Tatra Mt. Math. Publ. **24** (2002), 109–115

TATRA MOUNTAINS
Mathematical Publications

ON A BANACH SPACE VALUED OBSERVABLES

Beloslav Riečan

ABSTRACT. The theory of observables in some quantum structures is extended from the real valued case to the Banach space valued case. A representation theorem is proved on representation of Banach space valued observables by Banach space valued random elements.

1. Banach space valued observable

Let us consider a probability space $(\Omega, \mathcal{S}, \mu)$ and the MV algebra $\mathcal{F}$ of all $\mathcal{S}$-measurable functions $f\colon \Omega \to [0,1]$. Denote by m the mapping $m\colon \mathcal{F} \to [0,1]$ defined by the formula

$$m(f) = \int_{\Omega} f \, d\mu .$$

Recall that a random variable is a function $\xi\colon \Omega \to \mathbb{R}$ measurable with respect to $\mathcal{S}$, i.e., $\xi^{-1}(A) \in \mathcal{S}$ whenever $A \in \mathcal{B}(\mathbb{R})$. Therefore to any random variable ξ there exists a morphism $x\colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ defined by the formula $x(A) = \chi_{\xi^{-1}(A)}$. This is a motivation for the notion of an observable in a quantum model based on $\mathcal{F}$. On the other hand a random element in a Banach space X is an $\mathcal{S}$-measurable mapping ξ from Ω to X, i.e., $\xi^{-1}(A) \in \mathcal{S}$ whenever $A \in \mathcal{B}(X)$. This is our motivation for the following definition of an observable (more precisely Banach space valued observable).

Definition 1. Let X be a Banach space. Let $(\Omega, \mathcal{S})$ be a measurable space, $\Omega \in \mathcal{S}$, $\mathcal{F}$ be the set of all fuzzy events, i.e., $\mathcal{S}$-measurable mappings from Ω to $[0,1]$. An *observable* is a mapping $x\colon \mathcal{B}(X) \to \mathcal{F}$ ($\mathcal{B}(X)$ being the set of all Borel subsets of X) satisfying the following conditions:

(i) $x(X) = 1_{\Omega}$.

(ii) If $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

2000 Mathematics Subject Classification: 28B05, 28E10.
Keywords: Banach space, quantum mechanics, probability, observables.

The paper has been supported by the grant VEGA 2/5126/98.

(iii) If $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Definition 2. If $x, y\colon \mathcal{B}(X) \to \mathcal{F}$ are observables, then their *joint observable* is a mapping $h\colon \mathcal{B}(X^2) \to \mathcal{F}$ satisfying the following conditions:

(i) $h(X^2) = 1_\Omega$.

(ii) If $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.

(iii) If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.

(iv) $h(C \times D) = x(C) \cdot y(D)$ for any $C, D \in \mathcal{B}(C)$.

Let us mention that the notion of the joint observable corresponds to the notion of a random vector $T = (\xi, \eta)\colon \Omega \to \mathbb{R}$. Namely, the mapping h defined by the formula $h(A) = \chi_{T^{-1}(A)}$ is a morphism from $\mathcal{B}(\mathbb{R}^2)$ to $\mathcal{S}$. Moreover,

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D),$$

hence

$$h(C \times D) = \chi_{\xi^{-1}(C)} \cdot \chi_{\eta^{-1}(D)} .$$

Theorem 1. *For any observables $x, y\colon \mathcal{B}(X) \to \mathcal{F}$ there exists their joint observable.*

Proof. Fix $\omega \in \Omega$ and define $\mu_\omega, \nu_\omega\colon \mathcal{B}(X) \to [0,1]$ by the formulae

$$\mu_\omega(A) = x(A)(\omega), \quad \nu_\omega(B) = y(B)(\omega).$$

It is not difficult to see that μ_ω, ν_ω are probability measures, hence their product $\mu_\omega \times \nu_\omega$ can be constructed on the product σ-algebra $\mathcal{B}(X) \times \mathcal{B}(X) = \mathcal{B}(X^2)$. Define $h\colon \mathcal{B}(X^2) \to [0,1]^\Omega$ by the formula

$$h(A)(\omega) = \mu_\omega \times \nu_\omega(A).$$

By the definition of the product of measures

$$h(C \times D)(\omega) = \mu_\omega(C) \cdot \nu_\omega(D) = x(C)(\omega) \cdot y(D)(\omega) = x(C) \cdot y(D)(\omega),$$

hence

$$h(C \times D) = x(C) \cdot y(D).$$

Since $x(C) \in \mathcal{F}$, $y(D) \in \mathcal{F}$ and $h(C \times D)$ is the product of two measurable functions, we obtain $h(C \times D) \in \mathcal{F}$.

