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FOURIER TRANSFORMS OF VECTOR-VALUED MEASURES ON A CERTAIN COMPACT SEMIGROUP

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ABSTRACT. We present a characterization of Fourier transforms of vector-valued measures of bounded semivariation on a certain compact semigroup interesting from the point of view of fuzzy sets and systems.

Introduction

The purpose of this paper is to characterize Fourier transforms of vector-valued measures of bounded semivariation defined on the system of all Borel subsets of a certain compact semigroup G which was introduced in [8], cf. also [6]. The results concerning the semigroup G are of some interest in fuzzy systems theory [10]. The characterization for vector-valued measures on locally compact Abelian groups can be found, e.g., in [1] or [11], [12], [14].

The paper is divided into four sections. Sections 1 and 2 are introductory. Section 1 contains some basic facts about the considered semigroup G and its semicharacters. There is a necessary and sufficient condition recalled under which a complex-valued function defined on the semicharacter semigroup Γ of G is the Fourier transform of a complex-valued regular Borel measure on G [4]. In Section 2 a necessary and sufficient condition is given for a vector-valued function to be the Fourier transform of a vector-valued measure of finite variation [5]. Section 3 is devoted to the Fourier transforms of vector-valued measures of bounded semivariation. Finally, Section 4 deals with Fourier transforms of vector-valued measures of weakly compact semivariation.

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1. Preliminaries

Recall that G is a non-empty set, completely ordered by a transitive, irreflexive relation denoted by $<$ and made into a semigroup by the operation $xy = \max(x, y)$ for all $x, y \in G$. It is known that G is a topological semigroup in the interval topology. We will suppose that G is compact in the interval topology, hence, G has the least element denoted by α, and the greatest element denoted by ω. The intervals are defined as usual, so G can be written in the form $G = [\alpha, \omega]$.

An important role in the solution of our problem is played by semicharacters of the semigroup G. A semicharacter of the semigroup G is a bounded complex-valued function γ defined on G, not identically zero, and satisfying the equation

$$\gamma(xy) = \gamma(x)\gamma(y) \qquad \text{for each } \ x, y \in G\,.$$

It is known that G has semicharacters only of two types ([8], Th. 1.8.). Namely, the functions

$$\gamma_{a]}(x) = \begin{cases} 1, & x \le a\,, \\ 0, & x > a\,, \end{cases} \quad a \in G \ \text{ and } \ \gamma_{a)}(x) = \begin{cases} 1, & x < a\,, \\ 0, & x \ge a\,, \end{cases} \quad a \in (\alpha, \omega]$$

are all semicharacters of the semigroup G.

The set Γ of all semicharacters of G is a semigroup under pointwise multiplication ([8], Th. 1.12.).

As usual, we denote by $C(G)$ the space of all continuous complex functions defined on G with the sup (max) norm $\|f\|$, $f \in C(G)$. As it is well-known, the space of all continuous linear forms on $C(G)$ can be represented as the space $M(G)$ of all complex-valued regular Borel measures on G with the total variation norm. Let K denote the set of all complex numbers. The mappings

$$\boldsymbol{a}\colon M(G) \to K, \quad \boldsymbol{a}(\mu) = \mu\big([\alpha, a]\big) = \int\limits_G \gamma_{a]}(x)\, d\mu(x)\,, \quad a \in G\,, \tag{1}$$

$$\boldsymbol{a'}\colon M(G) \to K\,, \quad \boldsymbol{a'}(\mu) = \mu\big([\alpha, a)\big) = \int\limits_G \gamma_{a)}(x)\, d\mu(x)\,, \quad a \in (\alpha, \omega] \tag{2}$$

are all homomorphisms on $M(G)$.

The semigroup Γ of all semicharacters of G can be identified with the set of all complex homomorphisms on $M(G)$. If we denote the set of all homomorphisms of the type (1) by $\boldsymbol{G}$ and the set of all homomorphisms of the type (2) by $\boldsymbol{G'}$, we can write $\Gamma = \boldsymbol{G} \cup \boldsymbol{G'}$.

At the end of this section, let us recall the definition of the Fourier transform of a complex-valued measure $\lambda \in M(G)$.

Definition 1. Let $\lambda \in M(G)$. The function $\widehat{\lambda}\colon \Gamma \to K$ defined by

$$\widehat{\lambda}(\boldsymbol{a}) = \boldsymbol{a}(\lambda) = \lambda\big([\alpha, a]\big) = \int\limits_G \gamma_{a]}(x)\, d\lambda(x) \qquad \text{for } a \in G$$

and

$$\widehat{\lambda}(\boldsymbol{a'}) = \boldsymbol{a'}(\lambda) = \lambda\big([\alpha, a)\big) = \int\limits_G \gamma_{a)}(x)\, d\lambda(x) \qquad \text{for } a \in (\alpha, \omega]$$

is called the *Fourier transform of the measure* λ.

