Tatra Mt. Math. Publ. **23** (2001), 113–123

TATRA MOUNTAINS
Mathematical Publications

ALGEBRAIC LANGUAGE IN THE THEORY OF HARMONY

Anna Romanowska

ABSTRACT. The paper surveys the results of more than twenty years of discussion concerning the foundations of a formal language for the theory of harmony in equal temperament tone systems.

Introduction

In 1976, Rudolf Wille [W1] proposed seven theses that should underline mathematical music theory and help to answer the question:

What can mathematics do for contemporary music theory?

Three of these theses will play an essential role in this paper:

1. Mathematics is the basis of a formal language for music theory.
2. Mathematical language makes possible exact definitions of concepts in music theory.
3. Mathematics furnishes complete and effective notation systems for concepts in music theory.

These three theses address the following problems of music theory in general, and the theory of harmony in particular:

A. The lack of a precise language for concepts.

B. The ambiguity and vagueness of many concepts.

C. The lack of appropriate specifications, and even names, for many concepts that prove to be useful.

These deficiencies preoccupied many musicologists in the twentieth century. As the Polish musicologist Maciej Zalewski wrote in 1973 [Z]: "Even a fleeting glance at the analyses of the compositions of modern music of the twentieth century classics or of the twelve-tone school reveals what one might dare to call

2000 Mathematics Subject Classification: 93A30, 91F99, 08A99.
Keywords: tone system, pointed chord, formal language, harmonic structure.

powerlessness, a lack of conceptual or terminological apparatus. Indeed, many theoreticians and musicologists have expressed this."

Zalewski's work "Harmonia Teoretyczna" ("Theoretical Harmony") was one of several attempts to overcome these deficiencies. Mathematical formulations of some of his concepts were provided by Romanowska [R1, 2, 4]. Another attempt of a similar type was undertaken by Wille [W1, 2].

In this paper, we will propose a language for the theory of harmony, using and unifying concepts of these three authors. This will serve to illustrate the three theses presented above. The "harmonical" content of a chord is described by using certain sequences of natural numbers and their permutations, or else certain matrices of natural numbers and certain sets of similar matrices. They are obtained by introducing some equivalence relations on the set of all chords of a tone system. Only equal temperament tone systems are considered.

The paper only sets up a language describing an "harmonical content" of chords. It does not discuss the relations between chords nor their meaning in different styles of music.

Though practicing musicians currently do not seem to pay much attention to theoretical considerations, the development and importance of computers and the internet in the modern world may renew interest in a more precise foundation of the theory of harmony. The mathematics used in the paper is elementary but not without subtleties. Readers are referred to, e.g., [SR].

1. Tone systems and chords

The central concept of music theory is the concept of a *tone*. And one of its most important characteristics is its *pitch*. In music one uses only a small choice of possible sounds, usually with determined pitches. The pitch is a psychological concept depending on the frequency of the tones. A higher frequency is perceived as a higher pitch. The most essential are the differences of pitches of two tones, described physically as ratios of their frequencies, and usually called intervals. Later, in Section 3, we will describe such differences in another way, as is usually done in music theory, and in such a way that intervals really have the properties of a distance. In particular, an interval with a frequency ratio equal to 2 is called an *octave*. Two tones a multiple of an octave apart are perceived as "the same". A set of tones chosen according to some rules (generally governing the intervals between consecutive tones) is called a *tone system*. Mathematically, we can consider it as a pair (T, h) consisting of a set T of tones and an injective function, called the *pitch function*, $h\colon T \to \mathbb{R}^+$, assigning to each tone its frequency (in cycles per second). A system that contains along with each tone the tones an octave apart, and where the octave is "divided" into n equal parts, is called an *equal temperament n-tone system*, and will be denoted by $\mathcal{T}_n$. The 12-tone system $\mathcal{T}_{12}$ is the best known and most widely used. In this paper we will consider only such systems. For theoretical reasons, it is convenient to consider the set T to be isomorphic to the set $\mathbb{Z}$ of integers. We frequently will identify the set T with the set $\mathbb{Z}$. In practice, obviously one restricts T to a finite subset. For two tones t_1 and t_2 of a system (T,h), we say that t_2 is *higher* than t_1, and we write $t_1 < t_2$, if $h(t_1) < h(t_2)$. In what follows, we assume that the tones of a system (T,h) are ordered in this way.