Consider now the family $\mathcal{K} = \{A \in \mathcal{B}(X^2)\colon h(A) \in \mathcal{F}\}$. We have proved that $\mathcal{K} \supset \mathcal{D} = \{C \times D\colon C \in \mathcal{B}(X),\ D \in \mathcal{B}(X)\}$. Moreover, $\mathcal{K}$ is monotone, hence $\mathcal{K} \supset \mathcal{M}(\mathcal{D}) = \sigma(\mathcal{D}) = \mathcal{B}(X^2)$. Therefore $h(A) \in \mathcal{F}$ for any $A \in \mathcal{B}(X^2)$ and h is actually a mapping from $\mathcal{B}(X^2)$ to $\mathcal{F}$.

The property (iv) has been proved, (ii) and (iii) follow by the additivity, or the continuity of measures, respectively. Finally (ii) follows by

$$h(X^2) = h(X \times X) = x(X) \cdot y(X) = 1_\Omega \cdot 1_\Omega = 1_\Omega .$$

□

2. Representation

Now we shall construct to any sequence $(x_n)_n$ of observables a probability space and a sequence $(\xi_n)_n$ of random elements with the same probability distribution. Recall that a probability distribution of a random element $\xi\colon \Omega \to X$ is the mapping $P_\xi\colon \mathcal{B}(X) \to [0,1]$ defined by the formula $P_\xi(A) = P\big(\xi^{-1}(A)\big)$. Analogously we define the probability distribution $m_x\colon \mathcal{B}(X) \to [0,1]$ of an observable $x\colon \mathcal{B}(X) \to \mathcal{F}$ by the formula

$$m_x(A) = m\big(x(A)\big)\,.$$

So let $(x_n)_n$ be a sequence of observables $x_n\colon \mathcal{B}(X) \to \mathcal{F}$. First define $P_n\colon \mathcal{B}(X^n) \to [0,1]$ by the formula $P_n = m_n \circ h_n$, where h_n is the joint observable of $x_1,\dots,x_n$. Evidently P_n is a probability measure. Let $J \subset \mathbb{N}$, $J \neq \emptyset$ be any finite set, $n = \max\{i\colon i \in J\}$, $J_n = \{1,2,\dots,n\}$, $\pi_{J_n,J}$ be the corresponding projection $\pi_{J_n,J}\colon X^n \to X^{|J|}$. Define $P_J = P_n \circ \pi_{J_n,J}^{-1}\colon \mathcal{B}(X^{|J|}) \to [0,1]$. Then P_J is a probability measure and the family $\{P_J\colon\ J \subset \mathbb{N}\}$ is consistent. Therefore by the Kolmogorov theorem there exists a probability space $(X^{\mathbb{N}}, \sigma, P)$ with the following properties:

(i) σ is the σ-algebra generated by the family of all cylinders (i.e., sets of the form $\pi_J^{-1}(A)$, where $\emptyset \neq J \subset \mathbb{N}$, J is finite, $\pi_J\colon X^{\mathbb{N}} \to X^J$ is the projection and $A \in \mathcal{B}\big(X^{|J|}\big)$).

(ii) $P\big(\pi_J^{-1}(A)\big) = P_J(A)$ for any J and any A.

