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ONE OLD PROBLEM, ONE NEW, AND THEIR EQUIVALENCE

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ABSTRACT. During the *Real Analysis Problem Workshop* held in Lódź, Poland, 1994, two of the participants presented problems which proved to be quite closely related; even, in a sense, equivalent. One problem is rather old and famous, while the second is refreshingly new. The purpose of this note is to give a bit of background for each of the problems and show their relationship.

1. Introduction

As part of the 1994 *Lódź Real Analysis Problem Workshop*, participants were asked to present problems in real analysis which they felt held some significance for the field. One portion of the second author's lecture was devoted to describing the *Three Segment Problem*, what was known about it and why the author felt it should be of interest to real analysis. The following afternoon, the third author presented several interesting open problems; one in particular concerned intersections of linear spaces of measurable functions. Several days later, the first author pointed out an unexpected equivalence between the problems. The purpose of the present note is to describe some of the background for each of the two problems and to show in what sense they are equivalent.

2. The three segment problem

This is an old problem which had its origins in the theory of cluster sets back in the fifties. In order to provide a bit of context and to demonstrate a few elementary techniques of the theory, we present a few preliminary definitions and results before stating the main problem.

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* The authors dedicate this paper to the memory of Frederick Bagemihl.

Let $\mathbb{H}$ denote the open upper half-plane, $\mathbb{R}$ the real line, and $\mathbb{R}^*$ the extended real line. We identify the boundary of $\mathbb{H}$ with $\mathbb{R}$ so that the closure of $\mathbb{H}$, $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$. Let $f\colon \mathbb{H} \to \mathbb{R}$ be entirely arbitrary. Then a point $x \in \mathbb{R}$ is called an *ambiguous point* of f if there are arcs, α and β in $\overline{\mathbb{H}}$ such that

- $\alpha(1) = \beta(1) = x$,
- $\alpha\big|_{[0,1)} \cup \beta\big|_{[0,1)} \subset \mathbb{H}$,
- $\mathcal{C}(f,\alpha,x) \cap \mathcal{C}(f,\beta,x) = \emptyset$,

where the set $\mathcal{C}(f,\alpha,x) \equiv \{y \in \mathbb{R}^*\colon \text{there is a sequence, } \{t_n\} \to 1,\ t_n < 1 \text{ such that } \lim f \circ \alpha(t_n) = y\}$ denotes the *cluster set* of f at x along α. Restrictions on the nature of arcs allowed for the α and β are frequently made and such restrictions appear as adjectives for *ambiguous.* Thus for example, one can consider *rectifiably ambiguous* points of f where both α and β are required to be rectifiable arcs, or *rectilinear ambiguous* points where both α and β are required to be line segments, etc. If $E \subset \mathbb{H}$, then a point $x \in \mathbb{R}$ is an *ambiguous point of* E if x is an ambiguous point of the characteristic function of E.

In the 1930s, Henry Blumberg, Mabel Schmeiser, and Vojtěch Jarník studied rectilinearly ambiguous points of arbitrary functions $f\colon \mathbb{H} \to \mathbb{R}$. Then in 1955 in [2], Frederick Bagemihl, adapting the techniques of Blumberg and Schmeiser, first proved that there are at most enumerably many ambiguous points for a given a fixed set, $E \subset \mathbb{H}$ and then used this fact to prove the following theorem about ambiguous points of functions.

THEOREM 1. (F. Bagemihl, 1955, [2]) *An arbitrary function, $f\colon \mathbb{H} \to \mathbb{R}$ has at most enumerably many ambiguous points.*

The technique employed by Bagemihl has been often repeated and in fact there is a general method for generalizing a result about ambiguous points of sets to ambiguous points of functions. The connection is sufficiently close, that theorems about ambiguous points of closed sets correspond to theorems about ambiguous points of continuous functions, measurable sets to measurable functions, arbitrary sets to arbitrary functions, and so on. Moreover, given a real valued function $f\colon \mathbb{R} \to \mathbb{R}$ one can consider the difference quotient function $F\colon \mathbb{H}^* \to \mathbb{R}$ defined by $F(x,y) = \frac{f(x)-f(y)}{x-y}$ where $\mathbb{H}^* = \{(x,y)\colon y > x\}$. In this way, ambiguous point theorems (or more generally, cluster set theorems) can have something to say about derivates of real functions.

