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# REPETITION AND PSEUDO-PERIODICITY

William A. Sethares

ABSTRACT. Many phenomena exhibit great regularity without being periodic. This is modeled using the notion of "pseudo-periodic" functions and the related "$\rho$-norm", which allow complex repetitive phenomena to be represented as a periodic process plus a set of parameters that define the deviations of the process from true periodicity. Applications to sunspot data and to heartbeat analysis explore practical uses and limitations of the method.

## 1. Introduction

A data set often will exhibit great regularity without exactly repeating. For example, heartbeats always have the characteristic "lub-dub" pattern which occurs again and again, yet each recurrence differs slightly from each other. Some beats are faster, some slower, some are stronger and some weaker. Sometimes a beat may be "skipped". Nonetheless, the overriding regularity of the heartbeat is its most striking feature. This paper models such near repetition using the idea of a "pseudo-periodic" function. Coupled with the "$\rho$-norm", this provides a consistent framework with which to study and manipulate a large class of repetitious (but nonperiodic) phenomena.

The next section introduces the notion of pseudo-periodicity, and relates it to standard ideas of periodicity and almost-periodicity [3]. The basic idea is that each repetition of the pseudo-periodic function can be summarized by a pair of parameters $\alpha$ (which describes the amount of stretching or compression) and $\beta$ (which specifies where the repetition begins). An inner product and its associated norm are then used to solve a number of practical issues that arise in the study of pseudo-periodic functions.

Section 3 develops the theory behind pseudo-periodic functions in a series of results that demonstrate how the relevant parameters (the $\alpha$'s and $\beta$'s) can

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be reliably estimated from a given data waveform. Then Section 4 applies the method to a standard data base describing sunspot variations, and to a simulated heartbeat record. The application of the method to the rhythm tracking of musical signals is discussed, and the final section presents conclusions and suggestions for further study.

## 2. Periodicity and pseudo-periodicity

The function $s(t)$ is said to be periodic of period $T$ if $s(t) = s(t+T)$ for all $t$. (By convention, $T$ is the smallest value for which this holds.) As is well known, any continuous periodic function can be represented in terms of its Fourier Series, and this paper restricts attention to real periodic functions with Fourier coefficients that are absolutely summable. Let $\mathcal{P}_T$ be the set of all such $T$-periodic functions, which is closed under addition. With scalar multiplication defined in the usual way, $\mathcal{P}_T$ forms a linear vector space.

A more general class of functions is needed in order to study processes (such as the waveform of a heartbeat) which are repetitious but not periodic in the strict mathematical sense. Let $S(t)$ be a real valued continuous function with support in $[0, T]$. Given sequences $\alpha_i$ and $\beta_i$, let

$$R(t) = \sum_i S(\alpha_i t + \beta_i).$$

Such a function $R(t)$ is called *pseudo-periodic*, and $S(t)$ is called the *template* function for $R(t)$. The $\alpha_i$ are called the *stretching* parameters, and represent the lengthening or shortening of the periods. The $\beta_i$ are called the *translation* parameters, and allow nonuniform timing of the process, for instance, an acceleration or deceleration in the heartbeat rate. In a musical context, the $\alpha_i$ correspond to the pitch of the waveform while the $\beta_i$ correspond to the rhythm in which the waveforms appear. The $i$th *repetition* of the pseudo-periodic function $R(t)$ is designated $R_i(t) = S(\alpha_i t + \beta_i)$, which has support in $\left[-\frac{\beta_i}{\alpha_i}, \frac{T-\beta_i}{\alpha_i}\right]$. Usually, $\alpha_i \approx 1$ and $\beta_i \approx \beta_{i-1} + \frac{T}{\alpha_{i-1}}$, though successive repetitions may overlap. For the special case when $\alpha_i = 1$ and $\beta_i = iT \ \forall i$, $R(t)$ is periodic with period $T$, and is equal to $s(t)$, defined as the periodic extension of $S(t)$.

Because each repetition in $R(t)$ is simply related to the template function $S(t)$ via the stretching and translation parameters, it is possible to think of $R(t)$ as a member of an equivalence class represented by $s(t)$, thus reducing the complex mathematics of pseudo-periodic functions to the well understood periodic framework. The practical problem then arises of how to find (or estimate) the stretching and translation parameters for a given waveform $R(t)$.

To state this concretely, let $R_i(t)$ be the $i$th repetition of $R(t)$, and let $r_i(t)$ be the periodic extension of $R_i(t)$. Again, let $s(t)$ represent the periodic extension of the template function $S(t)$. Then the problem of identifying the stretching and translation parameters of the $i$th repetition can be reduced to the problem of finding $\alpha_i$ and $\beta_i$ to minimize a measure of the difference between $s(\alpha_i t+\beta_i)$ and $r_i(t)$, that is, to minimize $\| s(\alpha_i t+\beta_i)-r_i(t)\|$. But what norm is appropriate in this context? Since the functions are periodic (have infinite energy) the $L_2$ norm cannot be applied. Moreover, although both $s(t)$ and $r_i(t)$ are periodic, their periods may be different, and hence their sum (and difference) may not be periodic. This means that the minimization problem cannot be stated properly within the mathematics of periodic functions (those with Fourier Series containing only harmonics of a single underlying fundamental) and has entered the mathematics of functions with noninteger related Fourier Series. Such functions were first investigated by Bohr [3], who called them *almost periodic* functions. Later investigations by Besicovitch, by Weyl, and by Weiner (see [2] for a summary of this work) brought the field of almost periodic functions to maturity.

