ-algebra Σ of subsets of a non-empty set Ω, taking its values in a Banach space X. We begin with three obvious conditions which, nevertheless, are useful in practice.

Firstly, if the range $m(\Sigma) \subseteq X$ of m is not relatively compact, then I_m cannot be compact. Or equivalently, if I_m is compact, then $m(\Sigma)$ is necessarily relatively compact. Actually, condition (i) of Theorem A (see Section 2) *itself* already implies the relative compactness of $m(\Sigma)$. To see this, note that condition (i) is equivalent to the requirement that the integration operator I_m restricted to the Banach space $L^\infty(m) = L^\infty(|m|)$ is nuclear (and hence compact); see [7, Theorem VI. 4. 4], for example. Then apply the fact that $m(\Sigma)$ is relatively compact if and only if I_m restricted to $L^\infty(m) = L^\infty(|m|)$ is compact, [7, Theorem VI.2.18].

These comments show that the compactness of I_m and the relative compactness of $m(\Sigma)$ are intimately related. The indefinite integral of every Pettis integrable function (see [7, p. 53] for the definition) with respect to a finite perfect measure necessarily has relatively compact range, [10, Proposition 3J]. This provides a good source of vector measures with relatively compact range. It may be interesting to note that there exist Banach spaces X for which *every* X-valued vector measure (not necessarily of finite variation) has relatively compact range. Indeed, since the range of every vector measure is relatively weakly compact, [7, Corollary I. 2. 7], this is obviously true if X is a Schur space (i.e., relatively weakly compact and relatively compact sets coincide in X). For further examples, note that every continuous linear operator from $L^\infty(\nu)$ into $L^q(\mu)$ is compact whenever $\mu \geq 0$ is a purely atomic measure, $\nu \geq 0$ is a measure and $1 \leq q < 2$, [19, Remark 2, p. 211]. It follows from this fact that the Banach spaces ℓ^p, $1 \leq p < 2$, also have the property that every ℓ^p-valued vector measure m has relatively compact range. Indeed, since I_m restricted to $L^\infty(|m|) = L^\infty(m)$ is continuous, it is (by the above comments) also compact and hence, $m(\Sigma)$ is relatively compact in ℓ^p. The spaces ℓ^p, for $p \geq 2$, do not have this property, [7, p. 275].

A second useful condition is the observation that if m satisfies $L^1(m) \neq L^1(|m|)$, then I_m cannot be compact; see Theorem A. For an example of this phenomenon we refer to [14, Remark 2 (d), p. 296]. If it happens to be the case that $L^1(m) = L^1(|m|)$, then no general conclusion about the compactness of I_m is possible; see Proposition 5.1 (in Section 5) or [16, Example 3.2 and Remark 3.3 (i)].

If there exists a non-empty finite set $F \subseteq X'$ such that

$$|m|(E) \leq \sum_{x' \in F} \big|\langle m,\, x'\rangle\big|(E)\,, \qquad E \in \Sigma\,, \tag{4.1}$$

then necessarily $L^1(m) = L^1\big(|m|\big)$, [16, Lemma 2.6 (i)]. So, a third useful condition is that if (4.1) fails to hold for each non-empty finite set $F \subseteq X'$, then I_m cannot be compact. To see this, observe that if I_m is compact, then by Theorem A the $S[X]$-valued function $dm/d|m| \in \mathcal{B}\big(|m|, X\big)$ has $|m|$-essentially relatively compact range and hence ($|m|$-a.e.) this range is w-seminorm dominated, [16, Lemma 2.4]. Then Lemma 2.6 (ii) in [16] implies that (4.1) must be satisfied for some non-empty finite set $F \subseteq X'$. The following example illustrates how the failure of (4.1) can be checked in practice.

