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STATES AND IDEMPOTENTS OF PSEUDO MV-ALGEBRAS

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. We study states on pseudo MV-algebras in connection with idempotents. Such elements form a Boolean subalgebra. We present some assertions on states on the set of all idempotents of a pseudo MV-algebras satisfying general comparability. Such pseudo MV-algebras possess at least one state. In addition, some topological aspects of states are presented.

1. Introduction

Pseudo MV-algebras, introduced recently by Georgescu and Iorgulescu [GeIo] and independently by Rachůnek [Rac], are a non-commutative extension of MV-algebras which extend a two-valued reasoning. Non-commutative reasoning can be observed in every-day life and nowadays, there exists even a programming language [Bau] based on a non-commutative logic.

States on pseudo MV-algebras characterize some probability features of algebraic structures and they have been introduced in [Dvu 1]. In contrast to MV-algebras, pseudo MV-algebras can possess no state, [Dvu 1].

More deeper results on states can be obtained when we study idempotents of pseudo MV-algebras which form a Boolean subalgebra of ones, and pseudo MV-algebras satisfying general comparability. In particular, we show that such pseudo MV-algebras possess at least one state. Finally, some topological aspects of the sets of extremal states on pseudo MV-algebras and the set of idempotents, respectively, are proved.

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2. Pseudo MV-algebras

According to [GeIo], we say that a *pseudo MV-algebra* is an algebra $(M; \oplus, {}^-, {}^\sim, 0, 1)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim,$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

We shall assume that $0 \neq 1$. If we define $x \leq y$ iff $x^- \oplus y = 1$, then $\leq$ is a partial order such that M is a distributive lattice with $x \vee y = x \oplus x^\sim \odot y$ and $x \wedge y = x \odot (x^- \oplus y)$. For basic properties of pseudo MV-algebras see [GeIo].

For example, if u is a strong unit of a (not necessarily Abelian) ℓ-group G,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u\,, \\ x^- &:= u - x\,, \\ x^\sim &:= -x + u\,, \\ x \odot y &:= (x - u + y) \vee 0\,, \end{aligned}$$

then $\big(\Gamma(G, u); \oplus, {}^-, {}^\sim, 0, u\big)$ is a pseudo MV-algebra [GeIo].

We recall that the pseudo MV-algebra $\big(\Gamma(G, u); \oplus, {}^-, {}^\sim, 0, u\big)$ is in some sense the archetype of pseudo MV-algebras owing to the following basic representation theorem for pseudo MV-algebras proved in [Dvu, Thm 3.9, Thm 6.4] generalizing a famous result of M u n d i c i [Mun] for MV-algebras:

Theorem 2.1. *For any pseudo MV-algebra M, there exists a unique (up to isomorphism) unital ℓ-group G with a strong unit u such that $M \cong \Gamma(G, u)$. The functor Γ defines a categorical equivalence of the category of pseudo MV-algebras with the category of unital ℓ-groups. In addition, if h is a given isomorphism of M with $\Gamma(G, u)$, (G, h) is the universal group for M: that is,* *for any partially ordered group* $(K;+,\leq,0)$ *and any order* $-$ *and* $+$ *preserving mapping* $g\colon M\to K$ *there is a homomorphism of partially ordered groups* $g'\colon G\to K$ *such that* $g=g'\circ h$.[1]

A *state* on a pseudo MV-algebra M is a mapping $m\colon M\to[0,1]$ such that (i) $m(1)=1$, and (ii) $m(a+b)=m(a)+m(b)$ whenever $a+b$ is defined in M.

A mapping m from a pseudo MV-algebra M into the standard MV-algebra $[0,1]$ such that, for all $a,b\in M$, $m(a\oplus b)=m(a)\oplus_{\mathbb{R}} m(b)$, (ii) $m(a^-)=m(a^\sim)=1-m(a)$, and (iii) $m(1)=1$ is said to be a *state-morphism.* In other words, a mapping m is a state-morphism iff m is a homomorphism from M into the standard MV-algebra $[0,1]$. We recall that each state-morphism is a state.