In order to place this work in a modern mathematical framework, consider the inner product

$$\langle s, r\rangle_\rho = \lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} s(t)\, r(t)\, dt\,, \tag{1}$$

where $s(t)$ and $r(t)$ are periodic functions of periods $T_s$ and $T_r$ (which need not be the same). To see that (1) actually defines an inner product [9], there are four conditions that must be verified: commutativity, additivity, invariance under scalar multiplication, and positivity. The inner product (1) essentially finds the "average correlation" between $s(t)$ and $r(t)$ over all possible periods, and induces the $\rho$-norm

$$\| s\|_\rho = \sqrt{\langle s, s\rangle_\rho}\,.$$

In the discrete time case [16], an analogous inner product was used to induce the "periodicity norm". For the purposes of calculation, observe that if $s(t) \in \mathcal{P}_{T_s}$ then (1) is equal to the average over a single period, that is,

$$\| s\|_\rho^2 = \frac{1}{T_s} \int_0^{T_s} s^2(t)\, dt\,.$$

Accordingly, the problem of finding $\alpha$ and $\beta$ to best represent the $i$th repetition of the pseudo-periodic $R(t)$ can now be stated as the problem of minimizing

$$J(\alpha_i, \beta_i) = \frac{1}{2} \big\| s(\alpha_i t+\beta_i) - r_i(t)\big\|_\rho^2. \tag{2}$$

This mathematical setup is reminiscent of wavelet transforms [5], [18]. The template function plays a role analogous to that of the mother wavelet, while the stretching parameter is analogous to the scale factor. However, wavelet scale factors are often constrained to specific values which insure that the wavelet basis functions are orthogonal, while the stretching parameters assume arbitrary values and so the template functions need not be orthogonal. Hence, template functions do not form a basis, rather, they form a *frame* [4], a more-than-complete spanning set[1]. A more fundamental difference is that the goal of the pseudo-periodic analysis is quite different from the goal of most frame methods (such as [8], [10], [12]), as will become clear from the applications, which show how the pseudo-periodic analysis can directly provide information (such as the speed and frequency of repetitions of the template within a waveform) that other methods do not. From a computational point of view, the unknown parameters of standard orthogonal wavelet analysis enter in a linear fashion, which simplifies the calculations in comparison to (2), where the $\alpha$ and $\beta$ parameters enter in a nonlinear way.

## 3. Properties of the $\rho$-norm

The first result shows that periodic functions under the $\rho$-norm are insensitive to stretching by $\alpha$ and translation by $\beta$. A similar result is, of course, untrue for the standard $L_2$ norm.

**THEOREM 3.1.** *For any $s \in \mathcal{P}_{T_s}$ and any $\alpha, \beta \in \mathbb{R}$,* $\|s(t)\|_\rho = \|s(\alpha t + \beta)\|_\rho$.

P r o o f. Let $s(t) = r(\alpha t + \beta)$. Hence $r(t) \in \mathcal{P}_{T_r}$ where $T_r = \alpha T_s$. By definition,

$$\|r(t)\|_\rho^2 = \frac{1}{T_r} \int_{t=\hat{T}}^{t=\hat{T}+T_r} r^2(t)\, dt\,.$$

Let $t = \alpha\tau + \beta$, which implies that $dt = \alpha d\tau$. Using this change of variables, the lower limit of integration becomes $\tau = \frac{\hat{T}}{\alpha} \equiv T$ while the upper limit becomes $\tau = \frac{\hat{T}}{\alpha} + T_s = T + T_s$. Thus

$$\|r(t)\|_\rho^2 = \frac{1}{\alpha T_s} \int_{\tau=T}^{\tau=T+T_s} r^2(\alpha t + \beta)\alpha\, d\tau\,.$$

[1]Suitable conditions on $S(t)$ so that it forms a frame are given by (for instance), the Stone-Weierstrauss theorem [1], which also requires that the $i$th repetition be weighted $c_i S(\alpha_i t + \beta_i)$.

Canceling the $\alpha$'s and substituting the definition of $s(t)$ shows that this is exactly $\|s(t)\|_\rho$. □

The next result shows that the solution to the minimization problem (2) is the "same" whether the data is adjusted to match the template, or whether the template is adjusted to match to the data.