We will write it simply

$$\widehat{\lambda}(\gamma) = \int\limits_G \gamma(x)\, d\lambda(x)\,, \qquad \gamma \in \Gamma\,.$$

From now on, the semigroup Γ will be endowed with the weakest topology that makes all functions $\widehat{\lambda}$ continuous (the Gelfand topology on Γ). This topology is described in detail in [8].

For any $a, b \in G$ let us define the interval $[\boldsymbol{a}, \boldsymbol{b}] = \{\boldsymbol{c} \in \boldsymbol{G};\ a \le c \le b\}$ and let the other intervals be defined analogously.

Definition 2. Let p be a complex-valued function defined on Γ and let $a, b \in G$, $a < b$. Then the quantity

$$\sup_D \left\{ \sum_{i=1}^{n} \big| p(\boldsymbol{t}_i) - p(\boldsymbol{t}_{i-1}) \big| \colon D \subset G \right\},$$

where the supremum is taken over all sets $D = \{t_0, t_1, \dots, t_n \colon a = t_0 < t_1 < \dots < t_n = b\}$, is called the *variation of the function* p *on the interval* $[\boldsymbol{a}, \boldsymbol{b}]$ and is denoted by $V(p;\, \boldsymbol{a}, \boldsymbol{b})$. For $a = b$ we put $V(p;\, \boldsymbol{a}, \boldsymbol{b}) = 0$. The function p is said to be of bounded variation on the interval $[\boldsymbol{a}, \boldsymbol{b}]$ if $V(p;\, \boldsymbol{a}, \boldsymbol{b}) < \infty$.

The complex-valued function p defined on Γ is the Fourier transform of a measure λ in $M(G)$ if and only if p is continuous and of bounded variation on $\boldsymbol{G} = [\boldsymbol{\alpha}, \boldsymbol{\omega}]$.

2. Fourier transforms of vector-valued measures

Definition 3. Let X be a Banach space, $\mathcal{B}(G)$ be the system of all Borel subsets in G and let m be a vector-valued measure, $m\colon \mathcal{B}(G) \to X$. A function $\widehat{m}\colon \Gamma \to X$ defined by

$$\widehat{m}(\gamma) = \int\limits_G \gamma(x)\, dm(x) \tag{3}$$

is called the *Fourier transform of the measure* m.

Let us denote by X' the topological dual of X, i.e., the space of all continuous linear functionals on X. From Definition 3 and the properties of integrals of complex-valued functions it follows that $\widehat{m}(\gamma)$ is that element for which the following is true

$$\langle \widehat{m}(\gamma), x' \rangle = \int\limits_G \gamma(x)\, dx' \circ m(x), \qquad x' \in X'. \tag{4}$$

Since m is a vector-valued measure defined on $\mathcal{B}(G)$, $x' \circ m$ is a complex-valued Borel measure for all $x' \in X'$. By Definition 1,

$$\int\limits_G \gamma(x)\, dx' \circ m(x) = \widehat{x' \circ m}(\gamma) \tag{5}$$

and so, from (4) and (5) we get

$$\langle \widehat{m}(\gamma), x' \rangle = \widehat{x' \circ m}(\gamma) \qquad \text{for all } x' \in X' \text{ and } \gamma \in \Gamma. \tag{6}$$

If $\gamma = \gamma_{a]}$, $a \in G$, then

$$\widehat{m}(\gamma_{a]}) = \widehat{m}(\boldsymbol{a}) = m([\alpha, a])$$

and if $\gamma = \gamma_{a)}$, $a \in (\alpha, \omega]$, then

$$\widehat{m}(\gamma_{a}) = \widehat{m}(\boldsymbol{a'}) = m([\alpha, a)).$$

In the sequel, X will always denote a Banach space.

Definition 4. Let $m\colon \mathcal{B}(G) \to X$ be a vector-valued measure. The function $\|m\|_v$ defined by

$$\|m\|_v(E) = \sup \sum_{i \in I} \|m(E_i)\|, \qquad E \in \mathcal{B}(G),$$

where the supremum is taken over all finite disjoint systems $\{E_i\}_{i \in I}$, $E_i \in \mathcal{B}(G)$, such that $E = \bigcup\limits_{i \in I} E_i$, is called the *variation of the vector-valued measure* m.