We recall that if m is state on M, then, [Dvu, Prop 4.1], (i) $m(0)=0$, (ii) m is monotone, (iii) $m(a^-)=1-m(a)=m(a^\sim)$, (iv) $m(a^{=})=m(a)=m(a^{\approx})$, and (v) $m(a\oplus b)=m(b\oplus a)$.

Denote by $\mathscr{S}(M)$ the set of all states of a pseudo MV-algebra M. Then $\mathscr{S}(M)$ is a convex set, i.e., if $m_1,m_2\in\mathscr{S}(M)$, and $\lambda\in[0,1]$, then $\lambda m_1+(1-\lambda)m_2\in\mathscr{S}(M)$, and let $\operatorname{Ext}(\mathscr{S}(M))$ be the set of all its extremal states, i.e., of states m such that if $m=\lambda m_1+(1-\lambda)m_2$ for $\lambda\in(0,1)$, then $m=m_1=m_2$.

We recall, [Dvu, Prop 4.7], that the set $\operatorname{Ext}(\mathscr{S}(M))$ consists only from all state-morphisms.

According to [GeIo], we say that an *ideal* of a pseudo MV-algebra M is a subset I of M such that (i) $0\in I$, (ii) if $x,y\in I$, then $x\oplus y\in I$ and $y\oplus x\in I$, and (iii) if $x\in I$, $y\in M$, and $y\leq x$, then $y\in I$.

An ideal I of M is said to be (i) *normal* if $x\oplus I=I\oplus x$ for any $x\in M$,[2] (ii) *prime* if $x\wedge y\in I$ entails $x\in I$ or $y\in I$, (iii) *maximal* if I is a proper ideal of M and it is not included into any proper ideal of M different of I.

EXAMPLE 2.2. Let G be the group of all matrices of the form

$$A=\begin{pmatrix}\xi & \alpha\\ 0 & 1\end{pmatrix},$$

where ξ and α are rational (real) numbers such that $\xi>0$; the group-operation is the usual multiplication of matrices. We denote $A=(\xi,\alpha)$. Then $A^{-1}=(1/\xi,-\alpha/\xi)$, and $(1,0)$ is the neutral element. We define $G^+:=\big\{(\xi,\alpha)\colon$ where (i) $\xi>1$, or (ii) $\xi=1$ and $\alpha\geq 0\big\}$. Then G with the positive cone G^+ is a linearly ordered ℓ-group with a strong unit $U=(2,0)$. Define $M=\Gamma(G,U)$. Then $M=M_1\cup M_2\cup M_3$, where $M_1=\big\{(\xi,\alpha)\colon 1<\xi<2\big\}$, $M_2=\big\{(2,\alpha):\alpha\leq 0\big\}$, and $M_3=\big\{(1,\alpha)\colon \alpha\geq 0\big\}$. M_3 is a unique normal and maximal ideal of M, and there is a unique state-morphism m, namely $m\big((\xi,\alpha)\big)=\log_2(\xi)$, $(\xi,\alpha)\in M$.

[1] According to [Bae, Thm 1], we can assume that K is even a group not necessarily partially ordered.

[2] We define $x\oplus I:=\{x\oplus y\colon y\in I\}$; similarly we define $I\oplus x$.

3. Idempotents of pseudo MV-algebras

In the present section, we study idempotents of pseudo MV-algebras introduced in [GeIo]. Such elements form a Boolean subalgebra of the pseudo MV-algebra.

We say that an element e of a pseudo MV-algebra M is *idempotent* (or *Boolean*) if $e \oplus e = e$. We denote by $B(M)$ the set of all idempotents of M. If $M = \Gamma(G,u)$, we write also $B(G,u) := B\big(\Gamma(G,u)\big)$. It is clear that $0, 1 \in B(M)$. Moreover, as it was shown in [GeIo, Sect 4], $e \in B(M)$ iff $e \odot e = e$ iff $e \wedge e^- = 0$ iff $e \wedge e^\sim = 0$ iff $e \vee e^\sim = 1$. Moreover, (i) $e^- = e^\sim$ for any $e \in B(M)$; we write $e' := e^-$, (ii) if $x \in M$, then $e \oplus x = e \vee x = x \oplus e$, and $e \odot x = e \wedge x = x \odot e$ for any $e \in B(M)$, (iii) $\big(B(M); \wedge, \vee, ', 0, 1\big)$ is a Boolean algebra.