**Theorem 3.2.** *$\|r(\alpha t+\beta)-s(t)\|_\rho = \|r(t)-s(\bar{\alpha}t+\bar{\beta})\|_\rho$, where $\bar{\alpha}=1/\alpha$ and $\bar{\beta}=-\beta/\alpha$.*

P r o o f . By definition,

$$\langle r(\alpha t+\beta), s(t)\rangle_\rho = \lim_{k\to\infty}\frac{1}{2k}\int_{t=-k}^{t=k} r(\alpha t+\beta)s(t)\,dt\,.$$

Using the change of variables $\tau=\alpha t+\beta$, this becomes

$$\lim_{k\to\infty}\frac{1}{2k}\int_{\tau=-\alpha k+\beta}^{\tau=\alpha k+\beta} r(\tau)s\Big(\frac{\tau}{\alpha}-\frac{\beta}{\alpha}\Big)\,d\tau/\alpha\,.$$

Assuming $\alpha>0$, let $\bar{k}=\alpha k$. Using the definitions of $\bar{\alpha}$ and $\bar{\beta}$ gives

$$\lim_{\bar{k}\to\infty}\frac{\alpha}{2\bar{k}}\int_{\tau=-\bar{k}+\beta}^{\tau=\bar{k}+\beta} r(\tau)s(\bar{\alpha}\tau+\bar{\beta})\,d\tau/\alpha\,.$$

Canceling the $\alpha$'s gives

$$\langle r(\tau), s(\bar{\alpha}\tau+\bar{\beta})\rangle_\rho\,.$$

Hence

$$\|r(\alpha t+\beta)-s(t)\|_\rho^2 = \|r(\alpha t+\beta)\|_\rho^2 - 2\langle r(\alpha t+\beta), s(t)\rangle_\rho + \|s(t)\|_\rho^2\,.$$

Theorem 3.1 shows that the first and third terms on the right hand side are translation and stretch invariant, while the middle term follows directly from the argument above. Thus the last formula can be rewritten as

$$\|r(t)\|_\rho^2 - 2\langle r(t), s(\bar{\alpha}t+\bar{\beta})\rangle_\rho + \|s(\bar{\alpha}t+\bar{\beta})\|_\rho^2 = \|r(t)-s(\bar{\alpha}t+\bar{\beta})\|_\rho^2\,.$$

□

Consider the problem of choosing $\alpha$ and $\beta$ so as to minimize the cost $J(\alpha,\beta)$ of (2), which can be expanded as

$$\|s(t)-r(\alpha t+\beta)\|_\rho^2 = \|s(t)\|_\rho^2 - 2\langle s(t), r(\alpha t+\beta)\rangle_\rho + \|r(\alpha t+\beta)\|_\rho^2\,.$$

Theorem 3.1 guarantees that both $\|s\|_\rho$ and $\|r\|_\rho$ are independent of $\alpha$ and $\beta$, and hence minimizing $J(\alpha,\beta)$ is equivalent to maximizing

$$\bar{J}(\alpha,\beta) = \big\langle s(t), r(\alpha t+\beta)\big\rangle_\rho . \tag{3}$$

This has the form of a correlation with respect to the $\rho$-norm, and hence the optimization is effectively finding the parameters which best correlate $r(\alpha t+\beta)$ with $s(t)$.

The next result shows that the set of all possible maxima of $\bar{J}$ can be characterized simply in terms of the Fourier Series coefficients $s_m$ of $s(t) \in \mathcal{P}_{T_s}$ and $r_n$ of $r(t) \in \mathcal{P}_{T_r}$.

**Theorem 3.3.** *The maxima of $\bar{J}(\alpha,\beta)$ of (3) (equivalently, the minima of $J(\alpha,\beta)$ of (2)) can occur only at discrete values of $\alpha$ for which $\ell_1\alpha T_s = \ell_2 T_r$ where $\ell_1$ and $\ell_2$ are integers. Moreover, at these values, $\bar{J} = \sum_k r_{k\ell_1} s_{-k\ell_2}$.*

Proof. Express $s(t)$ in terms of its Fourier coefficients

$$s(t) = \sum_m s_m e^{jm\omega_s t}, \qquad \text{where} \quad \omega_s = \frac{2\pi}{T_s},$$

and expand $r(t)$ as

$$r(t) = \sum_n \bar{r}_n e^{jn\omega_r t}, \qquad \text{where} \quad \omega_r = \frac{2\pi}{T_r}.$$

Then

$$r(\alpha t+\beta) = \sum_n \bar{r}_n e^{jn\omega_r(\alpha t+\beta)} = \sum_n \big(\bar{r}_n e^{jn\omega_r\beta}\big) e^{jn\omega_r\alpha t} \equiv \sum_n r_n e^{jn\omega_r\alpha t}.$$

By the definition of the inner product (1),

$$\bar{J}(\alpha,\beta) = \lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k} s(t)r(\alpha t+\beta)\,dt .$$