Definition 5. A vector-valued measure m is of *bounded variation* [2] if

$$\|m\|_v(E) < \infty \qquad \text{for each } E \in \mathcal{B}(G).$$

Let $M_v(G)$ denote the set of all regular vector-valued measures $m\colon \mathcal{B}(G) \to X$ of bounded variation. Note that a vector-valued measure m is regular if a complex-valued measure $x' \circ m$ is regular for each $x' \in X'$.

Definition 6. Let $a, b \in G$, $a < b$. The *variation of a function* $q\colon \Gamma \to X$ *on the interval* $[\boldsymbol{a}, \boldsymbol{b}]$ is a quantity denoted by $V(q; \boldsymbol{a}, \boldsymbol{b})$ and defined by

$$\sup_{D \subset G} \sum \|q(\boldsymbol{t}_i) - q(\boldsymbol{t}_{i-1})\|,$$

where the supremum is taken over all sets $D = \{t_0, t_1, \dots, t_n;\ a = t_0 < t_1 < \dots < t_n = b\}$. If $a = b$ we put $V(q; \boldsymbol{a}, \boldsymbol{b}) = 0$.

Following the definition for complex-valued functions [9], we say that a function $q\colon \Gamma \to X$ is of bounded variation on the interval $[\boldsymbol{a}, \boldsymbol{b}]$ if $V(q; \boldsymbol{a}, \boldsymbol{b}) < \infty$, [9].

In [5], the following theorem for vector-valued functions was proved.

Theorem 1. *A function $q\colon \Gamma \to X$ is the Fourier transform of a vector-valued measure $m \in M_v(G)$ if and only if q is weakly continuous and of bounded variation on $\boldsymbol{G}$.*

Example 1. Let $G = [0, 1]$, $X = L_2(G, \mu)$, where μ is the Lebesgue measure on G. Let q be defined on $\Gamma = \mathbf{G} \cup \mathbf{G}'$ by

$$q(\mathbf{a}) = \chi_{[0,a]}, \qquad a \in [0, 1],$$
$$q(\mathbf{a}') = \chi_{[0,a)}, \qquad a \in (0, 1],$$

where χ stands for the characteristic function of a set. Then the variation $V(q; \mathbf{0}, \mathbf{1}) = \infty$.
Indeed, let

$$D_n = \{t_0, t_1, \dots, t_n\} = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\} \subset G.$$

Then

$$\begin{aligned}\sum_{i=1}^{n} \|q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\| &= \sum_{i=1}^{n} \|\chi_{[0,t_i]} - \chi_{[0,t_{i-1}]}\| \\ &= \sum_{i=1}^{n} \|\chi_{(t_{i-1},t_i]}\| = \sum_{i=1}^{n} \sqrt{\int_G |\chi_{(t_{i-1},t_i]}|^2 \, d\mu} \\ &= \sum_{i=1}^{n} \sqrt{\frac{1}{n}} = \sqrt{n}.\end{aligned}$$

It follows that

$$\begin{aligned}V(q; \mathbf{0}, \mathbf{1}) &= \sup_{D} \sum \|q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\| \\ &\geq \sup_{D_n} \sum \|q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\| = \infty,\end{aligned}$$

which is our claim.

The function q given in Example 1 is not of bounded variation on $\mathbf{G}$ but it is of bounded semivariation on $\mathbf{G}$ [9].

Definition 7. The *semivariation of a function* $q\colon \Gamma \to X$ *on the interval* $[\mathbf{a}, \mathbf{b}]$, the notation $s\text{-}V(q; \mathbf{a}, \mathbf{b})$, is the supremum of the numbers

$$\Big\| \sum d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big] \Big\| ,$$

where the supremum is taken over all divisions

$$D = \{t_0, t_1, \dots, t_n \colon a = t_0 < t_1 \dots < t_n = b\}$$

of the interval $[a, b]$ and over all finite systems $\{d_i\}_{i \in I}$ of complex numbers with the property $|d_i| \le 1$.

We say that a function $q\colon \Gamma \to X$ is of *bounded semivariation on the interval* $[\mathbf{a}, \mathbf{b}]$ if there exists a constant $K = K(a, b) < \infty$ such that $s\text{-}V(q; \mathbf{a}, \mathbf{b}) \le K$.

It is clear that if a function q is of bounded variation on the interval $[\mathbf{a}, \mathbf{b}]$ then it is of bounded semivariation on $[\mathbf{a}, \mathbf{b}]$.

Example 2. The function q given in Example 1 is of bounded semivariation on $\mathbf{G}$.