Let e be an idempotent. According to [Jak], the set $M_e := \{x \in M : x \le e\}$ can be organized into a pseudo MV-algebra $(M_e; \oplus, \odot, {}^{-_e}, {}^{\sim_e}, 0, e)$, where $x^{-_e} := e \odot x^-$ and $x^\sim_e := x^\sim \odot e$. In addition, M_e is a normal ideal of M, [GeIo, Lem 4.13].

The mapping $p^M_e : M \to M_e$ defined by

$$p^M_e(x) := x \wedge e\,, \quad x \in M\,,$$

is a homomorphism of M onto M_e with kernel $M_{e'}$. Moreover, as it was shown in [Jak], M is isomorphic, as a pseudo MV-algebra, to the direct product

$$M_e \times M_{e'}\,. \tag{3.1}$$

Proposition 3.1. *Let e be an idempotent of a pseudo MV-algebra M and m a state on M.*

(a) *If $0 < \lambda := m(e) < 1$, then $m_1 := \frac{1}{\lambda} m \circ p^M_e$ and $m_2 := \frac{1}{1-\lambda} m \circ p^M_{e'}$ are two distinct states on M, and $m = \lambda m_1 + (1-\lambda) m_2$.*

(b) *If m is an extremal state on M, then $m(e) \in \{0, 1\}$.*

P r o o f . (a) It is clear that m_1 and m_2 are states on M, and since $m_1(e) = 1$ and $m_2(e) = 0$, we see that $m_1 \ne m_2$. The rest is evident.

(b) It follows immediately from (a). □

Proposition 3.2. *Let (G,u) be a unital ℓ-group and let e be an idempotent of $\Gamma(G,u)$. Let G_e and G_{u-e} be the convex ℓ-subgroup of G generated by e and $u-e$, respectively. Then G_e and G_{u-e} are ℓ-ideals of G, and*

$$G = G_e \oplus G_{u-e}. \tag{3.2}$$

P r o o f . It is clear that $G_e = \big\{g \in G : |g| \le ne \text{ for some } n \ge 1\big\}$. Given $g \in G_e \cap G_{u-e}$, we have $|g| \in G_e \cap G_{u-e}$, and $|g| \le ne$ and $|g| \le k(u-e)$

for some integers $k, n \geq 1$. Due to the Riesz decomposition property, we have $|g| = g_1 + \cdots + g_n$ for some elements $g_i \in \Gamma(G,u)$ with $g_i \leq e$. Since $g_i \leq |g| \leq k(u-e)$, each $g_i = g_{i1} + \cdots + g_{ik}$ for some $g_{ij} \in \Gamma(G,u)$ such that $g_{ij} \leq u-e$. In addition, each $g_{ij} \leq g_i \leq e$, whence $0 \leq g_{ij} \leq e \wedge (u-e) = 0$. Then $g_{ij} = 0$ and $|g| = 0$ which proves that $G_e \cap G_{u-e} = \{0\}$.

Given any $g \in G^+$, $g \leq nu = ne + n(u-e)$, so that $g = g_1 + g_2$, where $g_1 \leq ne$ and $g_2 \leq n(u-e)$. Therefore, if $g_0 \in G_e$, then $g + g_0 = g_1 + g_2 + g_0 = g_1 + g_0 + g_2 = (g_1 + g_0 - g_1) + g_1 + g_2 = g_0' + g$, where $g_0' \in G_e$.

Similarly, if g is arbitrary, then $g = g^+ - g^- = g_1^+ + g_2^+ - g_2{}^- - g_1^-$, where $g_1^+, g_1{}^- \in G_e$, $g_2^+, g_2{}^- \in G_{u-e}$ which easily proves that G_e and G_{u-e} are ℓ-ideals of G, and, in addition, $G = G_e \oplus G_{u-e}$. $\square$

Define p_e to be the projection of G onto G_e with kernel G_{u-e}. Then (i) $p_e(e) = e$; (ii) $p_e(g+h) = p_e(g) + p_e(h)$, (iii) $p_e(-g) = -p_e(g)$ for $g, h \in G$, and (iv) $p_e(g) \geq 0$ if $g \in G^+$.