Now, suppose A, B and C are three sets in $\overline{\mathbb{H}}$ with $A \cap B \cap C = \mathbb{R}$. A point $x \in \mathbb{R}$ is said to have the *three arc* (*segment*) *property* with respect to A, B, and C, if there are arcs (segments) $\alpha \subset A$, $\beta \subset B$ and $\gamma \subset C$ with $x = \alpha(1) = \beta(1) = \gamma(1)$ and $\alpha, \beta, \gamma \subset \mathbb{H} \cup \{x\}$. The following examples are easy and enlightening.

Example 1. There are three closed sets A, B, C in $\overline{\mathbb{H}}$ such that every point $x \in \mathbb{R}$ has the three arc property.

P r o o f. First define a and b to be *consecutive binary fractions* if they are either of the form $.d_1 d_2 \dots d_n \overline{0}$ and $.d_1 d_2 \dots d_n^* \overline{0}$ or $.d_1 d_2 \dots d_n \overline{1}$ and $.d_1 d_2 \dots d_n^* \overline{1}$ where $d_n^* = 1 - d_n$. Let A denote the union of the sides of equilateral triangles with consecutive binary fraction bases. The sets B and C are similar, but using isosceles right triangles with right angle on the left for A and the right for B. $\square$

One can use transfinite induction together with a well-ordering of $\mathbb{R}$ to prove the existence of the following example; see [1].

Example 2. There are three sets A, B, C in $\overline{\mathbb{H}}$ such that every point $x \in \mathbb{R}$ has the three segment property.

Finally, there is the following example, essentially due to J a r n í k, [5].

Example 3. There are three closed sets A, B, C in $\overline{\mathbb{H}}$ such that every two sided limit point of the Cantor ternary set has the three segment property.

P r o o f. In this case, the sets, A, B, and C respectively consist of the union of all rays of angles $\pi/4$, $3\pi/4$ and $\pi/2$ emanating from points of the Cantor set. It is easy to see that any point of $\mathbb{H} \cap A \cap B$ lies above the closure of a contiguous interval. $\square$

The main question arising from these three examples is then:

Question 1. Are there three closed sets in $\overline{\mathbb{H}}$ such that every point of $\mathbb{R}$ has the three segment property?

However, not even the answer to the following question is known.

Question 2. Are there three measurable sets in $\overline{\mathbb{H}}$ such that every point of a set of positive measure has the three segment property?

These were two of the questions presented by the second author at the Łódź Workshop in 1994.

3. Intersections with linear spaces of measurable functions

In 1994, G. S z é k e l y asked the following

Question 3. Let L be a finite dimensional linear space of measurable functions on $\mathbb{R}$. Is it true that there exists a measurable function f such that for every $g \in L$ the set $\mathbb{R}(f = g) \equiv \{x \in \mathbb{R} : f(x) = g(x)\}$ is of measure zero?

Shortly thereafter, M. Laczkovich and G. Petruska proved that the answer is "yes" by proving the following theorem (unpublished).

Theorem 2. *Suppose L is a finite dimensional linear space of measurable functions on $\mathbb{R}$. Then there is a measurable function f such that $\mathbb{R}(f=g)$ is σ-porous.*

The third author of this paper then noticed that a straightforward transfinite construction yields the following result.

Theorem 3. *Suppose L is an arbitrary linear space of functions on $\mathbb{R}$ of dimension n. Then there is a function f (not necessarily measurable) such that $\left|\mathbb{R}(f=g)\right| \le n$.*

The natural question presented by the third author in Łódź, is:

Question 4. If the space L referred to in Theorem 3 consists of measurable functions, can the function f be chosen to be measurable?

L. Pósa proposed the following linear space as a counterexample. Let L^* denote the space generated by the functions $h_1(x) \equiv 1$ and $h_2(x) \equiv x - [x]$ where $[x]$ is the integer part of x. Pósa could show that for every measurable function $f\colon \mathbb{R} \to \mathbb{R}$ there is a $g \in L^*$ such that $\left|\mathbb{R}(f=g)\right| \ge 3$, but his argument rested on an affirmative answer to the following question, and to date, this question remains unresolved.

Question 5. Suppose f_1, f_2, and f_3 are measurable functions defined on an interval I. Does there exist a nonvertical line ℓ which intersects the graph of each of the functions?