Let $D = \{t_0, t_1, \dots, t_n \colon 0 = t_0 < t_i < \dots < t_n = 1\}$ be an arbitrary division of the interval $[0, 1]$ and let $\{d_i\}$ be any finite family of complex numbers with $|d_i| \le 1$ for every i. Then

$$\begin{aligned} \Big\| \sum_{i=1}^{n} d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big] \Big\| &= \Big\| \sum_{i=1}^{n} d_i \chi_{(t_{i-1}, t_i]} \Big\| \\ &= \sqrt{\int_G \Big| \sum_{i=1}^{n} d_i \chi_{(t_{i-1}, t_i]} \Big|^2 d\mu} \le 1 , \end{aligned}$$

because

$$\begin{aligned} \Big| \sum_{i=1}^{n} d_i \chi_{(t_{i-1}, t_i]} \Big|^2 &\le \Big(\sum_{i=1}^{n} |d_i| \chi_{(t_{i-1}, t_i]} \Big)^2 \\ &\le \Big(\sum_{i=1}^{n} \chi_{(t_{i-1}, t_i]} \Big)^2 = \chi_{(0,1]} . \end{aligned}$$

Since the inequality

$$\Big\| \sum_{i=1}^{n} d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big] \Big\| \le 1$$

holds for every division D and any family of complex numbers $\{d_i\}$ introduced above, we have

$$s\text{-}V(q; \mathbf{a}, \mathbf{b}) = \sup_{D, \{d_i\}} \Big\| \sum_{i=1}^{n} d_i \big[q(\mathbf{t}_{i-1}) - q(\mathbf{t}_{i-1})\big] \Big\| \le 1 .$$

Vector-valued functions of bounded semivariation on $\mathbf{G}$ are closely connected with Fourier transforms of vector-valued measures on Borel sets $B(G)$ into X that are of bounded semivariation [7].

DEFINITION 8. The *semivariation of a vector measure* $m\colon B(G) \to X$, [7], is a set function $\|m\|_s$ defined by

$$\|m\|_s(A) = \sup \Big\| \sum d_i m(A_i) \Big\| , \qquad A \in B(G) ,$$

where the supremum is taken over all finite decompositions $\{A_i\}$, $A_i \in B(G)$, of the set A, and all finite families of complex numbers $\{d_i\}$ with $|d_i| \le 1$.

In contrast with the variation of a vector-valued measure $m\colon B(G) \to X$, the semivariation is always a finite nonnegative set function.
We will use the following properties of $\|m\|_s$ [2], [13].

$$\|m\|_s(A) \ge \big\|m(A)\big\| \ge 0 , \qquad A \in B(G) ,$$

$$\|m\|_s(A) \le 4 \sup_{B \subset A} \big\|m(B)\big\| < \infty , \qquad A \in B(G) ,$$

$$\|m\|_s(A) \le \|m\|_s(B) , \qquad A, B \in B(G),\ A \subset B .$$

3. Fourier transforms of vector-valued measures of bounded semivariation

THEOREM 2. *If a function* $q\colon \Gamma \to X$ *is the Fourier transform of a measure* $m\colon B(G) \to X$ *then* q *is weakly continuous and of bounded semivariation on* $\mathbf{G}$.

P r o o f . Let $q = \widehat{m}$. The weak continuity can be proved as in [5]. Further, let D be any division of the set G and $\{d_i\}$ be any system of complex numbers as in Definition 8.
Then

$$\Big\| \sum d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big] \Big\| = \Big\| \sum d_i \big[\widehat{m}(\mathbf{t}_i) - \widehat{m}(\mathbf{t}_{i-1})\big] \Big\| ,$$

$$\Big\| \sum d_i \ m\big((t_{i-1}, t_i]\big) \Big\| \le \|m\|_s(G) < \infty .$$

Since the preceding inequality is true for any considered set D and system of complex numbers $\{d_i\}$,we have

$$s\text{-}V(q; \boldsymbol{\alpha}, \boldsymbol{\omega}) \le \|m\|_s(G) < \infty ,$$

which proves our theorem. □

Recall [3], [7] that if $m\colon B(G) \to X$ is a vector-valued measure then there exists a finite positive measure ν such that $\lim\limits_{\nu(E) \to 0} m(E) = 0$. This means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\nu(E) < \delta$ implies $\big\|m(E)\big\| < \varepsilon$.