Example 4.1. Let $X := \ell^p$, $1 < p < \infty$, and let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be the standard basis of X. Let $\Omega := \mathbb{N}$ and $\Sigma := 2^{\mathbb{N}}$. Define a vector measure (of finite variation) $m\colon \Sigma \to X$ by

$$m(E) := \sum_{n=1}^{\infty} 2^{-n} \delta_n(E)\, \mathbf{e}_n\,, \qquad E \in \Sigma\,,$$

where δ_n is the Dirac measure at each $n \in \Omega$. Then $|m| = \sum_{n=1}^{\infty} 2^{-n}\delta_n$. Suppose that there exists a finite subset $F := \{\xi_j \colon j = 1, \ldots, k\}$ of X' such that (4.1) holds. Since

$$\big|\langle m, \xi_j\rangle\big| = \sum_{n=1}^{\infty} 2^{-n} \big|\langle \mathbf{e}_n\,,\, \xi_j\rangle\big| \delta_n\,, \qquad 1 \leq j \leq k\,,$$

setwise on Σ, the inequality (4.1) implies that

$$\sum_{n=1}^{\infty} 2^{-n}\delta_n \leq \sum_{n=1}^{\infty} 2^{-n} \left(\sum_{j=1}^{k} \big|\langle \mathbf{e}_n\,,\, \xi_j\rangle\big| \right) \delta_n\,,$$

on Σ. Accordingly, $\Sigma_{j=1}^{k} \big|\langle \mathbf{e}_n\,,\, \xi_j\rangle\big| \geq 1$ for all $n \in \mathbb{N}$. But, this is impossible since $\langle \mathbf{e}_n\,,\, \xi_j\rangle \to 0$ as $n \to \infty$, whenever $1 \leq j \leq k$. This shows that (4.1) cannot hold for any finite set $F \subseteq X'$. In particular, I_m is not compact.

Since the closed unit ball of an infinite-dimensional Banach space is never compact, the following result is clear.

Proposition 4.2. *Let X be an infinite-dimensional Banach space and $m\colon \Sigma \to X$ be a vector measure such that $L^1(m)$ is infinite-dimensional. If the integration operator $I_m \colon L^1(m) \to X$ is an isomorphism onto its range, then I_m is not compact.*

Large classes of vector measures m whose integration operator I_m is an isomorphism onto its range $I_m\big(L^1(m)\big) \subseteq X$ occur in the theory of cyclic Banach spaces, as evaluations of spectral measures at certain points in X; see [9] and [14, Example 3]. Some of these measures m have finite variation while others do not. In all cases, Proposition 4.2 shows that the integration operator I_m cannot be compact whenever $L^1(m)$, or equivalently the associated cyclic subspace of X, is infinite-dimensional.

In many cases, Banach space properties of $L^1(m)$ and/or X play a crucial role in determining the compactness of I_m. The criteria of the next result were effectively applied in [14, Example 1].

PROPOSITION 4.3. *Let $m\colon \Sigma \to X$ be a vector measure such that the Banach space $L^1(m)$ has the Dunford–Pettis property. Suppose that there exists a sequence $\{f_n\}_{n=1}^{\infty} \subseteq L^1(m)$ which converges weakly to zero and satisfies $\inf_{n\in\mathbb{N}} \|I_m(f_n)\| > 0$. Then the integration operator $I_m\colon L^1(m) \to X$ is not weakly compact, and hence, also not compact.*

Proof. If I_m were weakly compact, then the Dunford–Pettis property of $L^1(m)$ would ensure that $\{I_m(f_n)\}_{n=1}^{\infty}$ is norm convergent to zero in X by the Brace–Grothendieck theorem, [7, p. 177]. This contradicts the hypotheses. □

REMARK 4.4. (i) If X is a Schur space and $m\colon \Sigma \to X$ is any vector measure with σ-finite variation, then $L^1(m)$ has the Dunford–Pettis property, [4, Theorem 4]. This is the case for the vector measures constructed in Section 5.

(ii) If $L^1(m)$ is an AL-space, then it has the Dunford-Pettis property, [1, Theorem 19.7]. Those vector measures m for which $L^1(m)$ is an AL-space have been investigated in [3].

Another effective criterion concerning the compactness of I_m is the following one; see [4, Claim, p.3800] and [16, Corollary 4.1].

