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ON THE HASHIMOTO TOPOLOGY WITH RESPECT TO AN EXTENSION OF THE LEBESGUE MEASURE

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ABSTRACT. We prove that the topology induced by any semi-lower density operator Φ^* in the family of all measurable sets with respect to any complete extension μ of the Lebesgue measure is formed by semi-operator Φ^* restricted to the family of Lebesgue measurable sets and the σ-ideal of μ-null sets.

Let ℓ be the Lebesgue measure in $\mathbb{R}$ and $\mathcal{L}$ denote the family of all Lebesgue measurable sets. Let Φ_d denote the classical density operator in $\mathcal{L}$ introduced in the following way:

$$\forall_{X\in\mathcal{L}}\ \Phi_d(X) = \left\{x \in \mathbb{R} \colon \lim_{h\to 0+} \frac{\ell(X\cap[x-h,x+h])}{2h} = 1\right\}.$$

By the Lebesgue density theorem $\Phi_d(X) \in \mathcal{L}$.

Let $\mathcal{T}_d = \{X \in \mathbb{R} \colon X \subset \Phi_d(X)\}$. It is well known that $\mathcal{T}_d$ is a topology in $\mathbb{R}$, usually called the density topology (cf. [3]).

Definition 1. We shall say that $\Phi\colon \mathcal{L}\to\mathcal{L}$ is a *semi-lower density operator* if

1° $\Phi(\emptyset)=\emptyset$, $\Phi(\mathbb{R})=\mathbb{R}$,

2° $\forall_{A,B\in\mathcal{L}}\ \Phi(A\cap B)=\Phi(A)\cap\Phi(B)$,

3° $\forall_{A,B\in\mathcal{L}}\ \big(\ell(A\triangle B)=0 \Rightarrow \Phi(A)=\Phi(B)\big)$,

4° $\forall_{A\in\mathcal{L}}\ \Phi(A)\subset\Phi_d(A)$.

Let $\mathit{\Phi}$ denote the family of semi-lower density operators in $\mathcal{L}$.

Theorem 2. *For every $\Phi\in\mathit{\Phi}$ the family*

$$\mathcal{T}_\Phi = \{X\in\mathcal{L}\colon X\subset\Phi(X)\}$$

is a topology in $\mathbb{R}$ such that $\mathcal{T}_\Phi\subset\mathcal{T}_d$.

Proof. It is a consequence of 1° and 2° and the fact that $\mathcal{T}_d$ is a topology. □

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Example 3. Let for every $X \in \mathcal{L}$

$$\Phi_0(X) = \begin{cases} \emptyset\,, & \ell(\mathbb{R} - X) = 0\,,\\ \mathbb{R}\,, & \ell(\mathbb{R} - X) > 0\,. \end{cases}$$

Then $\Phi_0 \in \mathit{\Phi}$ and $\mathcal{T}_{\Phi_0} = \big\{\emptyset,\, \mathbb{R} - A \colon \ell(A) = 0\big\}$.

Example 4. Let $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and nondecreasing function such that $\lim\limits_{x\to 0+} \psi(x) = 0$. Let C be the family of all such functions. Fix $\psi \in C$.

Namely, 0 is a ψ-density point of $X \in \mathcal{L}$ (cf. [5]) if and only if

$$\lim_{h\to 0+} \frac{\ell\big((\mathbb{R} - X) \cap [-h, h]\big)}{2h\,\psi(2h)} = 0\,.$$

Further, y is a ψ-density point of X if and only if 0 is a ψ-density point of $X - y = \{x - y\colon x \in X\}$. Putting $\Phi_\psi(X) = \{x \in \mathbb{R}\colon x$ is a ψ-density point of $X\}$. By Theorem 1.2 in [4] $\Phi_\psi(X) \in \mathcal{L}$ and moreover $\Phi_\psi \in \Phi$.

Let μ be an arbitrary complete extension of the Lebesgue measure. Let $S_\mu = \operatorname{dom}\mu$ and $\mathcal{I}_\mu$ — σ-ideal μ-null sets.

