Tatra Mt. Math. Publ. **24** (2002), 141–146

TATRA MOUNTAINS
Mathematical Publications

BAIRE CATEGORY AND STANDARD OPERATIONS ON PAIRS OF CONTINUOUS FUNCTIONS

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ABSTRACT. Let C denote the space of all real-valued continuous functions on $[0,1]$ with the supremum norm. Let Φ be any of the following operations on $C \times C$: addition, multiplication, and composition. We show that the set of all pairs $(f,g) \in C \times C$ such that $\Phi(f,g)$ has not a finite derivative at any point of $[0,1]$ is residual. In the proofs we follow some methods used in the classical theorem of Banach and Mazurkiewicz.

In what follows we denote by C the space of all continuous real functions on $[0,1]$ equipped with the supremum norm, by C^* the subspace of C consisting of functions valued in $[0,1]$, by $\mathbb{N}$ the set of positive integers. All formulations "almost every" are meant in the sense of category, i.e., the respective properties hold on a residual subset of some space. Through the paper a function $f \in C$ will be called *ugly* if it is nowhere differentiable on $[0,1]$. We consider three standard operations on C, that is, if $f, g \in C$ and $x \in [0,1]$ then $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x) \cdot g(x)$, and if $(f,g) \in C \times C^*$ then $(f \circ g)(x) = f\big(g(x)\big)$.

To prove our main results we need the following definition and lemmas.

Definition 1. A function $l \in C$ is called a *polygonal function* or briefly a *polygonal* if there exist a positive integer n_l and a partition $0 = x_0 < x_1 < \cdots < x_{n_l} = 1$ of $[0,1]$ such that for each $i \in \{1, \dots, n_l\}$ there exist real numbers a_i, b_i such that for each $x \in [x_{i-1}, x_i]$ we have $l(x) = a_i x + b_i$. The points $x_0, x_1, \dots, x_{n_l}$ are called *the partition points* of the polygonal l. For any polygonal $l \in C$ let $M_l = \max\limits_{i \in \{1,\dots,n_l\}} |a_i|$.

Lemma 1. (Well-known) *The set of all polygonals is a dense subset of C.*

Lemma 2. (Banach, Mazurkiewicz) *The set*

$$\left\{ f \in C \colon \underset{n\in\mathbb{N}}{\exists} \ \underset{x\in[0,1-1/n]}{\exists} \ \underset{0<h<1-x}{\forall} \left| \frac{f(x+h)-f(x)}{h} \right| \le n \right\}$$

2000 Mathematics Subject Classification: Primary 26A27; Secondary 54E52.
Keywords: Baire category, nowhere differentiable continuous function, algebraic operations on a set of functions.

is of the first category in C. (See [1] or [2] or [3])

In our paper we shall use the following stronger version of Lemma 2

Lemma 3. *The set*

$$\left\{f \in C\colon \underset{n\in\mathbb{N}}{\exists}\ \underset{x\in[0,1-1/n]}{\exists}\ \underset{0<h<\min(1/n,1-x)}{\forall}\ \left|\frac{f(x+h)-f(x)}{h}\right| \le n\right\}$$

is of the first category in C.

Remark 1. Lemma 2 immediately implies that almost every function f in C is ugly. Observe also that, since the intersection of any two residual sets is residual, Lemma 3 implies in fact that almost every function f in C possesses unbounded right-hand and left-hand difference quotients at every neighbourhood of any point of $[0,1]$ (of course at 0 we consider only right-hand difference quotients, at 1-only left-hand difference quotients).

It is easy to prove the following lemma.

Lemma 4. *If $p \in C$ is nowhere constant on any interval $[a,b] \subset [0,1]$ $(a<b)$ and $l \in C$ is a polygonal, then for each $r>0$ there exists a polygonal l_0 in an open ball $B(l,r)$ such that the condition $\underset{x\in[0,1]}{\forall}\big(p(x)=0 \Rightarrow l_0(x)\neq 0\big)$ holds.*

Proposition 1. *For almost every pair (f,g) in $C\times C$ the function $f+g$ is ugly.*

P r o o f. It suffices to show that the set

$$F=\left\{(f,g)\in C\times C\colon \underset{n\in\mathbb{N}}{\exists}\ \underset{x\in[0,1-1/n]}{\exists}\ \underset{0<h<1-x}{\forall}\ \left|\frac{(f+g)(x+h)-(f+g)(x)}{h}\right|\le n\right\}$$

is of the first category in $C\times C$.