PROPOSITION 1. *If a function $q\colon \Gamma \to X$ is the Fourier transform of a measure $m\colon B(G) \to X$, then q is continuous on Γ.*

Proof. Let $q = \widehat{m}$. According to the preceding remark, for any $\varepsilon > 0$ there exists a finite positive measure ν and a number $\delta > 0$ such that $\nu(E) < \delta$ implies $\|m(E)\| < \varepsilon$. It is known that the Fourier transform $\widehat{\nu}$ of ν is a continuous function. Let $x_0 \in [\alpha, \omega)$ and suppose that x_0 has no immediate successor. From the continuity of $\widehat{\nu}$ at $\mathbf{x}_0$ it follows that there exists $\sigma(\mathbf{x}_0) = [\mathbf{x}_0, \mathbf{c}) \cup (\mathbf{x}_0', \mathbf{c}']$ such that for any $\gamma \in \sigma(\mathbf{x}_0)$ there holds

$$\left|\widehat{\nu}(\gamma) - \widehat{\nu}(\mathbf{x}_0)\right| < \delta .$$

Let $\gamma = \mathbf{a} \in \sigma(\mathbf{x}_0)$. Then using $\nu\big((x_0, a]\big) = \left|\widehat{\nu}(\mathbf{a}) - \widehat{\nu}(\mathbf{x}_0)\right| < \delta$ we get

$$\left\|q(\mathbf{a}) - q(\mathbf{x}_0)\right\| = \left\|\widehat{m}(\mathbf{a}) - \widehat{m}(\mathbf{x}_0)\right\| = \left\|m\big((x_0, a]\big)\right\| < \varepsilon .$$

Similarly for $\gamma = \mathbf{a}' \in \sigma(\mathbf{x}_0)$. We omit the details of the proof for other points in Γ. □

We will look for the conditions under which a function $q\colon \Gamma \to X$ of bounded semivariation on $\mathbf{G}$ is the Fourier transform of a measure $m\colon B(G) \to X$.

DEFINITION 9. Let $q\colon \Gamma \to X$ be a function of bounded semivariation on $\mathbf{G}$. Then for every complex-valued function f on G and every set $D \subset G$,

$$D = \{t_0, t_1, \dots, t_n \colon \alpha = t_0 < t_1 < \dots < t_n = \omega\}$$

define

$$S_q(f, D) = \sum f(t_i)\big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big] .$$

THEOREM 3. *Let $q\colon \Gamma \to X$ be a function of bounded semivariation on $\mathbf{G}$. Then for each complex function $f \in C(G)$ there exists a unique element $T(f) \in X$ such that for every $\varepsilon > 0$ there exists a set $D_0 \subset G$ (of the form as in Definition 9) that*

$$\left\|S_q(f, D) - T(f)\right\| < \varepsilon$$

for every $D \supset D_0$. This property will be denoted by $T(f) = \lim\limits_{D} S_q(f, D)$.

Proof. Let $f \in C(G)$. If $s\text{-}V(q; \boldsymbol{\alpha}, \boldsymbol{\omega}) = 0$ then $S_q(f, D) = 0$ for every D. If $s\text{-}V(q; \boldsymbol{\alpha}, \boldsymbol{\omega}) = K$, $0 < K < \infty$, then from continuity of f it follows that for every $\varepsilon > 0$ there exists a set $D_\varepsilon \subset G$,

$$D_\varepsilon = \{t_0, t_1, \dots, t_n \colon \alpha = t_0 < t_1 < \dots < t_n = \omega\} ,$$

such that the oscillation of f on each interval $(t_{i-1}, t_i]$ is less than $\frac{\varepsilon}{K}$.

Let $D \supset D_\varepsilon$

$$D = \{\tau_0, \tau_1, \dots, \tau_m \colon \alpha = \tau_0 < \tau_1 < \dots < \tau_m = \omega\} .$$

and

$$\alpha = t_0 = \tau_0 < \tau_1 < \cdots < \tau_s = t_1 < \tau_{s+1} < \cdots < \tau_r = t_2 < \cdots < \tau_m = t_n = \omega .$$

Similar arguments as in [5] give

$$\begin{aligned}\|S_q(f,D) - S_q(f,D_\varepsilon)\| = \Bigg\| &\sum_{i=1}^{s} [f(\tau_i) - f(t_i)]\,[q(\boldsymbol{\tau}_i) - q(\boldsymbol{\tau}_{i-1})] \\ &+ \sum_{i=s+1}^{r} [f(\tau_i) - f(t_2)]\,[q(\boldsymbol{\tau}_i) - q(\boldsymbol{\tau}_{i-1})] + \dots \Bigg\|\end{aligned}$$