Similarly, p_{u-e} is a projection of G onto G_{u-e} with kernel G_e.

Proposition 3.3. *Let (G,u) be a unital ℓ-group and let e be an idempotent of $\Gamma(G,u)$. If $g \in G^+$ and $g \leq nu$ for some integer $n \geq 1$, then*

$$p_e(g) = g \wedge ne\,. \tag{3.3}$$

P r o o f. As $g \leq nu$, then $p_e(g) \leq p_e(nu) = ne$. Moreover, since $0 \leq g$, we obtain $0 \leq p_{u-u}(g) = g - p_e(g)$, whence $p_e(g) \leq g$. Given $h \in G$ such that $h \leq g$ and $h \leq ne$, we have $p_e(h) \leq p_e(g)$ and $h - p_e(h) = p_{u-e}(h) \leq p_{u-e}(ne) = 0$. Hence $h \leq p_e(h) \leq p_e(g)$ which proves $p_e(g) = g \wedge ne$. $\square$

Proposition 3.4. *Let $g \in \Gamma(G,u)$, $e \in B(G,u)$.*

(a) *$n(g \wedge e) = ng \wedge ne$ for all $n \geq 1$.*

(b) *If $f \in B(G,u)$, then $p_{e\wedge f} = p_f p_e = p_e p_f$.*

(c) *If $f \in B(G,u)$, $e \wedge f = 0$, then $p_{e+f} = p_{e\vee f} = p_{f+e} = p_e + p_f = p_f + p_e$.*

P r o o f. (a) Since $ng \leq nu$, we have by Proposition 3.3, $p_e(ng) = ng \wedge ne$. On the other hand, $p_e(ng) = np_e(g) = n(g \wedge e)$.

(b) Now let $f \in B(G,u)$. Given $g \in G^+$, find an integer $n \geq 1$ such that $g \leq nu$. By (3.3), $p_e p_f(g) = (g \wedge nf) \wedge ne = g \wedge n(f \wedge e) = p_{e\wedge f}(g)$. Thus $p_e p_f$ and $p_{e\wedge f}$ agree on G^+, so that $p_e p_f = p_{e\wedge f}$ on G. By symmetry, $p_f p_e = p_{e\wedge f}$.

(c) Since $e \wedge f = 0$, we have $ne \wedge nf = 0$ for any $n \geq 1$, so that $ne + nf = ne \vee nf$. Hence if $g \in G^+$ is such that $x \leq nu$, then

$$\begin{aligned} p_e(g) + p_f(g) &= (g \wedge ne) + (g \wedge nf) = (g \wedge ne) \vee (g \wedge nf) \\ &= g \wedge (ne \vee nf) = g \wedge (ne + nf) \\ &= g \wedge n(e+f) = g \wedge n(e \vee f) \\ &= p_{e\vee f}(g) = p_{e+f}(g)\,. \end{aligned}$$

Hence $p_e + p_f = p_{e+f} = p_{e\vee f} = p_{f+e} = p_f + p_e$. □

4. Pseudo MV-algebras satisfying General Comparability

In the present section, we will apply methods of Abelian unital ℓ-groups satisfying general comparability developed, e.g., in [Goo, Sect 8], to pseudo MV-algebras to describe the state space of pseudo MV-algebras satisfying general comparability. In particular, such pseudo MV-algebras possess at least one state.

We say that (i) a unital ℓ-group (G,u) satisfies *general comparability* if, for each $x,y \in G$, there exists some $e \in B(G,u)$ such that $p_e(x) \le p_e(y)$ while $p_{u-e}(x) \ge p_{u-e}(y)$, (ii) a pseudo MV-algebra $M = \Gamma(G,u)$ satisfies *general comparability* if (G,u) satisfies general comparability, (iii) a pseudo MV-algebra M satisfies *weak general comparability* if, for each $x,y \in M$, there exists some $e \in B(M)$ such that $p^M_e(x) \le p^M_e(y)$ while $p^M_{u-e}(x) \ge p^M_{u-e}(y)$.