For example, the system $\mathcal{T}_{12} = (\mathbb{Z},h)$ has its pitch function defined by $h\colon \mathbb{Z} \to \mathbb{R}^+;\ z \mapsto 440.2^{\frac{z}{12}}$. Moreover, for each $z \in \mathbb{Z}$, $h(z+1)/h(z) = 2^{\frac{1}{12}}$. One can also consider this system as a pair (T,h) with $T = \{-48,\dots,36\}$.

Let us note that for any system $\mathcal{T}_n = (\mathbb{Z},h)$ with $h(0) = h_0$, one has $h(z) = h_0 \cdot 2^{\frac{z}{n}}$ for any z in $\mathbb{Z}$.

By a "chord" in a system $\mathcal{T}_n$, one usually understands a set of several tones played together or in a sequence built according to certain rules. This justifies to define an *m-chord* C in a system $\mathcal{T}_n = (T,h)$ as a finite subset of T with m elements. Denote by $C^m(\mathcal{T}_n)$ the set of all m-chords of $\mathcal{T}_n$ and by $C(\mathcal{T}_n) = \bigcup_{m\in\mathbb{N}} C^m(\mathcal{T}_n)$ the set of all m-chords for all natural m (including the "empty" chord or a "rest"). The theory of harmony deals with chords of a given tone system and with the relationships between them.

Harmonical analysis also requires the possibility of speaking about chords with some special selected notes, like a key note or a note in a melody. Such chords will be called *pointed*, and mathematically defined as pairs (C,t), where t is a selected element of a chord C. The "point" t can then be interpreted according to need. In this paper we will interpret t always as a key note. The symbol $C_p^m(\mathcal{T}_n)$ will denote the set of all pointed m-chords of the system $\mathcal{T}_n$. Similarly the symbol $C_p(\mathcal{T}_n)$ will denote the set of all pointed m-chords for all possible $m > 0$.

2. The harmonical structure of chords and pointed chords

The fact that tones of a system $\mathcal{T}_n = (T,h)$ an octave apart are harmonically identical can be expressed mathematically by means of the equivalence relation ω defined on T as follows:

$$(t,t') \in \omega \iff \log_2 h(t) - \log_2 h(t') \in \mathbb{Z}\,.$$

One can extend this relation to an equivalence relation Ω on the set $C(\mathcal{T}_n)$ by defining

$$\begin{aligned}(C,C') \in \Omega \iff &\text{ for each } t \text{ in } C \text{ there is } t' \text{ in } C' \text{ such that } t\,\omega\,t' \\ &\text{ and for each } t' \text{ in } C' \text{ there is } t \text{ in } C \text{ such that } t\,\omega\,t'.\end{aligned}$$

The Ω-classes are called *harmonies.* The set of all harmonies is denoted by $C(\mathcal{T}_n)^{\Omega}$. Examples are given by the harmonies C major and A minor of the system $\mathcal{T}_{12}$. The representatives of the harmony C major consist of C major triads in all possible (closed or open) positions, and inversions. Let us note that each non-empty harmony has as a representative an ordered k-chord $(t_1, \ldots, t_k)$ such that for any t_i, t_j

$$\frac{1}{2} < \frac{h(t_i)}{h(t_j)} < 2\,,$$

i.e., the chord $(t_1, \ldots, t_k)$ is in a closed position. Let us call such a representative *ordered closed.*

An equivalence relation Ω_p similar to Ω can be defined on the set $C_p(\mathcal{T}_n)$ of pointed chords:

$$\big((C,t),(C',t')\big) \in \Omega_p \iff (C,C') \in \Omega \quad \text{and} \quad (t,t') \in \omega\,.$$

The Ω_p-classes are called *pointed harmonies.* The set of all pointed harmonies is denoted by $C_p(\mathcal{T}_n)^{\Omega_p}$. All C major triads in the first inversion, i.e., with a note E as the lowest tone, form representatives of the pointed harmony C major in the first inversion (i.e., with a third in the bass) in the system $\mathcal{T}_{12}$. As in the previous case, each pointed harmony has an ordered closed representative.

