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ON PRESERVING ($\mathbb{R}$, EUCL.) AND ALMOST PERIODIC FUNCTIONS

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ABSTRACT. In this paper we answer a question of J. Doboš on whether continuous functions f that take the Euclidean metric on $\mathbb{R}$ to another metric on $\mathbb{R}$ with

$$\liminf_{x\to\infty} f(x) = 0$$

are also almost periodic.

We begin with functions that preserve all metrics. Let $\mathbb{R}^+$ denote $[0,\infty)$. We say that $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is metric preserving if for every metric space (M,ρ)

$$f\circ\rho\colon M\times M \to \mathbb{R}^+$$

is still a metric on M. Our first example of this is $f(x) = \frac{x}{x+1}$ which turns any metric into a bounded metric. Some slightly more extreme examples include the Cantor function (see [2]), which means metric preserving functions can be singular, and the continuous, nowhere differentiable metric preserving function constructed in [5].

A more analytic description of metric preserving functions is due to F. Terpe [7]. He uses the notion of triangle triplets which are defined as follows:

Definition 1. Let a,b,c come from $\mathbb{R}^+$. Then (a,b,c) is a *triangle triplet* if $a\le b+c$, $b\le a+c$, and $c\le a+b$.

The following theorem then says that as long as f is zero only at the origin and f does not disturb triangle triplets, then f is a metric preserving function.

Theorem 1. *A function $f\colon \mathbb{R}^+\to\mathbb{R}^+$ is metric preserving if and only if $f^{-1}(0)=\{0\}$ and $\big(f(a),f(b),f(c)\big)$ is a triangle triplet whenever (a,b,c) is one.*

Proof. See [7]. □

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The following is one of several properties of metric preserving functions that we can derive using these triangle triplets. It says that given a point a we can find a lower bound on $f(x)$ for all $x > a$.

Theorem 2. *Let f be a metric preserving function and $a \in \mathbb{R}^+$. Then for $x > a$, $f(x) \geq \frac{1}{2} f(a)$.*

Proof. Let $x > a$, then (a, x, x) is a triangle triplet. Since f is metric preserving then $\big(f(a), f(x), f(x)\big)$ is also a triangle triplet. So $f(a) \leq f(x) + f(x)$ or $f(x) \geq \frac{1}{2} f(a)$. $\square$

For more on these types of functions, an interested reader is referred to [4] and [8].

In this note, we will concentrate on a larger collection of functions which preserve just the Euclidean metric ($d(a,b) = |a - b|$) on the real line. Of course, any function which is metric preserving will preserve ($\mathbb{R}$, Eucl.). Before we go on to examples which show that metric preserving functions are a proper subset of the functions that preserve ($\mathbb{R}$, Eucl.), we state a relationship between these functions and triangle triplets.

Theorem 3. *A function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ preserves ($\mathbb{R}$, Eucl.) if and only if $f^{-1}(0) = \{0\}$ and $\big(f(a), f(b), f(a+b)\big)$ is a triangle triplet.*

Example 1. [6] Let f be the function given by

$$f(x) = \begin{cases} 2x\,, & 0 \leq x \leq 1\,, \\ 3 - x\,, & 1 \leq x \leq 2\,, \end{cases}$$

and $\phi(x) = f(x - 2n) + n$ if $2n \leq x \leq 2(n+1)$, $n = 0, 1, 2, \dots$ Then ϕ preserves ($\mathbb{R}$, Eucl.), but is not metric preserving. The former is proved in the source and the latter can be seen using the triangle triplet $(2,\ 2,\ 3)$.

Example 2. [3] Let $g(x) = |\sin x| + \big|\sin\big(\sqrt{2}x\big)\big|$. This is a function which preserves ($\mathbb{R}$, Eucl.), but is not metric preserving. To see that it does not preserve all metrics, just note that $\liminf\limits_{x\to\infty} g(x) = 0$.