It is clear that the general comparability of M implies a weak general comparability; the converse implication seems to be open.

We recall that any linearly ordered pseudo MV-algebra trivially satisfies general comparability, such a pseudo MV-algebra is given by Example 2.2; we take $e = 0$ or $e = 1$. More generally, if M is a direct product of linearly ordered pseudo MV-algebras, then M satisfies general comparability. Moreover, if M is a σ-complete pseudo MV-algebra, then it is a σ-complete MV-algebra satisfying general comparability (see [Goo, Thm 9.9]).

We recall that a *state* on a unital ℓ-group (G,u) is a mapping $s\colon G \to \mathbb{R}$ such that (i) $s(g_1+g_2) = s(g_1)+s(g_2)$, $g_1,g_2 \in G$, (ii) $s(g) \ge 0$ for $g \in G^+$, and (iii) $s(u) = 1$. In view of Theorem 2.1 any state m on $\Gamma(G,u)$ can be uniquely extended to a state on (G,u), and conversely, the restriction of any state on (G,u) is a state on $\Gamma(G,u)$.

PROPOSITION 4.1. *Let $M = \Gamma(G,u)$ be a pseudo MV-algebra satisfying general comparability. Let m be a state on M and define $I = B(G,u) \cap \mathrm{Ker}(m)$. Then the following statements are equivalent:*

(i) *m is an extremal state.*

(ii) *If t is a state on M such that $\mathrm{Ker}(t) \supseteq I$, then $t = m$.*

(iii) *I is a maximal ideal of $B(G,u)$.*

Proof. It is easy to verify that I is an ideal of the Boolean algebra $B(G,u)$.

(i) $\Rightarrow$ (iii). For any $e \in B(G,u)$, Proposition 3.1 entails that either $m(e) = 0$ or $m(e) = 1$, whence either $e \in I$ or $u - e \in I$. Therefore I is a maximal ideal of $B(G,u)$.

(iii) $\Rightarrow$ (ii). If $e \in B(G,u) \setminus I$, then $u-e \in I$. Then $u-e \in \mathrm{Ker}(t)$ and so $t(e) = t(u) = 1$ which yields $e \notin \mathrm{Ker}(t)$. Thus $B(G,u) \cap \mathrm{Ker}(t) = I$.

We note that $\mathrm{Ker}(m)$ is a normal ideal of M, [Dvu 1, Prop 4.1].

Claim 1. $M/\mathrm{Ker}(m)$ is a linearly ordered pseudo MV-algebra.

Given $x, y \in M$, there exists $e \in B(G,u)$ such that $x \wedge e \le y \wedge e$ while $x \wedge e' \ge y \wedge e'$. As I is a maximal ideal of $B(G,u)$, either $e \in I$ or $e' \in I$. Then either $e \in \mathrm{Ker}(m)$ or $e' \in \mathrm{Ker}(m)$, so that either $p_e^M(M) \subseteq \mathrm{Ker}(m)$ or $p_{e'}^M(M) \subseteq \mathrm{Ker}(m)$. This entails either $x/\mathrm{Ker}(m) \ge y/\mathrm{Ker}(m)$ or $x/\mathrm{Ker}(m) \le y/\mathrm{Ker}(m)$. Thus $M/\mathrm{Ker}(m)$ is a linearly ordered pseudo MV-algebra (even an MV-algebra).

Claim 2. $\mathrm{Ker}(m) \subseteq \mathrm{Ker}(t)$.

If not, there exists an element $x \in \mathrm{Ker}(m)$ such that $t(x) > 0$. Let $\hat{m}$ and $\hat{t}$ be the states on (G,u) which are unique extensions of m and t, respectively. There exists an integer $n \ge 1$ such that $\hat{t}(nx) > 1$. Moreover, $\hat{m}(nx) = 0$. The general comparability applied to $y = nx$ and u entails that there exists $e \in B(G,u)$ such that $p_e(y) \le p_e(u)$ and $p_{u-e}(y) \ge p_{u-e}(u) = u-e$, and either $e \in I$ or $u-e \in I$.