PROPOSITION 4.5. *Let $m\colon \Sigma \to X$ be a vector measure such that the Banach space $L^1(m)$ is infinite-dimensional. If the integration operator $I_m\colon L^1(m) \to X$ is compact, then m has finite variation and $L^1(m)$ contains a complemented subspace isomorphic to ℓ^1. Equivalently, if $L^1(m)$ does not contain a complemented subspace isomorphic to ℓ^1, then I_m is not compact.*

It is worthwhile to make some comments on how Proposition 4.5 is used in practice. Banach spaces containing a *complemented* subspace isomorphic to ℓ^1 can be characterized by the Bessaga–Pelczyński theorem, [6, Theorem V.10]. Recalling the fact that $L^1(m)$ is also a Banach lattice, [2, p. 318], let us record some useful equivalences:

(a) *The Banach space $L^1(m)$ contains a complemented subspace isomorphic to ℓ^1.*

(b) *The Banach lattice $L^1(m)$ contains a sublattice which is lattice-isomorphic to ℓ^1.*

(c) *The dual Banach lattice* $\big(L^1(m)\big)'$ *does not have order continuous norm.*

The equivalence of these three statements, together with further equivalent conditions, can be found, for example, in [1, Theorem 14.21] or [13, Theorem 2.4.14]. It may be interesting to note that the equivalence of (a) and (c) is used for the proof of Proposition 4.5 above as given in [4].

Banach spaces containing a complemented subspace isomorphic to ℓ^1, of course, also contain an isomorphic copy of ℓ^1. Again by taking account of the fact that $L^1(m)$ is a Banach lattice, the following conditions are equivalent:

(d) *The Banach space* $L^1(m)$ *does not contain a subspace isomorphic to* ℓ^1.

(e) *The dual space* $\big(L^1(m)\big)'$ *has the Radon–Nikodým property.*

(f) *The dual space* $\big(L^1(m)\big)'$ *has the weak Radon–Nikodým property.*

(g) *The Banach space* $L^1(m)$ *is an Asplund space.*

The equivalence of (d) and (e) is given in [13, Theorem 5.4.14]. The definition of the Radon–Nikodým property can be found, for example, in [7, Definition III. 1.3]. The weak Radon–Nikodým property is the Pettis integral version of the Radon–Nikodým property; see [20, Definition 5.8]. For the equivalence of (d) and (f), see [20, Chapter IV]. The definition of Asplund spaces and the equivalence of (e) and (f) can be found, for example, in Definition 1.22 and Theorem 5.7 of [17], respectively. For further information on Banach spaces not containing a copy of ℓ^1 we refer to [6, Chapters XI–XIII] and [20].

By Proposition 4.5, any one of the equivalences (d) to (g) implies that I_m is not compact. In particular, I_m cannot be compact if any of the following conditions holds:

(h) $L^1(m)$ *is reflexive.*

(i) $L^1(m)$ *has a separable dual.*

(j) $L^1(m)$ *is isomorphic to the Banach sequence space* c_0.

(k) *Every separable subspace of* $L^1(m)$ *has a separable dual.*

In fact, either (h) or (i) implies (e); see [7, p. 218]. Clearly (j) implies (d). Moreover, (k) is equivalent to (g), [17, Corollary 2.15]. That (h) implies (d) can also be seen from the fact that a Banach lattice is reflexive if and only if it does not contain a subspace isomorphic to c_0 or ℓ^1, [13, Theorem 2.4.15].

We point out that the converse of Proposition 4.5 is not valid, in general. To see this, let $\Omega := \mathbb{N}$ and $\Sigma := 2^\Omega$. Then, with $X := \ell^1$ and $\{\mathbf{e}_n\}_{n=1}^\infty$ the standard basis of X, it is routine to verify that the vector measure $m\colon \Sigma \to X$ given by

$$m(E) := \sum_{n=1}^{\infty} n^{-2} \delta_n(E)\, \mathbf{e}_n\,, \qquad E \in \Sigma\,,$$

has finite variation and satisfies $L^1(m) = \left\{f \in \mathbb{C}^\Omega : \sum_{n=1}^{\infty} |f(n)| n^{-2} < \infty\right\}$. However, I_m turns out to be an isomorphism of $L^1(m)$ onto X and so, by Proposition 4.2 the operator I_m cannot be compact.