For $n\in\mathbb{N}$, let

$$F_n=\left\{(f,g)\in C\times C\colon \underset{x\in[0,1-1/n]}{\exists}\ \underset{0<h<1-x}{\forall}\ \left|\frac{(f+g)(x+h)-(f+g)(x)}{h}\right|\le n\right\}.$$

Of course $F=\bigcup_{n\in\mathbb{N}} F_n$. So, we need to show that, for each $n\in\mathbb{N}$, F_n is a nowhere dense subset of $C\times C$. It is easy to see that F_n is a closed subset of $C\times C$, thus we shall only prove that F_n has the empty interior. Suppose that it is not the case, i.e., there exist an $n_0\in\mathbb{N}$ and balls $B(t,\varepsilon)$, $B(s,\eta)$ in C, such that

$$B(t,\varepsilon)\times B(s,\eta)\subset F_{n_0}. \tag{1}$$

By Lemma 1 there exists a polygonal $l\in B(s,\eta)$. From Lemma 3 it follows that there is a function $p\in B(t,\varepsilon)$ such that for any $x\in[0,1)$ we can find an

$h_x \in (0, 1-x)$ such that $\left|\frac{p(x+h_x)-p(x)}{h_x}\right| > n_0+M_l$. Observe that $\left|\frac{l(x+h_x)-l(x)}{h_x}\right| \leq M_l$. Hence for all $x \in [0,1)$ we have:

$$\left|\frac{(p+l)(x+h_x)-(p+l)(x)}{h_x}\right| \geq \left|\frac{p(x+h_x)-p(x)}{h_x}\right| - \left|\frac{l(x+h_x)-l(x)}{h_x}\right|$$
$$> n_0 + M_l - M_l = n_0 .$$

Therefore $(p,l) \notin F_{n_0}$, which contradicts (1). So, F is of the first category in $C \times C$. □

Proposition 2. *For almost every pair (f,g) in $C \times C$ the function fg is ugly.*

Proof. It suffices to show that the set

$$F = \left\{(f,g) \in C \times C \colon \underset{n\in\mathbb{N}}{\exists} \ \underset{x\in[0,1-1/n]}{\exists} \ \underset{0<h<1-x}{\forall} \left|\frac{(fg)(x+h)-(fg)(x)}{h}\right| \leq n\right\}$$

is of the first category in $C \times C$.

For $n \in \mathbb{N}$, let

$$F_n = \left\{(f,g) \in C \times C \colon \ \underset{x\in[0,1-1/n]}{\exists} \ \underset{0<h<1-x}{\forall} \left|\frac{(fg)(x+h)-(fg)(x)}{h}\right| \leq n\right\}.$$

Of course $F = \bigcup_{n\in\mathbb{N}} F_n$. One can easily show that, for each $n \in \mathbb{N}$, F_n is a closed subset of $C \times C$, therefore we shall only show that F_n has the empty interior. Suppose to the contrary that there exist an $n_0 \in \mathbb{N}$ and balls $B(t,\varepsilon)$, $B(s,\eta)$ in C, such that

$$B(t,\varepsilon) \times B(s,\eta) \subset F_{n_0} . \tag{2}$$

We will follow a bit subtler than in the proof of Proposition 1. By Lemma 1 there exists a polygonal $l \in B(s,\eta)$. Without loss of generality we may assume that for each $i \in \{1,\dots,n_l\}$ we have $a_i \neq 0$. Let $A = \{x \in [0,1) \colon l(x) = 0\}$. There are two possible cases:

1) $A = \emptyset$. Since $l \in C$ then there exists $\delta > 0$ such that $|l(x)| > \delta$ for all $x \in [0,1]$. Let $x_0 \in [0,1)$. There exists an $h_{x_0} \in (0, 1-x_0)$ such that for all $x \in [x_0, x_0+h_{x_0}]$ we have

$$l(x) = a_{i_0}x + b_{i_0} , \qquad \text{for some} \quad i_0 \in \{1,\dots,n_l\} . \tag{3}$$

By Lemma 3 there exist a function $p \in B(t,\varepsilon)$ and a real number $h_0 \in (0, h_{x_0})$ such that the condition

$$\left|\frac{p(x_0+h_0)-p(x_0)}{h_0}\right| > \frac{n_0+KM_l}{\delta}$$

holds, where $K=\sup\limits_{x\in[0,1]}\left|p(x)\right|$. Then we have:

$$\left|\frac{(pl)(x_0+h_0)-(pl)(x_0)}{h_0}\right| \overset{(3)}{=} \left|(a_{i_0}x_0+b_{i_0})\frac{p(x_0+h_0)-p(x_0)}{h_0}+a_{i_0}p(x_0+h_0)\right|$$
$$\geq \left|(a_{i_0}x_0+b_{i_0})\right|\left|\frac{p(x_0+h_0)-p(x_0)}{h_0}\right|-\left|a_{i_0}p(x_0+h_0)\right| >$$
$$> \frac{\left|(a_{i_0}x_0+b_{i_0})\right|}{\delta}(n_0+KM_l)-KM_l \geq n_0+KM_l-KM_l=n_0\,.$$

Hence we infer that $(p,l_0)\notin F_{n_0}$ which contradicts (2), and we get the assertion.

2) $A\neq\emptyset$. Observe that $A=\{x_1,\dots,x_k\}$, where $k\in\mathbb{N}$ $(k\leq n_l)$ and $x_i\in[0,1]$ for $i\in\{1,\dots,k\}$. By Lemma 3 there exists a function $p\in B(t,\varepsilon)$ with an unbounded right-hand difference quotient at any point of $[0,1)$. Let x_i be a fixed element of $A\setminus\{1\}$. The function p fulfils the assumptions of Lemma 3, thus we may assume that $p(x_i)\neq 0$. Then there exist reals $h_{x_i}>0$ and $\delta_i>0$ such that for all $x\in[x_i,x_i+h_{x_i}]$ and some $j_i\in\{1,\dots,n_l\}$ the conditions $\left|p(x)\right|>\delta_i$, $l(x)=a_{j_i}x+b_{j_i}$ hold. Put $c_i=\frac{n_0+d_i}{\delta_i}$ where d_i is a positive real number chosen so that $c_i>\left|a_{j_i}\right|$. Let $z_0<\dots<z_{n_l}$ denote the partition points of the polygonal l and let $w_i=\min\{z_k\colon z_k>x_i,\ k=1,\dots,n_l\}$. Put $\lambda=\min\Big\{\inf\limits_{x\in[0,1]}\left|s(x)+\eta-l(x)\right|,\ \inf\limits_{x\in[k0,1]}\left|s(x)-\eta-l(x)\right|\Big\}$. It is clear that $\lambda>0$. Put also $h_i=\min\{\frac{h_{x_i}}{2},\frac{\lambda}{2(c_i-a_{j_i})}\}$. Define an auxiliary polygonal l_0 by the formula:

$$l_0(x)=\begin{cases} c_i(x-x_i) \text{ for } x\in[x_i,x_i+h_i], \text{ if } i=1,\dots,k \text{ and } a_{j_i}>0\,,\\ \frac{c_ih_i-a_{j_i}w_i-b_{j_i}}{x_i+h_i-w_i}(x-w_i)+a_{j_i}w_i+b_{j_i} \text{ for } x\in(x_i+h_i,w_i]\,,\\ \quad\text{if } i=1,\dots,k \text{ and } a_{j_i}>0\,,\\ -c_i(x-x_i) \text{ for } x\in[x_i,x_i+h_i]\,, \text{ if } i=1,\dots,k \text{ and } a_{j_i}<0\,,\\ -\frac{c_ih_i-a_{j_i}w_i-b_{j_i}}{x_i+h_i-w_i}(x-w_i)-a_{j_i}w_i-b_{j_i} \text{ for } x\in(x_i+h_i,w_i]\,,\\ \quad\text{if } i=1,\dots,k \text{ and } a_{j_i}<0\,,\\ l(x) \text{ for } x\in[0,1]\Big\backslash\bigcup\limits_{i\in\{1,\dots,k\}}[x_i,w_i]\,.\end{cases}$$