The harmonies and pointed harmonies of a system $\mathcal{T}_n$ are related as follows. Let $(t_1, \ldots, t_k) =: \mathrm{oc}(h)$ be an ordered closed representative of a pointed harmony $h = \big(\{t_1, \ldots, t_k\}, t_1\big)^{\Omega_p} \in C_p(\mathcal{T}_n)^{\Omega_p}$. Then the (ordered closed) chord $(t_i, \ldots, t_k, t_1, \ldots, t_{i-1})$ is called an $(i-1)$-*inversion* of $(t_1, \ldots, t_k)$ and the pointed harmony $h^{i-1} = \big(\{t_1, \ldots, t_k\}, t_i\big)^{\Omega_p}$ is called an $(i-1)$-*inversion of the harmony* h. It is easy to see that the relation Ψ defined on the set $C_p(\mathcal{T}_n)^{\Omega_p}$ by

$$\begin{aligned}(h,h') \in \Psi \iff\ & \text{for each } \mathrm{oc}(h') \text{ there is } \mathrm{oc}(h) \text{ such that}\\ & \mathrm{oc}(h')\ \Omega\ \big(\mathrm{oc}(h)\big)^{i-1} \text{ for some } i\\ & \text{and for each } \mathrm{oc}(h) \text{ there is } \mathrm{oc}(h') \text{ such that}\\ & \mathrm{oc}(h)\ \Omega\ \big(\mathrm{oc}(h')\big)^{j-1} \text{ for some } j\end{aligned}$$

is an equivalence relation and that

$$C(\mathcal{T}_n)^{\Omega} = \big(C_p(\mathcal{T}_n)^{\Omega_p}\big)^{\Psi}\,.$$

Let r be a positive real number. An injective function $\tau_r \colon T \to T$, such that for each t in T

$$\frac{h\big(\tau_r(t)\big)}{h(t)} = r\,,$$

is called a *transposition* of the system $\mathcal{T}_n$ by r. The transposition τ_r moves each tone of T the same interval up or down.

Using the concept of a transposition, we will define one more equivalence relation Φ on the set $C(\mathcal{T}_n)$:

$$(C, C') \in \Phi \iff \exists r \in \mathbb{R}^+ \text{ such that } \tau_r(C) = C'.$$

Similarly for (C, t) and (C', t') in $C_p(\mathcal{T}_n)$ we define

$$\big((C, t), (C', t')\big) \in \Phi_p \iff \exists r \in \mathbb{R}^+ \text{ such that } \tau_r(t) = t' \text{ and } \tau_r(C) = C'\,.$$

Elements of the sets $C(\mathcal{T}_n)^{\Phi}$ and $C(\mathcal{T}_n)^{\Phi_p}$ of classes are called *chordal forms* and *pointed chordal forms* respectively. Intervals, like thirds, fifths, etc. are examples of chordal forms of the system $\mathcal{T}_{12}$.

Now composition of the relations Ω and Φ will lead us to the central concept of this section. Since $\Omega \circ \Phi = \Phi \circ \Omega$ and $\Omega_p \circ \Phi_p = \Phi_p \circ \Omega_p$, the compositions $\Omega \circ \Phi$ and $\Omega_p \circ \Phi_p$ are equivalence relations on $C(\mathcal{T}_n)$ and $C_p(\mathcal{T}_n)$, respectively. Two chords of $C(\mathcal{T}_n)$ are $\Omega \circ \Phi$-related if one can be obtained from the other by the octave identification and transposition. Elements of $C(\mathcal{T}_n)^{\Omega \circ \Phi}$ are called *harmonic structures* and elements of $C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p}$ are called *pointed harmonic structures.* As examples of harmonic structures of $\mathcal{T}_{12}$, one has major and minor triads. The first inversion of the major triad is an example of a pointed harmonic structure. As in the case of harmonies, (pointed) harmonic structures have ordered closed representatives. For example, the first inversion of the major triad has as representatives chords E, G, C' or B, D', G'.