Substituting the Fourier coefficients into this expression yields

$$\begin{aligned}\bar{J}(\alpha,\beta) &= \lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k}\Big(\sum_m s_m e^{jm\omega_s t}\Big)\Big(\sum_n r_n e^{jn\omega_r\alpha t}\Big)\,dt\\ &= \lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k}\sum_m\sum_n s_m r_n e^{j(m\omega_s+n\alpha\omega_r)t}\,dt .\end{aligned}$$

Letting $\tau = t/k$, $d\tau = dt/k$, the limits of the integration are transformed from $t = \pm k$ to $\tau = \pm 1$ we obtain

$$\lim_{k\to\infty} \frac{1}{2k} \int_{-1}^{1} \sum_m \sum_n s_m r_n e^{j(m\omega_s + n\alpha\omega_r)k\tau} k \, d\tau \,,$$

and canceling the $k$'s yields

$$\lim_{k\to\infty} \frac{1}{2} \int_{-1}^{1} \sum_m \sum_n s_m r_n e^{j(m\omega_s + n\alpha\omega_r)k\tau} \, d\tau \,.$$

The bounded convergence theorem (Corollary 5.37 of [21]) can now be applied since $r_n$ and $s_m$ are absolutely summable and since the complex exponential is bounded by unity. Accordingly, the limit can be interchanged with the sums to give

$$\sum_m \sum_n s_m r_n \lim_{k\to\infty} \frac{1}{2} \int_{-1}^{1} e^{j(m\omega_s + n\alpha\omega_r)k\tau} \, d\tau \,.$$

Doing the reverse change of variables ($t = k\tau$) gives

$$\sum_m \sum_n s_m r_n \lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} e^{j(m\omega_s + n\alpha\omega_r)t} \, dt \,.$$

Let $\omega = m\omega_s + n\alpha\omega_r$. Any $k$ can be expressed uniquely as $k = \frac{2\pi n}{\omega} + \phi$ where $n$ is an integer and $\phi \in [0, 2\pi)$. For $\omega \neq 0$, the integral can be rewritten

$$\begin{aligned} \left| \int_{-k}^{k} e^{j\omega t} \, dt \right| &= \left| \int_{-\frac{2\pi n}{\omega}-\phi}^{\frac{2\pi n}{\omega}+\phi} e^{j\omega t} \, dt \right| \\ &= \left| \int_{-\frac{2\pi n}{\omega}}^{\frac{2\pi n}{\omega}} e^{j\omega t} \, dt + \int_{-\frac{2\pi n}{\omega}-\phi}^{-\frac{2\pi n}{\omega}} e^{j\omega t} \, dt + \int_{\frac{2\pi n}{\omega}}^{\frac{2\pi n}{\omega}+\phi} e^{j\omega t} \, dt \right| . \end{aligned} \tag{4}$$

The first of these three integrals is identically zero since it is the integral of a complex sinusoid over an integral number of periods. Hence

$$\left| \int_{-k}^{k} e^{j\omega t} \, dt \right| \leq \int_{-\frac{2\pi n}{\omega}-\phi}^{-\frac{2\pi n}{\omega}} \left| e^{j\omega t} \right| dt + \int_{\frac{2\pi n}{\omega}}^{\frac{2\pi n}{\omega}+\phi} \left| e^{j\omega t} \right| dt \leq 4\pi \,.$$

The argument of each of the remaining integrals is bounded by one, and is integrated over a region of length $\phi$, which is at most $2\pi$. Hence (4) can be bounded independently of $k$, and so

$$\lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} e^{j\omega t}\, dt = \lim_{k\to\infty} \frac{4\pi}{2k} = 0\,.$$

On the other hand, for $\omega = 0$, $e^{j\omega t} = 1$, and

$$\lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} e^{j\omega t}\, dt = \lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} dt = 1\,.$$

Accordingly,

$$\bar{J}(\alpha,\beta) = \begin{cases} \sum_m \sum_n s_m r_n & \text{if } \omega = 0\,, \\ 0 & \text{if } \omega \neq 0\,. \end{cases} \tag{5}$$

Using the definitions of $\omega_s$ and $\omega_r$, $\omega = 0$ can be rewritten as

$$\frac{m}{n} = -\alpha \frac{T_s}{T_r}\,.$$

Since $m$ and $n$ are integers, this requires that $\alpha T_s$ be rationally related to $T_r$, that is, there must be integers $\ell_1$ and $\ell_2$ such that $\ell_1 \alpha T_s = \ell_2 T_r$. For such $\alpha$,

$$\sum_m \sum_n s_m r_n = \sum_k r_{k\ell_1} s_{-k\ell_2}\,,$$

which was the desired result. □

Theorem 3.1 shows that maxima of $\bar{J}$ occur only at isolated values of $\alpha$. As we shall see, the global maximum will typically lie at the simplest relationship where $n = m$, that is, when $\alpha T_s = T_r$. Before proceeding, it is worthwhile looking at a pair of examples that show two ways that this may fail to be the global maximum.