Let

$$\forall_{X\in S_\mu}\ \Phi_\mu(X) = \Big\{x \in \mathbb{R}\colon \lim_{h\to 0+} \frac{\mu(X \cap [x-h, x+h])}{2h} = 1\Big\}$$

and

$$\mathcal{T}_\mu = \{X \in S_\mu \colon X \subset \Phi_\mu(X)\}\,.$$

Theorem 5. (cf. [2]) *The family $\mathcal{T}_\mu$ is a topology in $\mathbb{R}$.*

Definition 6. We shall say that $\Phi^*\colon S_\mu \to \mathcal{L}$ is a *semi-lower density operator in S_μ* if

1* $\Phi^*(\emptyset) = \emptyset$, $\Phi^*(\mathbb{R}) = \mathbb{R}$,

2* $\forall_{A,B\in S_\mu}\ \Phi^*(A\cap B) = \Phi^*(A) \cap \Phi^*(B)$,

3* $\forall_{A,B\in S_\mu}\big(\mu(A \triangle B) = 0 \Rightarrow \Phi^*(A) = \Phi^*(B)\big)$,

4* $\forall_{A\in S_\mu}\ \Phi^*(A) \subset \Phi_\mu(A)$.

By $\mathit{\Phi}^*$ we shall denote the family of all Φ^*—semi-lower operators in S_μ. It is clear that $\Phi_\mu \in \mathit{\Phi}^*$.

Example 7. It is easy to observe that if in the definition of Φ_ψ-operator we replace Lebesgue measure by its extension μ then we get an operator called Φ^*_ψ such that (following the proof of Thorem 1.2 in [4]) $\Phi^*_\psi(X) \in \mathcal{L}$ for every $X \in S_\mu$. Finally, we conclude that $\Phi^*_\psi \in \mathit{\Phi}^*$.

By the same arguments as in Theorem 1 we get

THEOREM 8. *For every $\Phi^* \in \varPhi^*$ the family $\mathcal{T}_{\Phi^*} = \{X \in S_\mu : X \subset \Phi^*(X)\}$ is a topology in $\mathbb{R}$ such that and $\mathcal{T}_{\Phi^*} \subset \mathcal{T}_\mu$.*

Let $\Phi^* \in \varPhi^*$. It is clear that $\Phi = \Phi^*|\mathcal{L} \in \varPhi$. In the sequel we show that

THEOREM 9. *If $\Phi^* \in \varPhi^*$ then*

$$\mathcal{T}_{\Phi^*} = \mathcal{T}_\Phi \ominus \mathcal{I}_\mu = \{X \subset \mathbb{R} : X = A - B,\ A \in \mathcal{T}_\Phi,\ B \in \mathcal{I}_\mu\}$$

($\mathcal{T}_{\Phi^*}$ is called Hashimoto topology generated by topology $\mathcal{T}_\Phi$ and σ-ideal $\mathcal{I}_\mu$ (cf. [1])).

First we present the following

LEMMA 10. *If $X \in \mathcal{T}_\mu$ then $X = A - B$, where $A \in G_{\delta\sigma}$, $B \in \mathcal{I}_\mu$.*

Proof. Let $X \in \mathcal{T}_\mu$. Then $X \subset \Phi_\mu(X)$. Suppose that X is bounded and K is an open interval containing X. Let $0 < \varepsilon < \ell(K)$. At the begining we show that there exists a set $X \in G_\delta$ such that $X \subset C$ and $\mu(C - X) < \varepsilon$. Putting

$$\mathcal{K} = \left\{\Delta \subset K : \mu(\Delta \cap X) > \left(1 - \frac{\varepsilon}{\ell(K)}\right)\ell(\Delta)\right\},$$

where Δ denotes an interval, we obtain a covering of set X in Vitali's sense. Hence we have a sequence $\{\Delta_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ of pairwise disjoint intervals such that $\ell\left(X - \bigcup\limits_{n=1}^{\infty} \Delta_n\right) = 0$.