Harmonic structures and pointed harmonic structures are related to each other similarly as harmonies and pointed harmonies. Let $(t_1, \ldots, t_k) =: \mathrm{oc}(s)$ and $(t'_1, \ldots, t'_l) =: \mathrm{oc}(s')$ be ordered closed representatives of the pointed harmonic structures $s = \big(\{t_1, \ldots, t_k\}, t_1\big)^{\Omega_p \circ \Phi_p}$ and $s' = \big(\{t'_1, \ldots, t'_l\}, t'_1\big)^{\Omega_p \circ \Phi_p}$. Then the $(i{-}1)$-*inversion* of s is the pointed harmonic structure $s^{i-1} = \big(\{t_1, \ldots, t_k\}, t_i\big)^{\Omega_p \circ \Phi_p}$. Define the following equivalence relation ψ on $C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p}$ similarly as Ψ:

$$\begin{aligned}(s, s') \in \psi \iff\ & \text{for each } \mathrm{oc}(s') \text{ there is } \mathrm{oc}(s) \text{ such that} \\ & \mathrm{oc}(s')\ \Omega \circ \Phi\ \big(\mathrm{oc}(s)\big)^{i-1} \text{ for some } i \\ & \text{and for each } \mathrm{oc}(s) \text{ there is } \mathrm{oc}(s') \text{ such that} \\ & \mathrm{oc}(s)\ \Omega \circ \Phi\ \big(\mathrm{oc}(s')\big)^{j-1} \text{ for some } j\,.\end{aligned}$$

Then, as in the previous case,

$$C(\mathcal{T}_n)^{\Omega \circ \Phi} = \big(C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p}\big)^{\psi}\,.$$

3. Representations of harmonic structures

The definitions of harmonic structures provided in the previous section capture what is essential in a chord from the harmonic point of view. Nevertheless they would be rather awkward to use. In this section we provide some simple ways of representing harmonic structures.

First we define an *interval* (or a *distance*) between two tones. Let (t_1, t_2) be an ordered closed representative of a pointed harmony of a system $\mathcal{T}_n$. Then the interval between t_1 and t_2 is the number

$$d(t_1, t_2) = \log_{2^{\frac{1}{n}}} \frac{h(t_2)}{h(t_1)} = n\big(\log_2 h(t_2) - \log_2 h(t_1)\big)\,.$$

If $\mathcal{T}_n = (\mathbb{Z}, h)$ and $h(0) = h_0$, then

$$d(t_1, t_2) = \log_{2^{\frac{1}{n}}} \frac{h(t_2)}{h(t_1)} = \log_{2^{\frac{1}{n}}} 2^{\frac{t_2 - t_1}{n}} = t_2 - t_1$$

is a positive integer. For two consecutive tones $z, z+1$ of the system $\mathcal{T}_{12}$ one has

$$d(z, z+1) = \log_{2^{\frac{1}{12}}} 2^{\frac{1}{12}} = 1\,.$$

Hence in $\mathcal{T}_{12}$, the interval 1 corresponds to the equal temperament semitone. For any two t_i, t_j of an ordered closed chord with $t_i < t_j$, $d(t_i, t_j)$ is the number of semitones between t_i and t_j.

Now let $(t_1, \ldots, t_k)$ be an ordered closed representative of a pointed harmonic structure $(C, t_1)^{\Omega_p \circ \Phi_p}$ of a system $\mathcal{T}_n$. Let t_0 be a tone of T an octave higher than t_1. And finally let S be the set of finite sequences of natural numbers. Define the following mapping

$$f\colon C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p} \longrightarrow S;$$

$$(C, t_1)^{\Omega_p \circ \Phi_p} \mapsto \left(\log_{2^{\frac{1}{n}}} \frac{h(t_2)}{h(t_1)}, \ldots, \log_{2^{\frac{1}{n}}} \frac{h(t_k)}{h(t_{k-1})}, \log_{2^{\frac{1}{n}}} \frac{h(t_0)}{h(t_k)}\right),$$

Let $S_n := f\big(C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p}\big)$.