For an example of a different kind, consider the Volterra vector measure (of finite variation) m given in Example 3.2 of [16]; see also Example 2 of [14]. It is shown in [14] that I_m is not compact, while $L^1(m) = L^1(|m|)$ with $|m|(E) = \int_E (1-s)\,ds$, for E a Borel subset of $[0,1]$. Since the dual Banach lattice $\left(L^1(m)\right)' = L^\infty([0,1])$ does not have order continuous norm, it follows from the equivalence (a) $\iff$ (c) that $L^1(m)$ contains a complemented copy of ℓ^1.

5. Construction of compact integration operators

It is shown in Theorem 2 of [16] that in every infinite-dimensional Banach space X there exists a non-trivial X-valued vector measure m whose integration operator $I_m \colon L^1(m) \to X$ is compact. We now exhibit a construction (quite different from that used in the proof of [16, Theorem 2]) of ℓ^1-valued vector measures of finite variation whose integration operators can be "arranged" to be either compact or non-compact!

Let $\Omega := [0,1]$ and λ be Lebesgue measure on the Borel σ-algebra Σ of Ω. Let $X := \ell^1$ and $\{\mathbf{e}_n\}_{n=1}^{\infty}$ denote its standard basis of unit vectors. The norms on X and its dual space $X' = \ell^\infty$ are denoted by $\|\cdot\|_X$ and $\|\cdot\|_{X'}$, respectively.

Let $\{h_n\}_{n=1}^{\infty}$ be any unconditionally summable sequence in the Banach space $L^\infty(\lambda)$. Since the natural injection $\Phi \colon L^\infty(\lambda) \to L^1(\lambda)$ is absolutely 1-summing (see [8, Example 2.9 (d)]), the sequence $\left\{\Phi(h_n)\right\}_{n=1}^{\infty}$ is absolutely summable in $L^1(\lambda)$, [7, Proposition VI. 3.2]. The norms on $L^\infty(\lambda)$ and $L^1(\lambda)$ are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_1$, respectively. For simplicity we write $h_n = \Phi(h_n)$ for all $n \in \mathbb{N}$. Since $\{h_n\}_{n=1}^{\infty}$ is absolutely summable in $L^1(\lambda)$, there is a λ-null set $E_0 \in \Sigma$ such that $\Sigma_{n=1}^{\infty} |h(\omega)| < \infty$ for each $\omega \in \Omega \setminus E_0$. This allows us to define a function $H \colon \Omega \to X$ by

$$H(\omega) = \sum_{n=1}^{\infty} h_n(\omega)\,\mathbf{e}_n \tag{5.1}$$

for every $\omega \in \Omega \backslash E_0$ and by $H(\omega) = 0$ for every $\omega \in E_0$. Since $\langle H(\cdot), x' \rangle$ is clearly Σ-measurable for each $x' \in X'$ and X is separable, the function H is strongly λ-measurable by the Pettis measurability theorem, [7, Theorem II.1.2].

Moreover, we have

$$\int_\Omega \big\|H(\cdot)\big\|_X \, d\lambda = \int_\Omega \sum_{n=1}^{\infty} |h_n| \, d\lambda = \sum_{n=1}^{\infty} \|h_n\|_1 < \infty \,,$$

which shows that $H \in \mathcal{B}(\lambda, X)$.

(I) We claim that $H(\Omega \setminus F_0)$ is bounded in X for some λ-null set $F_0 \in \Sigma$, and accordingly, we may assume that H has bounded range. To see this, choose a countable set $\{z_k \colon k = 1, 2, \ldots, \} \subseteq B[X']$ which is norming for X, that is, $\|x\|_X = \sup_{k \in \mathbb{N}} \big|\langle x,\, z_k\rangle\big|$ for every $x \in X$. Since $\{h_n\}_{n=1}^{\infty}$ is unconditionally summable in $L^\infty(\lambda)$, the set function $\nu \colon 2^{\mathbb{N}} \to L^\infty(\lambda)$ defined by

$$\nu(A) := \sum_{n \in A} h_n \,, \qquad A \in 2^{\mathbb{N}},$$

is a vector measure. If $h_n \neq 0$ (the zero element of $L^\infty(\lambda)$), for every $n \in \mathbb{N}$, which we now assume, then ν has no null sets other than the empty set.