which can also be written as

$$\begin{aligned}\|S_q(f,D) - S_q(f,D_\varepsilon)\| &= \frac{\varepsilon}{K}\Bigg\| \sum_{i=1}^{s} \frac{K}{\varepsilon}[f(\tau_i) - f(t_1)] \cdot [q(\boldsymbol{\tau}_i) - q(\boldsymbol{\tau}_{i-1})] \\ &\qquad + \sum_{i=s+1}^{r} \frac{K}{\varepsilon}[f(\tau_i) - f(t_2)]\,[q(\boldsymbol{\tau}_i) - q(\boldsymbol{\tau}_{i-1})] + \dots \Bigg\| \\ &= \frac{\varepsilon}{K}\Bigg\| \sum_{i=1}^{m} e_i\,[q(\boldsymbol{\tau}_i) - q(\boldsymbol{\tau}_{i-1})] \Bigg\|\end{aligned}$$

where

$$\begin{aligned} e_i &= \frac{K}{\varepsilon}[f(\tau_i) - f(t_1)] \qquad \text{for } i = 1, 2, \dots, s, \\ e_i &= \frac{K}{\varepsilon}[f(\tau_i) - f(t_2)] \qquad \text{for } i = s+1, \dots, r, \end{aligned}$$

and so on.

Since for $i = 1, 2, \dots, s$ the points τ_i belong to the interval $(t_0, t_1]$, for $i = s+1, \dots, r$, $\tau_i \in (t_1, t_2]$, and so on, we get $|e_i| \le 1$ for every i. Therefore

$$\|S_q(f,D) - S_q(f,D_\varepsilon)\| < \varepsilon$$

for any $D \supset D_\varepsilon$. From this property it follows that for any positive integer n there exists a set D_n such that for every $D \supset D_n$ there holds

$$\|S_q(f,D) - S_q(f,D_n)\| < \frac{1}{n} .$$

Let us define the sequence $\{S_q(f, D'_n)\}_{n=1}^{\infty}$, where $D'_n = \bigcup_{i=1}^{n} D_i$. It can be shown that this sequence is Cauchy. Since X is a Banach space, there exists

$$\lim_{n\to\infty} S_q(f, D'_n) .$$

Let us denote it by $T(f)$. Similarly to [5] it can be proved that $T(f)$ is the required element and that it is unique. □

THEOREM 4. *The mapping T defined by*

$$T(f) = \lim_D S_q(f, D)\,, \qquad f \in C(G)\,,$$

is a continuous linear operator on $G(G)$ into X.

Proof. The assertion can be proved in the same way as Proposition 3 in [5]. To prove the continuity of T it suffices to use the boundedness of the semivariation of q on $\mathbf{G}$. We omit the details of the proof. $\square$

According to [7], and [13], to every continuous linear operator $T\colon C(G) \to X$ there corresponds a unique set function $\overline{m}$ defined on $B(G)$ with values in the second dual X'' of X such that

(a) $x' \circ \overline{m}$ is a complex regular measure on $B(G)$ for any $x' \in X'$,

(b) $x' \circ T(f) = \int\limits_G f\,dx' \circ \overline{m}$, $f \in C(G)$, $x' \in X'$,

(c) $\|T\| = \|\overline{m}\|_s(G)$.

This means that for any operator $T\colon C(G) \to X$ there exists a unique vector measure $\overline{m}$, whose values, however, belong in general to the second dual X''. Under some additional conditions we obtain a regular vector-valued measure with values in X, e.g., if X is a reflexive space. We will deal with this problem in the next section.

4. Vector-valued measures of weakly compact semivariation

DEFINITION 10. A function $q\colon \Gamma \to X$ is said to be of *weakly compact semivariation on the interval* $[\mathbf{a}, \mathbf{b}]$, if the set W of all elements of the form

$$\sum d_i\big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]$$

for all possible finite subsets $D \subset G$, $D\{t_0, t_1, \dots, t_n \colon a = t_0 < t_1 < \dots < t_n = b\}$ and for all finite families of complex numbers $\{d_i\}$, $|d_i| \le 1$, is relatively weakly compact.

LEMMA 1. *If $q\colon \Gamma \to X$ is a function of weakly compact semivariation on $[\mathbf{a}, \mathbf{b}]$ then it is of bounded semivariation on $[\mathbf{a}, \mathbf{b}]$.*

Proof. By assumption, the set W defined in Definition 10 is relatively weakly compact, and therefore it is (weakly) bounded. For any element of the set W there holds

$$\Big\|\sum d_i\big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]\Big\| \le K < \infty$$

and hence $s\text{-}V(q; \mathbf{a}, \mathbf{b}) \le K$. $\square$

Lemma 2. *If $q\colon \Gamma \to X$ is a function of bounded semivariation on the interval $[\mathbf{a}, \mathbf{b}]$ and X is a reflexive space, then q is of weakly compact semivariation on $[\mathbf{a}, \mathbf{b}]$.*

Proof. If $s\text{-}V(q; \mathbf{a}, \mathbf{b}) \le K < \infty$ then the set W is bounded. Since X is a reflexive space, W is a weakly compact set, and, in consequence, q is of weakly compact semivariation. □

Example 3. The function q from Example 2 is of weakly compact semivariation on the set $[\mathbf{0}, \mathbf{1}]$.