If $e \in I$, then $\hat{m}(ne) = 0$ and $\hat{m}(y) = \hat{m}(y \wedge ne) + \hat{m}(y \wedge ne') = \hat{m}(y \wedge ne') = \hat{m}\big(p_{u-e}(y)\big) \ge \hat{m}\big(p_{u-e}(u)\big) = m(e') = 1$, which contradicts the fact that $\hat{m}(y) = 0$.

If $u - e \in I$, then $t(e) = 1$ and

$$t(y) = t(y \wedge ne) = t\big(p_e(y)\big) \le t\big(p_e(u)\big) = t(e) = 1\,,$$

which contradicts the fact that $t(y) > 1$. Thus $\mathrm{Ker}(m) \subseteq \mathrm{Ker}(t)$.

Now let $\tilde{m}$ and $\tilde{t}$ be states on $M/\mathrm{Ker}(m)$ defined by $\tilde{m}\big(a/\mathrm{Ker}(m)\big) := m(a)$ and $\tilde{t}\big(a/\mathrm{Ker}(m)\big) := t(a)$ $(a \in M)$. As $M/\mathrm{Ker}(m)$ is linearly ordered, by [Dvu 1, Thm 5.5], $M/\mathrm{Ker}(m)$ possesses a unique state, so that $\tilde{m} = \tilde{t}$. Therefore $m = t$.

(ii) $\Rightarrow$ (i). Consider $m = \lambda m_1 + (1-\lambda) m_2$, where $0 < \lambda < 1$, and $m_1, m_2 \in \mathscr{S}(M)$. For any $a \in I$, we have

$$\lambda m_1(e) + (1-\lambda) m_2(e) = m(e) = 0\,,$$

whence each $m_i(e) = 0$. Then $I \subseteq \mathrm{Ker}(m_1) \cap \mathrm{Ker}(m_2)$, and hence $m_1 = m_2$ by (ii) which implies m is extremal. □

As we have already mentioned, any linearly ordered pseudo MV-algebra possesses a unique state. Now we extend this result.

Proposition 4.2. *Let $M = \Gamma(G,u)$ be a linearly ordered pseudo MV-algebra and m a unique state on it. Then*

$$\begin{aligned} m(x) &= \inf\{\, l/n \colon l,n \ge 1\,,\ nx \le lu\} \\ &= \sup\{\, k/i \colon k \ge 0,\ i \ge 1,\ ku \le ix\} \end{aligned}$$

for all $x \in M$.

Proof. Let $x \in M$ and define $f^*(x) := \inf\{\, l/n \colon l,n \ge 1,\ nx \le lu\}$ and $f_*(x) = \sup\{\, k/i \colon k \ge 0,\ i \ge 1,\ ku \le ix\}$. Then $0 \le f_*(x) \le m(x) \le f^*(x) \le 1$. Suppose that $f_*(x) < f^*(x)$. Hence we may choose two integers k and n such that $f_*(x) < k/n < f^*(x)$. Since $k/n < f^*(x)$, we have $nx \not\le ku$, and consequently $ku \le nx$. But then $k/n \le f_*(x)$, a contradiction. ☐

Proposition 4.3. *Let $M = \Gamma(G,u)$ be a pseudo MV-algebra satisfying weak general comparability, and let I be a maximal ideal of $B(G,u)$. Then there exists a unique state m on M such that $B(G,u) \cap \mathrm{Ker}(m) = I$. In addition, if M satisfies general comparability, then*

$$\begin{aligned} m(x) &= \inf\bigl\{l/n \colon l,n \ge 1\,,\ n(e \wedge x) \le le\,,\ e \in B(G,u) \setminus I\bigr\} \\ &= \sup\bigl\{k/i \colon k \ge 0\,,\ i \ge 1\,,\ ke \le i(e \wedge x)\,,\ e \in B(G,u) \setminus I\bigr\} \end{aligned}$$

for all $x \in M$.