Fix $k = 1, 2, \ldots$ The bounded function z_k on $\mathbb{N}$ clearly belongs to $L^\infty(\nu)$, with ν-essential sup-norm equal to $\|z_k\|_{X'}$, and hence also belongs to $L^1(\nu)$. It then follows from (2.1), with ν in place of m, that

$$\left\| \sum_{n=1}^{\infty} z_k(n) h_n \right\|_\infty = \left\| \int_{\mathbb{N}} z_k \, d\nu \right\|_\infty \le \|z_k\|_{X'} \cdot \|\nu\|(\mathbb{N}) \le \|\nu\|(\mathbb{N}) \,. \tag{5.2}$$

Choose a λ-null set $E_k \in \Sigma$ such that

$$\left| \sum_{n=1}^{\infty} z_k(n) h_n(\omega) \right| \le \left\| \sum_{n=1}^{\infty} z_k(n) h_n \right\|_\infty , \qquad \omega \in \Omega \setminus (E_0 \cup E_k) \,. \tag{5.3}$$

Let $F_0 := \bigcup_{k=0}^{\infty} E_k$ which is a λ-null set. Then H is bounded on $\Omega \setminus F_0$ because it follows from (5.2) and (5.3) that

$$\begin{aligned} \big\|H(\omega)\big\|_X = \sup_{k \in \mathbb{N}} \big|\langle H(\omega),\, z_k\rangle\big| &= \sup_{k \in \mathbb{N}} \left| \sum_{n=1}^{\infty} z_k(n) h_n(\omega) \right| \\ &\le \|\nu\|(\mathbb{N}) < \infty, \qquad \omega \in \Omega \setminus F_0 \,. \end{aligned} \tag{5.4}$$

(II) Let m denote the Bochner indefinite integral $H \cdot \lambda$ of the bounded function $H \in \mathcal{B}(\lambda, X)$; that is,

$$m(E) := (\mathrm{B})\text{-}\!\int_E H \, d\lambda = \sum_{n=1}^{\infty} \left(\int_E h_n \, d\lambda \right) \mathbf{e}_n \,, \qquad E \in \Sigma \,.$$

By [15, Lemma 3.1], we have $L^1(\lambda) \subseteq L^1(m)$ and

$$I_m f = (\mathrm{B})\text{-}\int_\Omega f \cdot H \, d\lambda = \sum_{n=1}^{\infty} \left(\int_\Omega f h_n \, d\lambda \right) \mathbf{e}_n , \qquad f \in L^1(\lambda) . \tag{5.5}$$

Since $|m| = \|H(\cdot)\|_X \cdot \lambda$, [7, Theorem II. 2.4 (iv)], it follows from (5.4) that the inequalities

$$\left(\operatorname{ess\,inf} \|H(\cdot)\|_X\right)\lambda \le |m| \le \left(\operatorname{ess\,sup} \|H(\cdot)\|_X\right)\lambda \le K\lambda \tag{5.6}$$

hold setwise on Σ, where $K := \|\nu\|(\mathbb{N})$ and the essential infimum and supremum (which are constants) are taken with respect to the measure λ. Accordingly,

$$L^1(\lambda) \subseteq L^1\big(|m|\big) \subseteq L^1(m) .$$

It is clear that $\operatorname{ess\,inf} \|H(\cdot)\|_X > 0$ if and only if $L^1(\lambda) = L^1\big(|m|\big)$.

We remark that if all the functions h_n, $n \in \mathbb{N}$, happen to be non-negative-valued, then $H(\Omega)$ is contained in the positive cone ℓ^1_+ of X and so is w-seminorm dominated by [16, Example 2.3 (ii)]. In this case Lemma 2.6 of [16] implies that $L^1(|m|) = L^1(m)$.

(III) Suppose that $\operatorname{ess\,inf} \|H(\cdot)\|_X > 0$; that is, $L^1(\lambda) = L^1\big(|m|\big)$ by part (II). The claim in this case is that the integration operator $I_m \colon L^1(m) \to X$ associated with $m = H \cdot \lambda$ is compact. Indeed, since $\{h_n\}_{n=1}^{\infty}$ is unconditionally summable in $L^\infty(\lambda)$, it follows from [6, Exercise VII.3 (ii)] that the operator I_m restricted to $L^1(|m|)$, which has the form (5.5), is compact. Now Corollary 1.1 of [16] applies to conclude that I_m, with domain $L^1(m)$, is compact.