If we take into account that $X = L_2(G, \mu)$ is a reflexive space, the assertion follows immediately from Lemma 2.

Recall that the linear operator $T\colon C(G) \to X$ is said to be weakly compact if it maps the unit ball of $C(G)$ into a weakly compact subset of X.

Theorem 5. *Let $q\colon \Gamma \to X$ be a function of weakly compact semivariation on $\mathbf{G}$. Then the operator $T\colon C(G) \to X$, $T(f) = \lim_D S_q(f, D)$ is weakly compact.*

Proof. Let $D = \{t_0, t_1, \dots, t_n\} \subset G$, $\alpha = t_0 < t_1 < \dots < t_n = \omega$. Let $f \in G(G)$, $0 \le f(t) \le 1$ for every $t \in G$. Form an integral sum

$$S_q(f, D) = \sum_{i=1}^{n} f(t_i)\bigl[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\bigr]$$

and order the indexes $1, 2, 3 \dots, n$ into such a sequence $i_1, i_2, \dots, i_n$ that

$$0 \le f(t_{i_1}) \le f(t_{i_2}) \le f(t_{i_3}) \le \dots \le f(t_{i_n}) \le 1 .$$

Then we get

$$S_q(f, D) = \sum_{i=1}^{n} f(t_i)\bigl[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\bigr] = \sum_{k=1}^{n} f(t_{i_k})\bigl[q(\mathbf{t}_{i_k}) - q(\mathbf{t}_{i_k - 1})\bigr] ,$$

which may be written in the form

$$\begin{aligned} S_q(f, D) = {} & f(t_{i_1}) \sum_{k=1}^{n} \bigl[q(\mathbf{t}_{i_k}) - q(\mathbf{t}_{i_k - 1})\bigr] \\ & + \sum_{s=2}^{n} \biggl\{ \bigl[f(t_{i_s}) - f(t_{i_{s-1}})\bigr] \sum_{k=s}^{n} \bigl[q(\mathbf{t}_{i_k}) - q(\mathbf{t}_{i_k - 1})\bigr] \biggr\} . \end{aligned}$$

Since $f(t_i) \ge 0$ and $f(t_{i_s}) - f(t_{i_{s-1}}) \ge 0$, it holds

$$\bigl|f(t_{i_1})\bigr| + \bigl|f(t_{i_2}) - f(t_{i_1})\bigr| + \dots + \bigl|f(t_{i_n}) - f(t_{i_{n-1}})\bigr| = f(t_{i_n}) \le 1 .$$

This means that $S_q(f,D)$ and hence also $T(f) = \lim\limits_{D} S_q(f,D)$ belong to the closed absolutely convex hull $\mathcal{K}$ of the set W. Since W is weakly relatively compact, according to the Krein–Smulian Lemma [7], [15], $\mathcal{K}$ is weakly compact.

Let $f \in C(G)$, $\|f\| \le 1$. The values of f can be expressed in the form

$$f(t) = f_0(t) - f_2(t) + i\big[f_1(t) - f_3(t)\big]\,, \tag{7}$$

where $0 \le f_i(t) \le 1$ for $i = 0,1,2,3$ and every $t \in G$.

By the preceding part, $T(f_i) \in \mathcal{K}$, and due to (7), the operator T maps the set $\big\{f \in C(G)\colon \|f\| \le 1\big\}$ into a weakly compact set

$$\left\{\sum_{r=0}^{3} i^r x_r \colon x_r \in \mathcal{K}\right\}$$

which proves our theorem. □

Since T is a weakly compact operator on $C(G)$ into X, there exists (see, e.g., [7], [13]) a unique regular measure $m\colon B(G) \to X$ such that

$$T(f) = \int\limits_G f\,dm\,, \qquad f \in C(G)\,.$$

In order to prove the main result we need the following two lemmas.