Proof. Let K be the ideal of M generated by I. Then $K = \{x \in M \colon x \le e,\ e \in I\} = \bigcup_{e \in I} [0,e]$.

We assert that K is a normal ideal of M. Indeed, take $y \in M$ and $x \in K$. Then there exists $e \in I$ such that $x \le e$. We show that $(y \oplus x) \odot y^- \le e$. Calculate and use [GeIo, Prop 1.17]

$$\begin{aligned} \bigl((y \oplus x) \odot y^-\bigr) \wedge e &= (y \oplus x) \odot y^- \odot e = e \odot (y \oplus x) \odot y^- \\ &\le e \wedge (y \oplus x) \odot y^- \le \bigl((e \wedge y) \oplus (e \wedge x)\bigr) \odot y^- \\ &\le (e \oplus e) \odot y^- = (e \vee e) \odot y^- = e \wedge y^- \le e\,, \end{aligned}$$

which proves $(y \oplus x) \odot y^- \in K$. In a similar way we show that $y^\sim \odot (x \oplus y) \in K$.

Given any $f \in K \cap B(G,u)$, it follows that $e \wedge f = f$ for some $e \in I$. Hence $f \le e$ and so $f \in I$. Thus $K \cap B(G,u) = I$.

Next, we show that M/K is a linearly ordered pseudo MV-algebra. Given $x,y \in M$, there exists $e \in B(G,u)$ such that $e \wedge x \le e \wedge y$ and $e' \wedge x \ge e' \wedge y$. Either $e \in I$ or $u - e \in I$. If $e \in I$, then $x/K = (x \wedge e)/ + (x \wedge e')/K = (x \wedge e')/K \ge (y \wedge e')/K = (y \wedge e)/K + (y \wedge e')/K = y/K$.

Similarly, if $e' \in I$, then $x/K \le y/K$. Thus M/K is a linearly ordered pseudo MV-algebra, and by [Dvu 1, Thm 5.5], it possesses a unique state, say μ. Define $m(a) := \mu(a/K)$, $a \in M$. Then m is a state on M and $\mathrm{Ker}(m) = K$.

If now m' is a state on M such that $B(G,u) \cap \mathrm{Ker}(m') = I$, then $I \subseteq \mathrm{Ker}(m')$, which implies $\mathrm{Ker}(m) = K \subseteq \mathrm{Ker}(m')$. Hence m' induces a state μ' on M/K such that $\mu'(a/K) = m'(a)$. Since $\mu' = \mu$, we have $m' = m$.

The rest follows from Proposition 4.2 and general comparability. □

We know that there exist pseudo MV-algebras possessing no state, see [Dvu 1, Cor 7.4]. The following statement extends the class of pseudo MV-algebras possessing at least one state to ones satisfying general comparability.

COROLLARY 4.4. *Any nontrivial pseudo MV-algebra satisfying general comparability possesses at least one state.*

P r o o f. This follows from Proposition 4.3. □

We denote by $\mathrm{Max}(M)$ or $\mathrm{Max}\big(B(G,u)\big)$ the set of all maximal ideals of the Boolean algebra $B(M)$ or $B(G,u)$ of all idempotents of $M = \Gamma(G,u)$ and equip it with the usual hull-kernel topology whose basis consists of clopen sets. We recall that a net $\{m_\alpha\}$ of states on M *converges weakly* to a state m iff $m_\alpha(a) \to m(a)$ for every $a \in M$. If $\mathscr{S}(M) \neq \emptyset$, then a compact convex Hausdorff space with respect to the weak topology, and the space of all state-morphisms on M is a non-void compact Hausdorff space. Any state is a weak limit of convex linear combinations of the set of extremal points of $\mathscr{S}(M)$, see [Dvu 1, Thm 4.8].

THEOREM 4.5. *Let $M = \Gamma(G,u)$ be a pseudo MV-algebra satisfying general comparability. Then the rule*

$$\theta_b(m) := B(G,u) \cap \mathrm{Ker}(m)\,, \qquad m \in \mathrm{Ext}\big(\mathscr{S}(M)\big)\,, \tag{4.1}$$

defines a homeomorphism θ_b of nonempty $\mathrm{Ext}\big(\mathscr{S}(M)\big)$ onto $\mathrm{Max}\big(B(M)\big)$.