Aside from this example, in [3] Doboš gives a more general construction starting with a continuous metric preserving function f and ending with a continuous function F which preserves ($\mathbb{R}$, Eucl.) for which $\liminf\limits_{x\to\infty} F(x) = 0$. He then notes that both the function $g(x)$ above and the result $F(x)$ are examples of almost periodic functions. This leads him to ask the following question:

Conjecture 1. Are all continuous functions f which preserve ($\mathbb{R}$, Eucl.) and have $\liminf\limits_{x\to\infty} f(x) = 0$ almost periodic?

This is the question we wish to answer here.

First, we give a pertinent definition and theorem concerning almost periodic functions.

Definition 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. We say f is *almost periodic* if for every $\varepsilon > 0$ there exists an $h > 0$ such that for every x there exists a $t \in (x, x+h)$ such that

$$\left| f(y+t) - f(y) \right| < \varepsilon$$

for all $y \in \mathbb{R}$.

Theorem 4. *If f is almost periodic, then f is bounded.*

Proof. See [1]. ◻

We are now ready for the main result. We shall construct an example which contradicts the theorem above to show that there is no connection between preserving ($\mathbb{R}$, Eucl.) and being almost periodic.

Example 3. There exists a continuous f, which

1. preserves ($\mathbb{R}$, Eucl.), and
2. has $\liminf\limits_{x\to\infty} f(x) = 0$, but
3. is unbounded.

Proof. In our construction of f we will have points s_n and t_n with the properties $s_n < t_n < s_{n+1}$, $f(s_n) \to 0$, and $f(t_n) \to \infty$. Let $s_0 = 0 = f(s_0)$, $t_0 = 1 = f(t_0)$, $s_1 = 3/2$, and $f(s_1) = 1/2$ with f linear between these points.

Now we must find the remaining s_n and t_n, where

$$f(s_n) = \frac{1}{n+1} \qquad \text{and} \qquad f(t_n) = \sum_{k=1}^{n+1} \frac{1}{k}.$$

In order to do this we will use both translations and reflections of portions of the function that will have already been found.

Let $t_1 = s_1 + t_0$. On the interval $[s_1, t_1]$ we use translation and define f by

$$f(x) = f(x - s_1) + f(s_1).$$

We will use reflections to define f from t_1 to s_2. If we reflect the graph we have about $x = t_1$ the result is three new line segments and a function which no longer preserves $(\mathbb{R}, Eucl.)$ because the value is 0 at $2t_1$. To work around this, we take the second to last line segment (whose domain is the interval $[7/2, 4]$) and extend its domain by $1/3$. Thus the newest sections of the function are a line with slope -1 from $x = 5/2$ to $x = 7/2$, a line segment of slope 1 from $x = 7/2$ to $x = 4 + 1/3 = 13/3$, and a line segment with slope -1 from $x = 13/3$ to $x = 16/3$. Now this means $s_2 = 2t_1 + 1/3$ and $f(s_2) = 1/3$ (as needed).

Now we are able to generalize. Suppose f has been defined on $[0, s_n]$. Let $t_n = s_n + t_{n-1}$ and define f from s_n to t_n via translation as

$$f(x) = f(x - s_n) + f(s_n).$$

Next we define s_{n+1} to be $2t_n + \frac{1}{n+1}$. The graph of f on $[t_n, s_{n+1}]$ will not be the reflection of f about t_n. We can see from the definition of s_{n+1} that this interval is a little longer than just a reflection. If we did reflect f about t_n, the interval $[t_n, 2t_n]$ can be viewed as 3^n sub-intervals $I_1, \ldots, I_{3^n-1}, I_{3^n}$ each of whose images would be a line segment with slope 1 or -1. To arrive at the correct value for s_{n+1}, and get the right graph, we take the segment I_2 and I_{3^n-1} (both of whose images are a line with slope 1) and increase the length of each by $\frac{1}{2} \cdot \frac{1}{n+1}$ leaving the lengths of the other intervals untouched, just shifting their placements to the right. Obviously $\lim_{n\to\infty} f(s_n) = 0$ and $\lim_{n\to\infty} f(t_n) = \infty$.