It is obvious that elements of S_n are sequences $(a_1, \ldots, a_k)$ of positive integers less than n. Moreover, for each $(a_1, \ldots, a_k)$ in S_n

$$\begin{aligned}\sum_{i=1}^{k} a_i &= \log_{2^{\frac{1}{n}}} \left(\frac{h(t_2)}{h(t_1)} \cdot \frac{h(t_3)}{h(t_2)} \cdots \frac{2h(t_1)}{h(t_k)}\right)\\ &= \log_{2^{\frac{1}{n}}} 2 = n\,.\end{aligned}$$

The image $f\big((C, t_1)^{\Omega_p \circ \Phi_p}\big)$ is called the *pointed interval structure* of the pointed harmonic structure $(C, t_1)^{\Omega_p \circ \Phi_p}$. The elements of S_n are called *pointed interval structures of the system* $\mathcal{T}_n$. Additionally, we may assume that the empty harmonic structure has an empty pointed interval structure denoted by $1 = (\ )$. For example, the 1st inversion of the dominant seventh in $\mathcal{T}_{12}$ has the pointed interval structure $(3, 3, 2, 4)$.

The following propositions are easy consequences of the definitions above.

PROPOSITION 3.1. *A sequence $(a_1, \ldots, a_k)$ of positive integers with $k \leq n$ is a pointed interval structure of a system $\mathcal{T}_n$ if and only if for all $i = 1, \ldots, k$*

$$a_i \in \{1, \ldots, n\} \qquad \text{and} \qquad \sum_{i=1}^{k} a_i = n\,.$$

PROPOSITION 3.2. *There is a bijective correspondence between the set $C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p}$ of pointed harmonic structures and the set S_n of pointed interval structures.*

The next step provides a passage from pointed to non-pointed interval structures.

Let $a = (a_1, \ldots, a_k)$ be in S_n. Then for each $i = 1, \ldots, k$, the sequence $a^{i-1} := (a_i, \ldots, a_k, a_1, \ldots, a_{i-1})$ is also a pointed interval structure of $\mathcal{T}_n$, and is called *the $(i-1)$-inversion of* a. It is easy to see that the relation (ψ) defined on the set S_n by

$$(a, b) \in (\psi) \iff b = a^{i-1} \quad \text{for some } i$$

is an equivalence relation. The elements $((a_1, \ldots, a_k)) := (a_1, \ldots, a_k)^{(\psi)}$ of the set $(S_n) := S_n^{(\psi)}$ of (ψ)-classes are called *interval structures* of the system $\mathcal{T}_n$. The following theorem establishes the final relationship between harmonic structures and interval structures of the system $\mathcal{T}_n$.

THEOREM 3.3. *There is a bijective correspondence between the set $C(\mathcal{T}_n)^{\Omega \circ \Phi}$ of harmonic structures and the set (S_n) of interval structures of the system $\mathcal{T}_n$.*

We omit the rather standard proof of this theorem. Note, however, the following:

$$C(\mathcal{T}_n)^{\Omega \circ \Phi} = (C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p})^{\psi} \cong S_n^{(\psi)}.$$

Harmonic and pointed harmonic structures can also be represented in a different way. First define the following mapping:

$$h \colon S_n \to S;\ (a_1, \ldots, a_k) \mapsto \left(a_1, a_1 + a_2, \ldots, \sum_{i=1}^{k} a_i\right) =: (A_1, A_2, \ldots, A_k)\,.$$

For $a = (a_1, \ldots, a_k) \in S_n$, the sequence $h(a)$ is called the *form of the pointed interval structure* a and the set $F_n = h(S_n)$ is the set of forms of pointed interval structures of the system $\mathcal{T}_n$. One has obvious counterparts to Propositions 3.1 and 3.2.

Proposition 3.4. *A sequence $(A_1, \dots, A_l)$ of positive integers with $l \leq n$ is the form of a pointed interval structure if and only if the sequence is strictly increasing and $A_k = n$.*

Proposition 3.5. *There is a bijective correspondence between the set $C_p(\mathcal{T}_n)^{\Omega_p \circ \Phi_p}$ of pointed harmonic structures and the set F_n of forms of pointed interval structures.*

P r o o f. The required bijection is given by the composition of the mappings f and h. □

The definition of an interval structure shows that we can consider it as the set of all cyclic permutations of any representative (a pointed interval structure). Sometimes it is more convenient to work with all representatives at the same time. One can do this by putting all of them in a matrix. However, one has much nicer properties, and an easier passage between interval structures and corresponding matrices, if instead of pointed interval structures one considers their forms arranged in a suitable way.