(III)$'$ Consider the special case when h_1 is the function $\mathbf{1}$ (constantly equal to 1 on Ω), so that $\operatorname{ess\,inf} \|H(\cdot)\|_X \ge 1$. It follows from part (III) that the integration operator $I_m \colon L^1(m) \to X$ is compact. This has already been obtained in [15, Proposition 3.6 (i)].

(IV) Both possibilities concerning compactness, or not, of I_m can occur in the case when $\operatorname{ess\,inf} \|H(\cdot)\|_X = 0$. To see this, let $\{x_j\}_{j=1}^{\infty}$ be any sequence in $S[X]$. Let $E(j) := \big[(j+1)^{-1}, j^{-1}\big)$ for every $j = 1, 2, \dots$ Then the functions $h_n := \Sigma_{j=1}^{\infty} j^{-1} x_j(n) \chi_{E(j)}$, defined pointwise, for $n = 1, 2, \dots$, form an unconditionally summable sequence in $L^\infty(\lambda)$. Define $H \colon \Omega \to X$ by (5.1) for every $\omega \in \Omega$. Then

$$H(\omega) = \sum_{j=1}^{\infty} j^{-1} \chi_{E(j)}(\omega) x_j \qquad \text{and} \qquad \big\|H(\omega)\big\|_X = \sum_{j=1}^{\infty} j^{-1} \chi_{E(j)}(\omega) ,$$

for every $\omega \in \Omega$. Let $G \colon \Omega \to X$ be a function satisfying $G(\omega) = H(\omega)/\|H(\omega)\|_X$ for every $\omega \in \Omega \setminus \{0, 1\}$. For the measure $m := H \cdot \lambda$ (of finite variation), we have

$G \in \mathcal{B}(|m|, X)$ and $m = G \cdot |m|$. So, by Theorem A (see Section 2) the integration operator I_m is compact if and only if $G(\Omega \setminus \{0,1\}) = \{x_j : j = 1, 2, \dots\} \subseteq S[X]$ is relatively compact in X.

For example, if the vectors $x_j \in S[X]$, for $j = 1, 2, \dots$, form a convergent sequence, say, then I_m is compact. On the other hand, by [16, Lemma 2.5] we can choose a subset $\{x_j : j = 1, 2, \dots\}$ of $S[X]$ which is w-seminorm dominated but not relatively compact. Then I_m is not compact, yet $L^1(|m|) = L^1(m)$ and $G := dm/d|m| \in \mathcal{B}(|m|, X)$ exists.

It is perhaps worthwhile to summarize the essential features of the above construction.

Proposition 5.1. *Let $\Omega := [0,1]$ and λ be Lebesgue measure on the Borel σ-algebra Σ of Ω. Let $\{h_n\}_{n=1}^{\infty}$ be any unconditionally summable sequence in $L^\infty(\lambda)$. Then the bounded function $H : \Omega \to \ell^1$ given by (5.1) is Bochner λ-integrable, and the vector measure $m := H \cdot \lambda$ has finite variation and satisfies (5.6). In particular,*

$$L^1(\lambda) \subseteq L^1(|m|) \subseteq L^1(m)$$

with the equality $L^1(\lambda) = L^1(|m|)$ holding if and only if

$$\operatorname{ess\,inf} \left\| H(\cdot) \right\|_{\ell^1} > 0 . \tag{5.7}$$

If (5.7) is satisfied, then the associated integration operator $I_m : L^1(m) \to \ell^1$ is compact.

There exist choices of $\{h_n\}_{n=1}^{\infty}$ such that $\operatorname{ess\,inf} \left\| H(\cdot) \right\|_{\ell^1} = 0$ and $I_m : L^1(m) \to \ell^1$ is compact, and other choices of $\{h_n\}_{n=1}^{\infty}$ such that $\operatorname{ess\,inf} \left\| H(\cdot) \right\|_{\ell^1} = 0$ and $I_m : L^1(m) \to \ell^1$ is not compact, yet $L^1(m) = L^1(|m|)$ and $dm/d|m| \in \mathcal{B}(|m|, \ell^1)$ exists.

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