Tatra Mt. Math. Publ. **24** (2002), 11–12

TATRA MOUNTAINS
Mathematical Publications

WEAK AND STRONG DIFFERENTIABILITY ARE NEARLY THE SAME FOR REAL FUNCTIONS

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ABSTRACT. In this note we prove that each continuous real function $f\colon \mathbb{R} \to \mathbb{R}$ which is weakly differentiable is strongly differentiable on a residual set.

There are many approaches how to define the differentiability of a real function $f\colon \mathbb{R} \to \mathbb{R}$. The strong differentiability at a point $x_0 \in \mathbb{R}$ means that there exists the finite double limit

$$\lim_{\substack{x \to x_0 \\ h \to 0}} \frac{f(x+h) - f(x)}{h} =: f^*(x_0)\,,$$

and the real number $f^*(x_0)$ is called the strong derivative of f at x_0. We say that f is *strongly differentiable* if it is strongly differentiable at each point of the definition domain. This notion is strong enough to imply some other types of differentiability, like the usual one or the so called weak differentiability, which is defined as follows.

Definition. Function $f\colon \mathbb{R} \to \mathbb{R}$ is weakly differentiable at a point $x_0 \in \mathbb{R}$ if there exist the finite limits

$$\lim_{n\to\infty} \frac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{f(x_0 - \frac{1}{n}) - f(x_0)}{-\frac{1}{n}}\,,$$

and their common value we shall denote by $f^w(x_0)$ and we call it *weak derivative of f at the point x_0*. If f has the weak derivative at each point of its definition domain, we say that f is *weakly differentiable* and f^w will denote the weak derivative of the function f.

Note that this is the sequential derivative with respect to the sequence $(h_n)_{n\ge 1}$, where $h_{2n} = \frac{1}{n}$, $h_{2n-1} = -\frac{1}{n}$ in the sense of A. Bruckner and

2000 Mathematics Subject Classification: 26A24.
Keywords: strongly differentiable, weakly differentiable, residual set.
Supported by the grant VEGA 1/7173/20.

used by M. Laczkovich and G. Petruska in the paper [2]. We remark that it would be possible to define yet the weaker derivative at x supposing only the existence of a sequence $(h_n)_{n\geq 1}$ convergent to x which contains two subsequences $(k_n > 0)_{n\geq 1}, (l_n < 0)_{n\geq 1}$ such that $\lim\limits_{n\to\infty} \frac{k_n}{k_{n+1}} = \lim\limits_{n\to\infty} \frac{l_n}{l_{n+1}} = 1$, and the next theorem would still be true.

In the paper [3] it was shown that each differentiable function $f\colon [a,b] \to \mathbb{R}$ is strongly differentiable on a dense G_δ subset of $[a,b]$ (hence on a residual set). Using a result of the paper [2] we can prove the strong differentiability of f supposing only the continuity and weak differentiability of f.

THEOREM. *Let $f\colon (a,b) \to \mathbb{R}$ be a continuous weakly differentiable function. Then f is strongly differentiable on a dense G_δ set.*

Proof. For each $n \geq 1$ let us define the functions

$$f_n(x) := n\big(f(x+\tfrac{1}{n}) - f(x)\big)\,.$$

They are all continuous and the sequence $(f_n)_{n\geq 1}$ converges pointwise on (a,b) to the weak derivative f^w. Following [3] Lemma 2 and Theorem 4, there is a dense G_δ set $U \subset (a,b)$ such that the sequence $(f_n)_{n\geq 1}$ converges uniformly at each point of U. Let us denote by D the set of all differentiability points of f. Owing to [2] Corollary 9, there is an open set $G \subset D$ which is dense in (a,b) and such that

$$f'(x) = f^w(x) \qquad \text{on } G\,.$$

Since $G \cap U \subset D \cap U$ and according to Theorem 5 of [3] we have $D \cap U = D^*$—the set of all points of strong differentiability of the function f. The fact that $G \cap U$ is dense G_δ set ends the proof. □

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Received September 21, 2000

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