Supposing the answer to Question 5 is "yes", it is easy to see that L^* has the required property. Indeed, if $f\colon \mathbb{R} \to \mathbb{R}$ is measurable, then let $f_1(x) = f(x)$, $f_2(x) = f(x+1)$, and $f_3(x) = f(x+2)$ for $x \in [0,1)$. Let g be a linear function such that $g(x_i) = f_i(x_i)$ for some distinct $x_1, x_2, x_3 \in [0,1)$, and extend g periodically to all of $\mathbb{R}$. Then, $g \in L^*$ and $R(f=g)$ contains $\{x_1,\ x_2+1,\ x_3+2\}$. From this it follows that the answer to Laczkovich's Question 4 is "no" if the answer to Pósa's Question 5 is "yes".

4. An equivalence of sorts

It should be apparent that there are many forms and variations of these questions which might be asked. For example, in Question 1 we may wish to consider the case when the three sets are measurable, whether the sets themselves can be composed only of line segments, or even lines, etc. Similarly, in Question 5 we may wish to worry about such issues as whether the functions can be replaced by sets in the plane, or whether the line can be chosen to have positive slope, or whether there is a non-vertical line which meets the closure of the graphs of each function, etc. Let us loosely refer to questions which are just variations of Question 1, as Type 1 questions, and those which are variations of Question 5 as Type 2 questions. In this section we are not going to be concerned so much with each possible variation. Instead, we wish to present a simple and general technique which will turn any question of Type 1 into an equivalent question of Type 2.

The technique is called *Besicovich Duality* (see [4, p. 101]) and it is this that Author 1 pointed out in Łódź. Duality, in this sense, simply relates lines to points via some specified parameterization. In this case, if ℓ is a line in the plane with non-zero slope $m(\ell)$ and x-intercept $x(\ell)$, then we define $T(\ell)$ to be the point in the plane given by

$$T(\ell) = \begin{cases} \left(x(\ell), \frac{1}{m(\ell)}\right) & \text{if } m \neq \infty \text{ or } 0\,, \\ \left(x(\ell), 0\right) & \text{if } m = \infty. \end{cases}$$

Then T defines a bijection from the set of lines with non-zero slope to the set of points in the plane.

In order for a line (with non-zero slope) to pass through a point (a,b), it is necessary and sufficient that $x(\ell) = -b \cdot \left(1/m(\ell)\right) + a$, where, as before, we make the convention that for a vertical line, $1/m(\ell) = 0$. Therefore, the map T takes such a line to a point (x,y) in the plane, where $x = -by + a$. If we consider the 'pencil' of all lines with nonzero slope which pass through (a,b), then T maps this pencil to the single line $x = -by + a$ with x-intercept a, and with non-zero slope $-1/b$.

Finally, if f is a function (or more generally a set) as taken from the context of Question 5, then define

$$A(f) = \{\ell \colon \ell \cap f \neq \emptyset\} \qquad \text{and} \qquad A^*(f) = \left\{T(\ell) \colon \ell \in A(f)\right\}.$$

It follows from the previous discussion that $A^*(f)$ is a set in the plane which consists of the union of a collection of non-horizontal lines. Furthermore, a point a is in the domain of f if and only if there is a line in $A^*(f)$ which crosses the x -axis at $(a,0)$. That is, $(a,0)$ is linearly accessible via $A^*(f)$.

Now suppose that f_1, f_2 and f_3 are three sets such that every vertical line meeting an interval I hits all of these, and every non-vertical line misses at least one of these. Then the sets $A^*(f_1)$, $A^*(f_2)$, $A^*(f_3)$ have the property that every point of I is linearly accessible via each of these sets, but no point of $\mathbb{H}$ is in all three. It is easy to see that by taking the union of countably many suitable affine copies of these three sets we can answer a Type 1 question in the affirmative.

REFERENCES

[1] BAGEMIHL, F.—PIRANIAN, G.—YOUNG, G. S.: *Intersections of cluster sets*, Bull. Inst. Politehn. Iaşi (N. S.) **5** (1959), 29–34.

[2] BAGEMIHL, F.: *Curvilinear cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. U. S. A. **41** (1955), 379–382.

[3] BLUMBERG, H.: *A theorem on arbitrary functions of two variables*, Fund. Math **16** (1930), 17–44.

[4] FALCONER, K. J.: *The Geometry of Fractal Sets*, Cambridge Univ. Press, Cambridge, 1985.

[5] JARNÍK, V.: *Sur les fonctions de deux variables réelles*, Fund. Math. **27** (1936), 147–150.

[6] SCHMEISER, M.: *Some properties of arbitrary functions*, Fund. Math. **22** (1934), 70–76.

[7] ZAJÍČEK, L.: *Porosity and σ-porosity*, Real. Anal. Exchange **13** (1987–1988), 314–350.

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