Let $\mathcal{M}$ be the set of square matrices of non-negative integers. Consider the following mapping

$$m \colon S_n \to \mathcal{M};$$

$$a = (a_1, \dots, a_k) \mapsto \begin{bmatrix} 0 & a_1 & a_1 + a_2 & \dots & \sum_{i=1}^{k-1} a_i \\ \sum_{i=2}^{k} a_i & 0 & a_2 & \dots & \sum_{i=2}^{k-1} a_i \\ \dots & \dots & \dots & \dots & \dots \\ a_k & a_k + a_1 & a_k + a_1 + a_2 & \dots & 0 \end{bmatrix}.$$

The image $m(a)$ is called the *matrix of the pointed interval structure* a. The set $\mathcal{M}_n := m(S_n)$ is the set of matrices of the pointed interval structures of the system $\mathcal{T}_n$.

Proposition 3.6. *A $k \times k$-matrix $\alpha = (a_{ij})$ with $k \leq n$ of non-negative integers belongs to the set $\mathcal{M}_n$ if and only if:*

(a) *it has zeros down the main diagonal;*

(b) *for each $i = 1, \dots, k$, the sequence $(a_{ii}, a_{ii+1}, \dots, a_{ik}, a_{i1}, \dots, a_{ii-1})$ is strictly increasing;*

(c) *$a_{ij} + a_{ji} = n$ for $j \neq i$;*

(d) *$a_{ij} = a_{1j} - a_{1i}$ for $j > i \neq 1$.*

We omit the easy proof based on elementary properties of matrices. Note only that for a given matrix α in the set $\mathcal{M}_n$, the corresponding pointed interval structure is recovered from its first row as $(a_{12}, a_{13}-a_{12}, \ldots, a_{1k}-a_{1k-1}, n-a_{1k})$.

Inversions of pointed interval structures have an easy counterpart when dealing with their matrices. Indeed, given a pointed interval structure $a = (a_1, \ldots, a_k)$ in S_n and its matrix $m(a)$ in $\mathcal{M}_n$, its i-inversion, $a^i = (a_{i+1}, \ldots, a_k, a_1, \ldots, a_i)$ has as its matrix the matrix α^i obtained from $m(a)$ by moving the top row down to the bottom and the left column across to the right i times. Mathematically, it can be expressed as follows

$$\alpha^i = E_{(i)}^{-1} \alpha E_{(i)} ,$$

whose $E_{(i)}$ is a suitable permutation $k \times k$-matrix. This leads to the last concept we need — the matrix counterpart of an interval structure. In the set $\mathcal{M}_n$, define the relation $\widetilde{\psi}$ as follows

$$(\alpha, \beta) \in \widetilde{\psi} \quad \Longleftrightarrow \quad \text{there is } i \leq n \text{ such that } \beta = \alpha^i .$$

The relation $\widetilde{\psi}$ is an equivalence relation and one easily proves that for a, b in S_n, $a\ (\psi)\ b$ if and only if $m(a)\ \widetilde{\psi}\ m(b)$. This implies the following.

Theorem 3.7. *There is a one-to-one correspondence between the set (S_n) of interval structures of the system $\mathcal{T}_n$ and the set of $\widetilde{\psi}$-classes of $\mathcal{M}_n$.*

4. Concluding remarks

As we have seen in the previous sections, interval structures and their matrices provide a precise mathematical language to describe the harmonical aspects of chords in equal temperament tone systems $\mathcal{T}_n$. Such concepts can also be defined for other tone systems, as for example systems (T, h) such that for any t_2, t_1 in T, one has $h(t_2)/h(t_1) \in \mathbb{Q}^+$. (See, e.g., [R3] and [F].)