Lemma 3. *If q is a scalar function defined on Γ then*

$$s\text{-}V(q;\mathbf{a},\mathbf{b}) = V(q;\mathbf{a},\mathbf{b})\,.$$

P r o o f . It is clear that $s\text{-}V(q;\mathbf{a},\mathbf{b}) \le V(q;\mathbf{a},\mathbf{b})$. Further, for any division $D \subset [a,b]$, $D = \{t_0, t_1, \dots, t_n \colon a = t_0 < t_1 < \dots < t_n = b\}$ it holds

$$\begin{aligned}\sum_{i=1}^{n} \big|q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big| &= \left|\sum_{i=1}^{n} \big|q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big|\right| \\ &= \left|\sum_{i=1}^{n} k_i\big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]\right|,\end{aligned}$$

where k_i are complex numbers with the property $|k_i| = 1$.

Hence

$$\begin{aligned}V(q;\mathbf{a},\mathbf{b}) &= \sup_D \sum \big|q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big| = \sup_D \Big|\sum k_i\big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]\Big| \\ &\le \sup_{D,\{d_i\}} \Big|\sum d_i\big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]\Big| = s\text{-}V(q;\mathbf{a},\mathbf{b})\,,\end{aligned}$$

where the last supremum is taken over all divisions D of the interval $[a,b]$ and over all finite families $\{d_i\}$ of complex numbers $\{d_i\}$, $|d_i| \le 1$. □

Lemma 4. *If a function $q\colon \Gamma \to X$ is of bounded semivariation on $\mathbf{G}$, then a scalar function $x' \circ q$ is of bounded variation on $\mathbf{G}$ for every $x' \in X'$.*

Proof. By assumption,

$$\sup \Big\| \sum d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big] \Big\| \le K < \infty \,,$$

where the supremum is taken over all sets $D \subset G$ and systems of complex numbers $\{d_i\}$ with properties as above. Consequently, the set W of all elements of the form

$$\sum d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]$$

is (weakly) bounded. Hence for every $x' \in X'$ there exists a constant $A_{x'} < \infty$, such that $\big|x'(W)\big| \le A_{x'}$. This gives

$$\sup_{D,\{d_i\}} \Big\| x'\Big(\sum d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]\Big) \Big\| \le A_{x'} \,.$$

Since

$$\begin{aligned} V(x' \circ q; \boldsymbol{\alpha}, \boldsymbol{\omega}) &= s\text{-}V(x' \circ q; \boldsymbol{\alpha}, \boldsymbol{\omega}) \\ &= \sup_{D,\{d_i\}} \Big| \sum d_i \big[x' \circ q(\mathbf{t}_i) - x' \circ q(\mathbf{t}_{i-1})\big] \Big| \\ &= \sup_{D,\{d_i\}} \Big| x'\Big(\sum d_i \big[q(\mathbf{t}_i) - q(\mathbf{t}_{i-1})\big]\Big) \Big| \end{aligned}$$

we have for every $x' \in X'$,

$$V(x' \circ q; \boldsymbol{\alpha}, \boldsymbol{\omega}) \le A_{x'} < \infty \qquad \text{for each } x' \in X'.$$

□

Summarize the previous results. For any function $q\colon \Gamma \to X$ of weakly compact semivariation on $\mathbf{G}$ it is possible to construct a weakly compact operator $T\colon C(G) \to X$ which can be uniquely represented as an integral with respect to a regular measure $m\colon B(G) \to X$. The following theorem establishes the relation between a function q and a vector measure m.

Theorem 6. *Let $q\colon \Gamma \to X$ be a weakly continuous function of weakly compact semivariation on the set $\mathbf{G}$. Then q is the Fourier transform of a regular measure $m\colon B(G) \to X$.*

Proof. Since q is of weakly compact semivariation on $\mathbf{G}$, the operator T obtained by means of q in the same way as in Theorem 9 is weakly compact. According to the previous results, there exists a unique regular measure $m\colon B(G) \to X$ representing T, namely

$$T(f) = \int\limits_G f \, dm \,, \qquad f \in C(G) \,.$$

By Lemma 1 and Lemma 4, a function $x' \circ f$ is of bounded variation on $\mathbf{G}$ for every $x' \in X'$. The weak continuity of q implies the continuity of $x' \circ q$ for every $x' \in X'$. By (5), the function $x' \circ q$ is the Fourier transform of a complex-valued regular measure $\lambda_{x'}$ defined on $B(G)$. and it holds $\widehat{\lambda}_{x'} = x'q - x'q(\boldsymbol{\alpha})$ on the set Γ. Moreover, $\lambda_{x'} = x' \circ m$ (see [5]). Combining the last two properties we get

$$\widehat{m} = q - q(\boldsymbol{\alpha}) .$$

From this we conclude that q is the Fourier transform of the measure

$$m + q(\boldsymbol{\alpha})\varepsilon_{\alpha} ,$$

where ε_{α} is the Dirac measure concentrated in α. □

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