One can easily check that $l_0\in B(s,\eta)$. Let $x_0\in[0,1)$. If $x_0\in A$, there exists $i\in\{1,\dots k\}$ such that $x_0=x_i$. Then we have:

$$\left|\frac{(pl_0)(x_i+h_i)-(pl_0)(x_i)}{h_i}\right|=\left|c_ip(x_i+h_i)\right|>\frac{n_0+d_i}{\delta_i}\cdot\delta_i>n_0\,.$$

Hence $(p,l_0)\notin F_{n_0}$, a contradiction. If $x_0\notin A$ then by the considerations analogous to those in 1) we also get a contradiction. $\square$

Proposition 3. *For almost every pair $(f,g) \in C \times C^*$ the function $f \circ g$ is ugly.*

Proof. It suffices to show that the set

$$F = \left\{(f,g) \in C \times C^* \colon \underset{n\in\mathbb{N}}{\exists} \ \underset{x\in[0,1-1/n]}{\exists} \ \underset{0<h<1-x}{\forall} \left|\frac{(f\circ g)(x+h) - (f\circ g)(x)}{h}\right| \le n\right\}$$

is of the first category in $C \times C^*$. For $n \in \mathbb{N}$, let

$$F_n = \left\{(f,g) \in C \times C^* \colon \underset{x\in[0,1-1/n]}{\exists} \ \underset{0<h<1-x}{\forall} \left|\frac{(f\circ g)(x+h) - (f\circ g)(x)}{h}\right| \le n\right\}.$$

As in the proofs of Propositions 1 and 2, we show only that, for each $n \in \mathbb{N}$, F_n has the empty interior. Suppose that it is not the case, i.e., there exist an $n_0 \in \mathbb{N}$ and balls $B(t,\varepsilon) \subset C$, $B(s,\eta) \subset C^*$ such that $B(t,\varepsilon) \times B(s,\eta) \subset F_{n_0}$. From Lemma 1 it follows that there exists a polygonal $l \in B(s,\eta)$. We may assume that for $i \in \{1,\dots,n_l\}$ we have $a_i \neq 0$. By Remark 1 there is a function $p \in B(t,\varepsilon)$ with unbounded right-hand and left-hand differential quotients at every point of $[0,1]$. Let $x_0 \in [0,1)$. There exist an $h_{x_0} \in (0, 1-x_0)$ and reals a_{i_0}, b_{i_0} such that $l(x) = a_{i_0}x + b_{i_0}$ for all $x \in [x_0, x_0 + h_{x_0}]$ and some $i_0 \in \{1,\dots n_l\}$. By Remark 1 there exists an $h_0 \in (0, h_{x_0})$ such that

$$\left|\frac{p\big((a_{i_0}x_0 + b_{i_0}) + a_{i_0}\mathbf{h}_0\big) - p(a_{i_0}x_0 + b_{i_0})}{a_{i_0}\mathbf{h}_0}\right| > \frac{n_0}{|a_{i_0}|},$$

so, we have a contradiction. □

Now we can formulate the main result of the paper.

Theorem 1. *Let Φ be any of the three operations in $C \times C$: addition, multiplication, and composition. Then for almost every pair (f,g) in $C \times C$ the function $\Phi(f,g)$ is nowhere differentiable on $[0,1]$.*

Acknowledgements. The author would like to thank Marek Balcerzak for his helpful remarks and ideas.

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Received September 29, 2000

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