The set of (pointed) harmonic structures, as well as the corresponding isomorphic set of (pointed) interval structures and the set of their matrices, surprisingly all have quite rich (isomorphic) algebraic structures. It was fully described in [R1, 2]. Not all these algebraic aspects have an interesting interpretation in music theory, so we will not repeat it here. We mention only some of them, and restrict ourselves mainly to the sets S_n and (S_n). (It is not difficult to describe isomorphic "algebras" of harmonic structures or matrices.)

Both the sets S_n and (S_n) have naturally defined partial orders. For $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_l)$ in S_n define $a \leq b$ if there is a sequence

$0 = t_0 < t_1 < \cdots < t_l = k$ such that for each $j = 1, \ldots, l$

$$b_j = \sum_{i=t_{j-1}+1}^{t_j} a_i \,.$$

It is easy to see that $\leq$ is a partial order on S_n. Moreover the following holds.

PROPOSITION 4.1. *The ordered sets $(S_n, \leq)$ of (non-empty) pointed interval structures, and $(CgC_n, \leq)$ of the congruence relations of the n-element chain considered as a lattice, are order-isomorphic.*

The proof is rather standard, and we omit it here. As a consequence, we get the following.

COROLLARY 4.2. *The ordered set $(S_n, \leq)$ is a Boolean lattice 2^{n-1}.*

The corresponding order relation on M_n is also easy to describe. For $a = (a_1, \ldots, a_k) \leq b = (b_1, \ldots, b_l)$, the matrix $m(a)$ is obtained form the matrix $m(b)$ by cancelling some rows and columns.

Though the Boolean operations are not preserved under the relation (ψ) on S_n, the inversions and some further unary operations [as for example $-a = -(a_1, a_2, \ldots, a_k) = (a_k, a_{k-1}, \ldots, a_1)$] are preserved.

The order relation $\leq$ on the set (S_n) of interval structures is defined by

$$\text{(a)} \leq \text{(b)} \iff \text{for each } c \text{ in (a) there is } d \text{ in (b) with } c \leq d.$$

REFERENCES

[F] FOKS, M.: *Elements of mathematical music theory*, Master's Thesis, Warsaw Technical University, 1994. (In Polish)

[Re] REISS, J. W.: *A Shorter Music Encyclopedia*, PWM, Warsaw, 1960. (In Polish)

[R1] ROMANOWSKA, A.: *An attempt to investigate the algebraic structure of a tone system*, Doctoral Thesis, Warsaw Technical University, 1973. (In Polish)

[R2] ROMANOWSKA, A.: *Algebraic structure of the tone system*, Demonstratio Math. **7** (1974), 525–542.

[R3] ROMANOWSKA, A.: *Über algebraische Aspekte der Theorie der Harmonie*, unpublished manuscript, 1975.

[R4] ROMANOWSKA, A.: *On some algebras connected with the theory of harmony*, Colloq. Math. **41** (1979), 181–185.

[S] SIKORSKI, K.: *Harmony*, PWM, Warsaw, 1952. (In Polish)

[SR] SMITH, J. D. H.—ROMANOWSKA, A.: *Post-modern Algebra*, Wiley, New York, 1999.

[Ś] ŚLEDZINSKI, S. (ed): *A Shorter Music Encyclopedia*, PWM, Warsaw, 1981. (In Polish)

[W1] WILLE, R.: *Mathematik and Musiktheorie*, in: Musik und Zahl (G. Schnitzler, ed.), Verlag für systematische Musik-Wissenschaft, Bonn-Bad Godesberg, 1976, pp. 233–264.

[W2] WILLE, R.: *Mathematische Sprache in der Musiktheorie*, Preprint 505, TH Darmstadt, 1979.

[Z] ZALEWSKI, M.: *Theoretical Harmony*, PWSM, Warsaw, 1973. (In Polish)

Received May 11, 2000

Faculty of Mathematics
and Information Sciences
Warsaw University of Technology
Plac Politechniki 1
PL–00-661 Warsaw
POLAND

E-mail: aroman@alpha.mini.pw.edu.pl

Department of Mathematics Iowa State University
Ames, Iowa 50011
U. S. A.