We show that $\mu(\bigcup\limits_{n=1}^{\infty} \Delta_n - X) < \varepsilon$. We have

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} \Delta_n - X\right) &= \mu\left(\bigcup_{n=1}^{\infty} (\Delta_n - X)\right) \le \sum_{n=1}^{\infty} \mu(\Delta_n - X) \\
&= \sum_{n=1}^{\infty} \big(\mu(\Delta_n) - \mu(\Delta_n \cap X)\big) < \sum_{n=1}^{\infty} \left[\mu(\Delta_n) - \left(1 - \frac{\varepsilon}{\ell(K)}\right)\mu(\Delta_n)\right] \\
&= \frac{\varepsilon}{\ell(K)} \sum_{n=1}^{\infty} \mu(\Delta_n) \le \varepsilon .
\end{aligned}$$

Let $C^* \in G_\delta$ be a such set that $X - \bigcup\limits_{n=1}^{\infty} \Delta_n \subset C^*$ and $\ell(C^*) = 0$. Then $C = C^* \cup \bigcup\limits_{n=1}^{\infty} \Delta_n \in G_\delta$, $X \subset C$ and $C - X \subset C^* \cup \left(\bigcup\limits_{n=1}^{\infty} \Delta_n - X\right)$. Hence $\mu(C - X) < \varepsilon$. Since ε was an arbitrary small number, it implies that there exists a set $C \in G_\delta$ such that $X \subset C$ and $\mu(C - X) = 0$. If X is a σ-bounded set then by the standard arguments we obtain a set $C \in G_{\delta\sigma}$ containing X and $\mu(C - X) = 0$. This equality ends the proof. $\square$

Lemma 11. *For every $\Phi^* \in \varPhi^*$ the following condition is satisfied:*

$$\forall_{X \in \mathcal{T}_{\Phi^*}} \ \mu\big(\Phi^*(X) \triangle X\big) = 0 .$$

P r o o f. Let $X \in \mathcal{T}_{\Phi^*}$. Then $\mu\big(\Phi^*(X) \triangle X\big) = \mu\big(\Phi^*(X) - X\big) \le \mu\big(\Phi_\mu(X) - X\big)$. By Lemma 10 $X = A - B$, where $A \in \mathcal{L}$ and $B \in \mathcal{I}_\mu$. Hence we have $\mu\big(\Phi_\mu(X) - X\big) = \mu\big(\Phi_\mu(A-B) - (A-B)\big) = \mu\big(\Phi_\mu(A-B) - A\big) \le \mu\big(\Phi_\mu(A) - A\big) = \ell\big(\Phi_d(A) - A\big) = 0$. □

P r o o f o f T h e o r e m 9. Let $X \in \mathcal{T}_\Phi \ominus \mathcal{I}_\mu$. Then $X = A - B$, where $A \in \mathcal{T}_\Phi$, $\mu(B) = 0$. Hence $\Phi^*(X) = \Phi^*(A - B) = \Phi^*(A) = \Phi(A) \supset A \supset X$. Therefore $X \in \mathcal{T}_{\Phi^*}$. Let $X \in \mathcal{T}_{\Phi^*}$ then $X \subset \Phi^*(X)$. Hence $X = \Phi^*(X) - \big(\Phi^*(X) - X\big)$. By Lemma 11 $\Phi^*(X) - X \in \mathcal{I}_\mu$. By condition 3^* $\Phi^*(X) = \Phi^*\big(\Phi^*(X)\big)$. Then $\Phi^*(X) = \Phi\big(\Phi^*(X)\big)$ and we conclude that $\Phi^*(X) \in \mathcal{T}_\Phi$. It implies that $X \in \mathcal{T}_\Phi \ominus \mathcal{I}_\mu$. □

Corollary 12. $\mathcal{T}_\mu = \mathcal{T}_d \ominus \mathcal{I}_\ell = \{X \colon X = A - B,\ A \in \mathcal{T}_d,\ \ell(B) = 0\}$.