Example 3.1. Let $s_{2k} = \frac{1}{2^{|k|-1}}$ and $r_{3k} = \frac{1}{2^{|k|-1}}$ for $k = \pm 1, \pm 2, \pm 3, \ldots$ and zero otherwise. Then the global maximum will occur at $\alpha = \frac{3T_r}{2T_s}$.

In this example, the period of $s(t)$ is "really" twice the nominal $T_s$ while the period of $r(t)$ is three times the nominal $T_r$, so it should be unsurprising that the optimal adjusts to these values. Of course, it is possible to replace the zeroes with values $\varepsilon_k$ (justifying the assumed periods for $T_s$ and $T_r$) but this will not change the location of the optimal for suitably small $\varepsilon_k$.

EXAMPLE 3.2. Let $s_k = -r_k$, i.e., $s(t) = -r(t)$. In this case, $\bar{J}$ is negative, and $\alpha = \frac{T_r}{T_s}$ is a minimum, rather than a maximum.

This second example emphasizes the importance of having the template $s(t)$ 'close to' the data waveform $r(t)$. To make this more concrete, suppose that the Fourier coefficients $s_k$ of $s(t)$ are approximately equal to the Fourier coefficients $\bar{r}_k$ of $\bar{r}(t)$. Then $\bar{r}(t)$ can be decomposed as $\bar{r}(t) = r(t)+\eta(t)$ where $\bar{r}_k = r_k+\eta_k$, the Fourier coefficients $r_k$ of $r(t)$ are assumed equal to the $s_k$, and $\eta_k$ are the Fourier coefficients representing the (small) difference. For technical reasons, assume that the sequence $\eta_k$ is absolutely summable, with $\sum\limits_k |\eta_k| < \eta^*$.

Now consider the problem of finding $\alpha$ and $\beta$ to minimize

$$J(\alpha,\beta) = \frac{1}{2}\left\| s(t) - \bar{r}(\alpha t+\beta)\right\|_\rho^2 = \frac{1}{2}\left\| s(t) - r(\alpha t+\beta) - \eta(\alpha t+\beta)\right\|_\rho^2. \tag{6}$$

The next result shows that the answer to the minimization problem is insensitive to such perturbations or disturbances, assuming they are sufficiently small.

**THEOREM 3.4.** *There is an $\eta^* > 0$ such that the unique global minimum of* (6) *is the same as the unique global minimum of $J(\alpha,\beta)$ of* (2).

P r o o f . There are two parts to the proof. First we show that $\alpha = \frac{T_r}{T_s}$ (i.e., $\ell_1 = \ell_2 = 1$) gives the unique global minimum of (2). Since $r_k = s_k$, the optimal cost is zero, and it is achieved at $\alpha = \frac{T_r}{T_s}$. To see that it is unique, recall that minimizing (2) is the same as maximizing $\bar{J}$ of (3) which was translated by Theorem 3.1 into maximizing $\sum\limits_k r_{k\ell_1} s_{k\ell_2}$. In this case

$$\sum_k r_{k\ell_1} r_{k\ell_2} \le \frac{1}{2}\sum_k r_{k\ell_1} r_{k\ell_1} + \frac{1}{2}\sum_k r_{k\ell_2} r_{k\ell_2} \tag{7}$$

$$\le \sum_k r_k r_k\,. \tag{8}$$

The inequality in (7) is a consequence of the fact that $\|x\|_2^2+\|y\|_2^2 \ge 2\|x\|_2\|y\|_2$ (here we use the standard two norm $\|x\|_2 = xx^*$), and so strict equality occurs only when $\ell_1 = \ell_2$. The inequality in (8) holds because $\sum_k r_k r_k$ contains all the terms in (7), plus a sum of nonnegative terms that do not appear in (7). When $\ell_1 = \ell_2 \neq 1$, these terms cannot all be zero, since this would violate the assumption that $T_r$ and $T_s$ are the (smallest possible) periods of $r(t)$ and $s(t)$. Hence, when $\ell_1 \neq \ell_2$, inequality (7) must be strict and when $\ell_1 = \ell_2 \neq 1$, inequality (7) must be strict. When $\ell_1 = \ell_2 = 1$, both (7) and (8) are equal, which corresponds to the desired global maximum.

The second part of the proof shows that the presence of the term $\eta(t)$ in (6) does not change the value of $\alpha$ at which this maximum occurs. Theorem 3.1

shows that any possible maximum must occur at $\alpha$ which correspond to integer values of $\ell_1$ and $\ell_2$. Therefore, consider the maximization of

$$\sum_k r_{k\ell_1}(s_{k\ell_2} + \eta_{k\ell_2}).$$

Since $s_k = r_k$ is assumed, this can be rewritten into form

$$\sum_k r_{k\ell_1} r_{k\ell_2} + \sum_k r_{k\ell_1} \eta_{k\ell_2}.$$

For all possible maxima with $\ell_1 \neq \ell_2$, the inequality in (7) is strict and there is an $\eta_1 > 0$ with