What remains to be shown is that for $a, b > 0$, $\big(f(a), f(b), f(a+b)\big)$ is a triangle triplet. In this argument, we will use

$$\big| f(a+b) - f(b) \big| \le f(a)$$

and

$$f(a) \le f(b) + f(a+b)$$

as the criteria to be met. We shall assume that $0 < a \le b$ and do a proof by induction. We know that $\big(f(a), f(b), f(a+b)\big)$ is a triangle triplet if $a, b \in [0, s_1]$ since this piece of the function was used in [5] (and in a slightly altered form in [6]). Assume that we have proven this on the interval $[0, s_n]$. We then will consider two cases: (i) on the interval $[s_n, t_n]$, and (ii) on the interval $[t_n, s_{n+1}]$.

First we look at $[s_n, t_n]$. We will consider two possibilities. If $s_n \le b < a+b \le t_n$ then since the graph on $[s_n, t_n]$ is just a translation of the graph on $[0, t_{n-1}]$ we see that $\big(f(a), f(b), f(a+b)\big)$ is a triangle triplet. That is, $\big(f(a), f(b), f(a+b)\big)$ is a triangle triplet because $\big(f(a - s_n), f(b - s_n), f(a + b - s_n)\big)$ is a triangle triplet by the induction hypothesis. On the other hand, if $b < s_n \le a + b \le t_n$ we begin with the triplet $(a, x, a + x)$ and look at what happens to the images of these points as x goes from $x = s_n$ to $x = b$. When $x = s_n$, we have

$$f(a) + f(x) = f(a + x),$$

whose three terms make a triangle triplet. Note that $f(s_n)$ is in fact a local minimum. At first, as x shifts to the left, either the distance between $f(x)$ and $f(x + a)$ remains at $f(a)$ (if both $f(x)$ and $f(a + x)$ are increasing as we go to the left) or the points get closer together (here $f(x)$ increases while $f(a + x)$ decreases but both change at the same rate). As x continues toward b, $\big| f(x) - f(x + a) \big|$ will never get above $f(a)$. The third part of the triangle triplet will be easily satisfied since if $x = s_n$

$$f(a) = f(a + x) - f(x) \le f(a + x) + f(x)$$

and it is not possible for the sum on the right go below $f(a)$ as x goes to b due to the fact that the two terms on the right-hand side either both increase, or one increases while the other decreases, but do so at the same rate.

A similar situation holds if we look at $[t_n, s_{n+1}]$. If $t_n \le b < a+b \le s_{n+1}$ then

$$\left|f(a+b) - f(b)\right| \le \left|f(2t_n - a - b) - f(2t_n - b)\right| \le f(a)$$

and

$$f(a) \le f(2t_n - a - b) - f(2t_n - b) \le f(a+b) + f(b)$$

by taking advantage of the "reflective" nature of this section of graph. When $b < t_n \le a+b \le s_{n+1}$ we start with $(a,\, x-a,\, x)$ and let x go from $x = t_n$ (where

$$f(a) + f(t_n - a) = f(t_n)$$

holds) to $x = a+b$. As x shifts we can see the triangle triplets still hold. So our proof is done. □

The difference between our example and the examples of D o b o š is that both of his are bounded functions. So it is natural to ask if all continuous functions f which preserve ($\mathbb{R}$, Eucl.), *are bounded*, and have $\liminf\limits_{x\to\infty} f(x) = 0$ are almost periodic. At The Summer Symposium in Real Analysis XXIII in Łódź, Poland, the author presented the above example and asked this question. M. Z e l e n y answered it in the negative by considering the function

$$F(x) = \min\bigl\{f(x)\,, 1\bigr\}\,,$$

where f is our construction from above.

P r o o f . First we note that since f is continuous and unbounded, for any length h there is an interval $(x, x+h)$ so that $f(x) \ge 1$ $(F(x) = 1)$ on that interval. Choose any $\varepsilon < 1$. From the definition of almost periodic this ε leads us to a length h. Now for this h find an interval $(x, x+h)$, where $F(x)$ is constant at 1. Than for any $t \in (x, x+h)$

$$1 = \left|f(t) - f(0)\right| > \varepsilon\,.$$

Thus F is not almost periodic. The fact that a minimum of two functions which preserve ($\mathbb{R}$, Eucl.) will still preserve ($\mathbb{R}$, Eucl.) is well known. □

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