P r o o f. By Propositions 4.1 and 4.3 we see that θ_b is a bijective map of $\mathrm{Ext}\big(\mathscr{S}(M)\big)$ onto $\mathrm{Max}(M)$. The space $\mathrm{Max}\big(B(M)\big)$ has a basis consisting of clopen sets of the form

$$U(e) := \big\{I \in \mathrm{Max}(M) \colon e \notin I\big\}$$

for $e \in B(M)$. We have

$$\theta_b^{-1}(U) = \big\{m \in \mathrm{Ext}\big(\mathscr{S}(M)\big) \colon m(e) \neq 0\big\}\,,$$

which is an open set of $\mathrm{Ext}\big(\mathscr{S}(M)\big)$. Thus θ_b is continuous.

The space $\mathrm{Ext}\big(\mathscr{S}(M)\big)$ has a subbasis of open sets

$$V = \big\{m \in \mathrm{Ext}\big(\mathscr{S}(M)\big) \colon m(x) < 1/k\big\}$$

for $x \in M$ and $k \geq 1$. Define the set

$$X = \big\{e \in B(M) \colon n(e \wedge x) \leq le\,,\ \tfrac{l}{n} \leq \tfrac{1}{k}\big\}\,.$$

We recall that the set $\{I \in \mathrm{Max}(M)\colon X \not\subseteq I\} = \bigcup\limits_{e\in X} U(e)$ which means that it is an open set of $\mathrm{Max}(M)$.

We claim that $\theta_b(V) = \{I \in \mathrm{Max}(M)\colon X \not\subseteq I\}$. By Proposition 4.3, there exist $l,n \geq 1$ such that $l/n < 1/k$ and $n(e\wedge x) \leq le$ for some $e \in B(M)\setminus I$. Then $e \in X$, so that $X \not\subseteq I$. Conversely, given $I \in \mathrm{Max}(M)$ with $X \not\subseteq I$, choose $e \in X\setminus I$. There exist $l,n\geq 1$ such that $l/n < 1/k$ and $n(e\wedge x) \leq le$. By Proposition 4.3, $\theta_b(I)(x) < l/n < 1/k$ so that $\theta_b^{-1}(I) \in V$. Thus $\theta_b(V)$ is of the claimed form and hence it is open in $\mathrm{Max}(M)$ which implies that θ_b is an open mapping and so θ_b is a homeomorphism in question. □

We denote by $\mathscr{S}(B(M))$ and $\mathrm{Ext}\big(\mathscr{S}(B(M))\big)$ the spaces of all states and extremal states on the Boolean algebra $B(M)$ of idempotents of M.

COROLLARY 4.6. *Let M be a pseudo MV-algebra satisfying general comparability. The mapping $\theta_s\colon\ \mathrm{Ext}(\mathscr{S}(M)) \to \mathrm{Ext}\big(\mathscr{S}(B(M))\big)$ defined by*

$$\theta_s(m) := m|B(M)\,, \qquad m \in \mathrm{Ext}(\mathscr{S}(M))\,, \tag{4.2}$$

defines a homeomorphism of $\mathrm{Ext}(\mathscr{S}(M))$ onto $\mathrm{Ext}\big(\mathscr{S}(B(M))\big)$.

P r o o f. This follows from Theorem 4.5 and Theorem [DvKa, Thm. 4.4]. □

COROLLARY 4.7. *Let M be a pseudo MV-algebra satisfying general comparability. Then any two-valued state on $B(M)$ can be uniquely extended to an extremal state on M.*

P r o o f. This follows from Corollary 4.6. □

COROLLARY 4.8. *If M is a pseudo MV-algebra satisfying general comparability theorem, then $\mathrm{Ext}(\mathscr{S}(M))$ is a non-void compact and totally disconnected topological space.*

P r o o f. Apply Theorem 4.5. □

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