$$\sum_k r_k r_k = \sum_k r_{k\ell_1} r_{k\ell_2} + \eta_1.$$

For all possible maxima with $\ell_1 = \ell_2 \neq 1$, the inequality in (8) is strict and there is an $\eta_2 > 0$ with

$$\sum_k a_k a_k = \sum_k a_{k\ell_1} a_{k\ell_2} + \eta_2.$$

Since $\sum_k r_{k\ell_1}\eta_{k\ell_2} \leq \|r_k\|_2 \|\eta\|_2$, choose $\eta^* = \max(\frac{\eta_1}{\|r_k\|_2}, \frac{\eta_2}{\|r_k\|_2})$. Hence, with this $\eta^*$, $\ell_1 = \ell_2 = 1$ again defines the global maximum. □

Thus the minimization problem is inherently robust and unbiased even in the presence of (suitably small) noises and disturbances. Said another way, the closer the template $s(t)$ is to the data function $r(t)$, the larger the noise that can be accommodated without degradation in the estimates of the stretching parameters.

## 4. Applications

This section shows how the pseudo-periodic idea can be applied by examining a standard record of sunspot data and a synthetic record of ECG (heartbeat) data. Applying the method requires choosing a template function and then calculating the appropriate parameters $\alpha_i$ and $\beta_i$.

### 4.1 Sunspot data

A record of monthly sunspot activity from the 1870's to the present is available in [11] and on the web at [19]. This is a fitting example because the (approximately) 11-year cycle of sunspot activity is repetitious without being periodic, and because the data has been extensively analyzed. For instance, Marple's book [11] compares and contrasts several different methods of spectral analysis using the sunspot data. The number of sunspots (averaged per month) is shown in part (a) of Figure 1. Because the data is noisy, it was passed through a linear lowpass filter, resulting in the plot shown in (b). Since each repetition of the sunspot activity looks like a single bump starting at zero, rising to its maximum and then falling back down to zero, a Gaussian shaped template function was chosen for which one period of the template (with $\alpha = 1$) corresponds to 10.1 years. The $\rho$-inner product of this template with the data record was then calculated for all $\beta$ and over a suitable range of $\alpha$, resulting in the contour plot shown in Figure 2. The arrows point to the times (the $\beta$ values) where this measure is maximized, and the corresponding $\alpha$ values are tabulated here:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_i$ | 0.94 | 0.92 | 0.82 | 0.92 | 0.84 | 0.94 | 0.90 | 1.08 | 0.74 | 0.86 | 1.04 |
| $\beta_i$ | 30 | 156 | 310 | 446 | 561 | 688 | 808 | 928 | 1065 | 1200 | 1314 |

These parameters should be interpreted to mean that (for instance) the fourth sunspot cycle is centered at sample 446, and it corresponds to a "width" ($\alpha$ value) of 0.92, which defines a $\frac{10.1}{0.92} = 10.9$ year cycle. The average value of the $\alpha$'s is 0.91 which corresponds to an 11.1 year width, and the average $\beta$ is 128.4. These values are well within the normal range of variation of the methods presented in [11]. Observe that the $\alpha$'s and $\beta$'s can be used to make simple predictions. For instance, linear extrapolation of the parameters would provide an estimate of the extent of the next sunspot cycle.

These values were found by simply searching over all $\alpha$ and $\beta$ within a reasonable range (the endpoints for the searches are shown in the figures). More sophisticated methods would include using gradient (hill climbing) methods to ascend the surface, but for the present, the figures should be taken as a plausibility argument for the use of the $\rho$-norm, rather than as a concrete proposal for an algorithm.

### 4.2 Synthetic heartbeat data

A "heartbeat" signal was synthesized by beginning with the template shown in Figure 3(b), which mimics some of the real features of heartbeat signals. The signal was then stretched (or compressed), amplified (or attenuated), spaced irregularly, and then modified with a small amount of additive noise to form the signal shown in (a). Figure 4 depicts a mesh or surface plot of the $p$-inner product of the heartbeat as a function of the $\alpha$ and $\beta$ parameters. Since the signal was synthetic, the actual values of the parameters could be compared to the estimated values, which occur at the peaks of the ridges in the figure. The $\beta$'s were accurate to the sample, and the $\alpha$'s to two decimal places.

## 5. Discussions and conclusions

Perhaps the most useful aspect of the pseudo-periodic analysis is that the sequences $\alpha_i$ and $\beta_i$ can give valuable information about the underlying phenomenon represented by the data. For instance, if in the heartbeat application the $\beta_i$ are increasing faster than $iT$ then the heartbeat is slowing down, while if the $\alpha_i$ are increasing, then the speed of each beat is increasing. More generally, patterns within the stretching and translation sequences may be of interest, for instance, statistical deviations from a "steady beat" represented by $\alpha_i = 1$ and $\beta_i = iT$ may contain useful information.