Remark 8. Giving up condition 3° and 3^* in Definitions 1 and 6, respectively, the families $\mathcal{T}_\Phi$ and $\mathcal{T}_{\Phi^*}$ are still topologies but the equality $\mathcal{T}_{\Phi^*} = \mathcal{T}_\Phi \ominus \mathcal{I}_\mu$ established in Theorem 9 does not hold. But we have

Property 13. $\mathcal{T}_{\Phi^*} \subset \mathcal{T}_\Phi \ominus \mathcal{I}_\mu$*, where* $\Phi = \Phi^*|\mathcal{L}$.

P r o o f. Let $X \in \mathcal{T}_{\Phi^*}$. Then $X \subset \Phi^*(X)$. Hence $X = \Phi^*(X) - \big(\Phi^*(X) - X\big)$. By Lemma 10, $\Phi^*(X) - X \in \mathcal{I}_\mu$ and by the monotonicity of operator Φ^*, $\Phi^*(X) = \Phi^*\big(\Phi^*(X) - \big(\Phi^*(X) - X\big)\big) \subset \Phi^*\big(\Phi^*(X)\big) = \Phi\big(\Phi^*(X)\big)$. Therefore $\Phi^*(X) \in \mathcal{T}_\Phi$ and $X \in \mathcal{T}_\Phi \ominus \mathcal{I}_\mu$.

To show that $\mathcal{T}_{\Phi^*} \neq \mathcal{T}_\Phi \ominus \mathcal{I}_\mu$ it is sufficient to take $\Phi^* \colon S_\mu \to \mathcal{L}$ of the following form

$$\Phi^*(X) = \begin{cases} \mathbb{R}, & X = \mathbb{R}, \\ \emptyset, & X \neq \mathbb{R}. \end{cases}$$

Conditions 1^*, 2^* and 4^* are fulfilled. Let $\Phi = \Phi^*|\mathcal{L}$. Then $\mathcal{T}_{\Phi^*} = \{\emptyset, \mathbb{R}\}$, $\mathcal{T}_\Phi = \{\emptyset, \mathbb{R}\}$. Hence property $\mathcal{T}_\Phi \ominus \mathcal{I}_\mu \subset \mathcal{T}_{\Phi^*}$ does not hold for any complete extension μ of Lebesgue measure. □

Let $\mathcal{T}$ be the natural topology in $\mathbb{R}$ and $\varPhi' = \{\Phi \in \varPhi \colon \mathcal{T} \subset \mathcal{T}_\Phi\}$. We have, for example, that $\Phi_d \in \varPhi'$ and for every $\psi \in C$, $\Phi_\psi \in \varPhi'$. The following property provides that the topology which is an intersection of $\mathcal{T}_\Phi$, where Φ runs $\varPhi'$ is the Hashimoto topology generated by the natural topology and the σ-ideal Lebesgue null sets.

PROPERTY 14.

$$\bigcap_{\Phi\in\mathit{\Phi}'} \mathcal{T}_{\Phi} = \mathcal{T} \ominus \mathcal{I}_{\ell}\,.$$

Proof. Fix $\Phi \in \mathit{\Phi}'$ and let $X = V - Z$, where $V \in \mathcal{T}$ and $\ell(Z) = 0$. Since $V \in \mathcal{T}_{\Phi}$ and $\mathbb{R} - Z \in \mathcal{T}_{\Phi}$, we have that $X \in \mathcal{T}_{\Phi}$. Thus $X \in \bigcap_{\Phi\in\mathit{\Phi}'} \mathcal{T}_{\Phi}$. To prove that $\bigcap_{\Phi\in\mathit{\Phi}} \mathcal{T}_{\Phi} \subset T \ominus \mathcal{I}_{\ell}$ it is sufficient to use the result from [4] that $\bigcap_{\psi\in C} \mathcal{T}_{\Phi_{\psi}} \subset \mathcal{T} \ominus \mathcal{I}_{\ell}$. □

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