Define $\xi_n\colon X^{\mathbb{N}} \to X$ by the formula $\xi_n\big((t_i)_i\big) = t_n$, $T_n\colon X^{\mathbb{N}} \to X^n$ by the formula $T_n = (\xi_1,\dots,\xi_n) = \pi_{J_n}$. Then the following theorem holds:

THEOREM 2. *Let (X^n, σ, P) correspond to the sequence (x_n) of observables. Then*

(i) *ξ_n is a random element and $P_{\xi_n} = m_{x_n}$.*

(ii) *$P_{T_n} = m \circ h_n$.*

(iii) *If $g_n\colon X^n \to X$ is a Borel function and $\eta_n = g_n \circ T_n = g_n(\xi_1,\dots,\xi_n)$, then η_n is a random element such that*

$$P_{\eta_n} = m \circ h_n \circ g_n^{-1}\,.$$

P r o o f.

$$\begin{aligned}
\text{(i)}\quad P_{\xi_n}(A) &= P\big(\xi_n^{-1}(A)\big) = P\big(\pi_{\{n\}}^{-1}(A)\big)\\
&= P_{\{n\}}(A) = P_n \circ \pi_{J_n\{n\}}^{-1}(A)\\
&= P_n(R \times \cdots \times R \times A)\\
&= m\big(h_n(R \times \cdots \times R \times A)\big)\\
&= m\big(1_\Omega \cdot 1_\Omega \ldots 1_\Omega \cdot x_n(A)\big)\\
&= m\big(x_n(A)\big) = m_{x_n}(A)\,.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad P_{T_n}(B) &= P\big(T_n^{-1}(B)\big) = P\big(\pi_{J_n}^{-1}(B)\big)\\
&= P_{J_n}(B) = m \circ h_n(B)\,.
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}\quad P_{\eta_n}(C) &= P\big(\eta_n^{-1}(C)\big) = P\Big(T_n^{-1}\big(g_n^{-1}(C)\big)\Big)\\
&= P_{T_n}\big(g_n^{-1}(C)\big) = m \circ h_n \circ g_n^{-1}(C)\,.
\end{aligned}$$

□

3. Calculus with observable

Inspired by the Kolmogorov case (where $(\xi_1+\xi_2)^{-1} = (g\circ T_2)^{-1} = T_2^{-1}\circ g^{-1}$, where $g(u,v) = u + w$) we shall define the sum of two observables x_1, x_2 by the formula

$$x_1 + x_2 = h_2 \circ g^{-1}$$

and generally

$$x_1 + x_2 + \cdots + x_n = h_n \circ g_n^{-1},$$

where $g_n(u_1,\ldots,u_n) = u_1 + \cdots + u_n$, $g_n\colon X^n \to X$. Here $x_1 + \cdots + x_n$ has the same probability distribution as $\eta_n = \xi_1 + \cdots + \xi_n$. Indeed,

$$\begin{aligned}
P_{\xi_n}(A) &= P\big(\eta_n^{-1}(A)\big) = P\big((g_n \circ T_n)^{-1}(A)\big)\\
&= P_{T_n}\big(g_n^{-1}(A)\big) = m \circ h_n \circ g_n^{-1}(A)\\
&= m \circ (x_1 + \cdots + x_n)(A)\,.
\end{aligned}$$

We say that an observable is *integrable*, if there exists the integral $\int\limits_X t\,dm(t) = E(x)$.

Theorem 3. *If x_1, x_2 are integrable, then $x_1 + x_2$ is integrable, too, and $E(x_1 + x_2) = E(x_1) + E(x_2)$.*

Proof. Put $k(u,v) = u$, $\ell(n,v) = v$. Then

$$\begin{aligned}
E(x_1 + x_2) &= \int\limits_X t\,dm \circ h_2 \circ g_2^{-1}(t) = \iint_{X^2} g_2\,dm \circ h_2 \\
&= \iint_{X^2} (u+v)\,dm \circ h_2(u,v) \\
&= \iint_{X^2} u\,dm \circ h_2(u,v) + \iint_{X^2} v\,dm \circ h_2(u,v) \\
&= \iint_{X^2} k\,dm \circ h_2 + \iint_{X^2} \ell\,dm \circ h_2 \\
&= \int\limits_X t\,dm \circ h_2 \circ k^{-1}(t) + \int\limits_X t\,dm \circ h_2 \circ \ell^{-1}(t) \\
&= \int\limits_X t\,dm_{x_1}(t) + \int\limits_X t\,dm_{x_2}(t) = E(x_1) + E(x_2)\,,
\end{aligned}$$

since $m \circ h_2 \circ k^{-1}(A) = m\big(h_2(A \times R)\big) = m\big(x_1(A)\big)$. □