There have been several recent attempts to automatically identify the metric structure of musical pieces such as [13], [20], and [6]. In [17], we described a method of determining rhythmic structure from digitized audio that is based on the idea of a psychoacoustically motivated method of data reduction (which subsamples the recording at an effective rate of 100 to 200 samples per second) followed by application of a period finding technique such as the FFT or the periodicity transform [16]. As noted in [15], this method does not degrade gracefully when the underlying tempo varies. In this musical context, the pseudo-periodicity idea may be useful as a way to preprocess the data, where the $R_i(t)$ relate to the shape of the subsampled waveform, the $\alpha_i$ specify how quickly the waveshape evolves, and the $\beta_i$ define the tempo variations. This effectively transforms the (slowly varying) periodicities of the musical tempo into an underlying periodic signal and provides a concrete way of talking about changes in the "periodicity" over time. Thus the $\rho$-norm approach is not intended to apply to musical performances at the "note" level, but at the level of rhythm events.

Using the stretching and translation parameters to map the data back to the underlying template (i.e., back to a periodic function) allows comparisons to be made across data sets. For example, the heartbeat of one person at one time can be compared to another person at another time. More generally, this inverse mapping can be used to transform mathematical manipulations on pseudo-periodic functions to the far simpler mathematics of periodic functions. This could be used (for instance) to clean up data before application of frequency analysis methods, or as a preprocessor for methods such as the Periodicity Transforms of [16], which assume an underlying periodic process.

Template functions are not restricted to the simple form suggested here. For instance, more complex templates might consist of two (or more) alternating functions $S^1(\alpha_i^1 t + \beta_i^1)$ and $S^2(\alpha_i^2 t + \beta_i^2)$, or they might contain more complex transformations within the template, such as $S(\gamma_i t^2 + \alpha_i t + \beta_i)$. It is not completely clear exactly which parts of the theory continue to hold for such modified templates, but they may be useful for waveforms which have greater variation than can be readily captured in the simpler model.

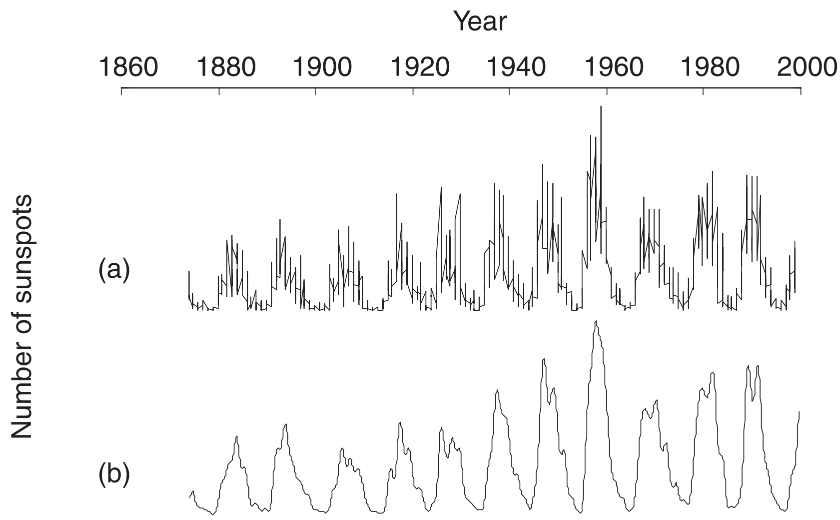


FIGURE 1. Number of sunspots (raw data) is presented in (a) and a (linearly) smoothed approximation is shown in (b).

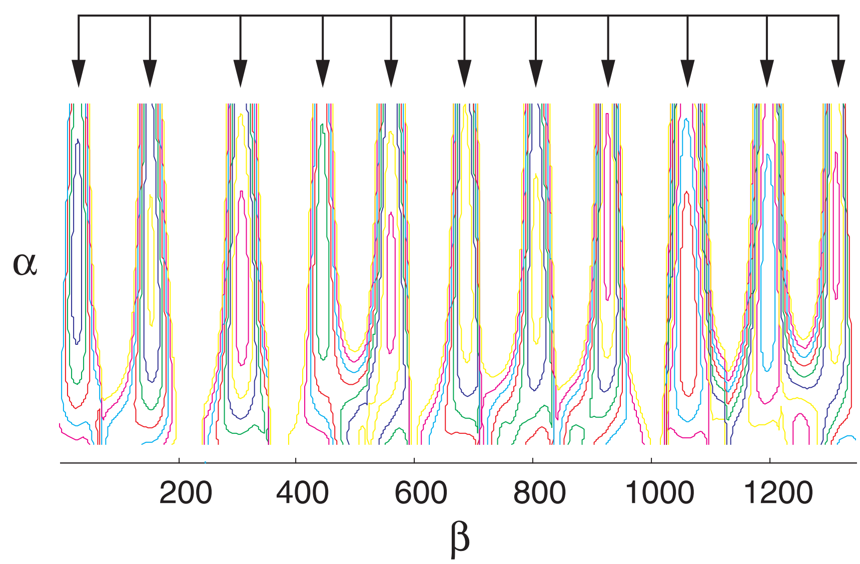


FIGURE 2. Contour plot of the sunspot data using a Gaussian shaped template function. Arrows point to times of maximum p-inner product.

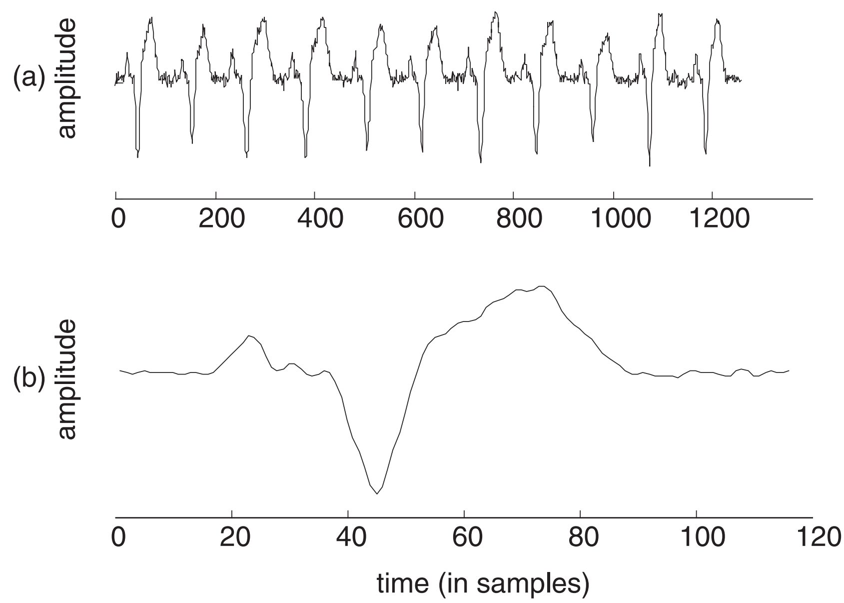


FIGURE 3. The synthetic heartbeat data in (a) is analyzed using the template function shown in (b).

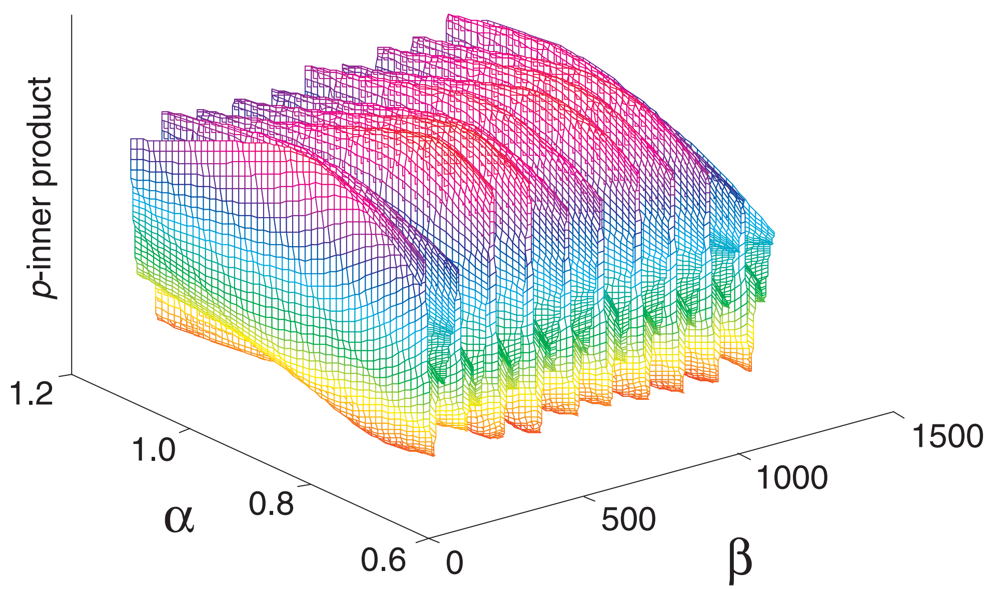


FIGURE 4. Mesh (surface) plot of the p-inner product of the synthetic heartbeat data as a function of a (stretching) and b (time shift).

Finally, note that the discussion here has focused on the problem of finding the stretching and translation parameters given the data $R(t)$ and the template $S(t)$. The presumption is that the generic behavior of the phenomenon is given by the template, while specific measurements are given by the data. But there is a sense in which it ought to be possible to determine the best template function $S(t)$ given just the pseudo-periodic function $R(t)$. This will likely involve some kind of alternating iteration between frequency estimation (or period detection) methods and the kinds of techniques presented here. Details of such a scheme are currently under investigation.

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*Department of Electrical*
*and Computer Engineering*
*University of Wisconsin-Madison*
*WI 53706–1691 Madison*
*U. S. A.*

*E-mail*: sethares@engr.wisc.edu