4. Convergence of observables

Let $y_n \colon \mathcal{B}(X) \to \mathcal{F}$ be observables, $n = 1, 2, \dots$. Denote $S_p = \big\{u \in X \colon \|u\| < \frac{1}{p}\big\}$. Then we say that

(i) (y_n) *converges in measure to* 0, if

$$\lim_{n\to\infty} m\big(y_n(X \setminus S_p)\big) = 0$$

for any $p \in \mathbb{N}$;

(ii) (y_n) *converges to* 0 *almost everywhere*, if

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} y_n(S_p)\right) = 1\,.$$

Theorem 4. *Let* $y_n \colon X^n \to X$ *be Borel functions,* $\eta_n = g_n(\xi_1, \dots, \xi_n)$, $y_n = h_n \circ g_n^{-1}$. *Then*

(i) (y_n) *converges in measure to* 0 *if and only if* (η_n) *converges in measure to* 0*;*

(ii) *if* (η_n) *converges* P*-almost everywhere to* 0*, then* (y_n) *converges* m*-almost everywhere to* 0*.*

P r o o f . (i) follows by the following equalities:

$$\begin{aligned} m\big(y_n(X \setminus S_p)\big) &= P_n(X \setminus S_p) \\ &= P\Big(\big\{u \in X \colon \|\eta_n(u)\| \geq \tfrac{1}{p}\big\}\Big) . \end{aligned}$$

Now we shall prove (ii).

$$\begin{aligned} &P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}(S_p)\right) \\ &= m\left(h_{k+i}\left(\bigcap_{n=k}^{k+i} \{(u_1,\dots,u_{k+i}) \colon (u_1,\dots,u_n) \in g_n^{-1}(S_p)\}\right)\right) \\ &\leq m\left(\bigwedge_{n=k}^{k+i} h_{k+i}\Big(\{(u_1,\dots,u_{k+i})\ (u_1,\dots,u_n) \in g_n^{-1}(S_p)\}\Big)\right) \\ &= m\left(\bigwedge_{n=k}^{k+i} h_{k+i}\Big(p_n^{-1}(g_n^{-1}(S_p))\Big)\right) , \end{aligned}$$

where $p_n \colon X^{k+i} \to X^n$ is the projection. Of course

$$\begin{aligned} &h_{k+i}(p_n^{-1}(A_1 \times \dots \times A_n)) \\ &= h_{k+i}(A_1 \times \dots \times A_n \times \mathbb{R} \times \dots \times \mathbb{R}) \\ &= h_n(A_1 \times \dots \times A_n) , \end{aligned}$$

hence $h_{k+i} \circ p_n^{-1} = h_n$. Therefore

$$P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}(S_p)\right) \leq m\left(\bigwedge_{n=k}^{k+i} h_n \circ g_n^{-1}(S_p)\right) .$$

But (η_n) converges P-a.e. to 0 if and only if

$$\begin{aligned} 1 &= P\left(\bigcap_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \eta_n^{-1}(S_p)\right) \\ &= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}(S_p)\right) . \end{aligned}$$

□

REFERENCES

[1] POTOCKÝ, R.—URBANÍKOVÁ, M.: *Strong law of large numbers for double sequences of random elements*, Math. Slovaca **49** (1999), 85–93.

[2] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure, and Ordering*, Kluwer, Dordrecht and Ister, Bratislava, 1997.

[3] TAYLOR, R. L: *Stochastic Convergence of Weighted Sums of Random Elements of Linear Spaces*, Lecture Notes in Math., Vol. 672, Springer-Verlag, Berlin, 1978.

Received January 25, 2000

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA
E-mail: riecan@mat.savba.sk

Department of Mathematics
Faculty of Natural Sciences
Matej Bel University
Tajovského 40
SK–974 01 Banská Bystrica
SLOVAKIA
E-mail: riecan@fpv.umb.sk