Tatra Mt. Math. Publ. **24** (2002), 65–77

**TATRA MOUNTAINS**
*Mathematical Publications*

# ON THE RESTRICTIONS OF FUNCTIONS. FINITELY CONTINUOUS FUNCTIONS AND PATH CONTINUITY

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ABSTRACT. In this paper we shall consider problems connected with the continuity of the restrictions of functions with respects to two aspects: finitely continuity and path continuity.

**Introduction.** The restrictions of functions are often unnoticeable, but essential tools for investigations of some mathematical problems. It is well known that the restrictions of functions are very useful by the considerations connected with the approximate continuity, Baire class one, various integrals...

Investigations connected with continuity of restrictions of functions have recently become especially interesting and intensive. In this paper we shall present this topic with respect to two aspects:

- Decomposition of the domain of a function in such a way that the restriction of this function to each composite of this decomposition is a continuous transformation.
- Possibility of choice, for a fixed point, of some subset of the domain of a function in such a way that the restriction to this set is a continuous function at this point.

The first direction of our investigations lead us to the considerations concerning the finitely continuous function ([6]). The second topic is connected with the theory of paths ([2] [3] [10]).

**Preliminaries.** We apply standard symbols and notions. By $\mathbb{R}$ we denote the set of real numbers (we use the same letter to denote the topological space $\mathbb{R}$ with the natural topology) and by $\mathbb{Q}$ ($\mathbb{Z}$) we denote the set of rational (integer) numbers.

If $a, b \in \mathbb{R}$, then by $I_{(a,b)}$ we denote the open interval $\big(\min(a,b), \max(a,b)\big)$.

2000 Mathematics Subject Classification: 26A15, 26A21.
Keywords: path, finitely continuous function, Darboux function, classes $B_1$, $B_1^*$ and $B_1^{**}$.

The symbol $\mathrm{int}(A)$ ($\mathrm{cl}(A)$, $\mathrm{bd}(A)$, $\mathrm{der}(A)$, $\ell(A)$) stands for the interior (closure, boundary, the set of all accumulation points, the family of all components, respectively) of a set $A$.

In this paper we consider only real-valued functions. No distinction is made between a function and its graph.

By $C_f(D_f)$ we denote the set of all continuity (discontinuity) points of the function $f$. If $f \in Y^X$ and $A \subset X$, then the symbol $f \restriction A$ denotes the restriction of the function $f$ to the set $A$.

The set of all right-hand limit numbers (all left-hand limit numbers) of $f$ at $x$ will be denoted by $L^+(f,x)$ ($L^-(f,x)$).

Let $X$ be a metric space. The function $f \in \mathbb{R}^X$ is said to be:

- *Darboux* if the image $f(C)$ is a connected set, for every connected subset $C \subset X$;
- *almost continuous* (*in the sense of Stallings*) if each open subset of $X \times \mathbb{R}$ containing the graph of $f$ contains also the graph of some continuous function mapping $X$ into $\mathbb{R}$.

## 1. Finitely continuous functions

The starting point of our considerations connected with the main topic is the following definition (e.g., [6]).

We say that a function $f \in Y^X$, $X$ and $Y$ being metric spaces, is *$n$-continuous* (*finitely continuous*) if there exists a cover $\mathcal{A}$ of $X$ such that $\mathrm{card}(\mathcal{A}) = n$ ($\mathcal{A}$ is a finite cover) and the restriction $f \restriction A$ is a continuous function, for each $A \in \mathcal{A}$ (we say also that $f$ is $\mathcal{A}$-continuous function).

Of course, even 2-continuous functions can be "not nice", for example, they can be nonmeasurable functions. But if we restricted our considerations to the case of Darboux functions, then the properties of such functions would be nicer. In particular, in the paper [6] the following property has been proved:

*Let $X$ be a locally connected metric space and let $f \in \mathbb{R}^X$ be a Darboux function. Then $f$ is 2-continuous if and only if $f \in B_1^{**}$.*[1]

It is interesting to remark that there exists a 3-continuous Darboux function $f \in \mathbb{R}^{[0,1]}$ such that $f \notin B_1$.

The set of all continuity points of any finitely continuous Darboux function is a big set in the topological sense (let us note that there exists a finitely

[1] A function $f$ belongs to the class $B_1^{**}$ if either $D_f = \emptyset$ or $f \restriction D_f$ is the continuous function [8].

continuous Darboux and nonmeasurable function). This fact follows from the following theorem ([6]):

*Let $X$ be locally connected metric space. If $f \in \mathbb{R}^X$ is finitely continuous Darboux function, then $\operatorname{int}(C_f)$ is a dense set in $X$.*

R. O'Malley showed ([5]) that in the case of functions from the class $DB_1^*$ also the image of the interior of the set of all continuity points is a large set. Of course, finitely continuous Darboux functions need not belong to $B_1^*$, but the theorem analogous to the O'Malley's result is also true:

**THEOREM 1.** *If $f \in \mathbb{R}^{\mathbb{R}}$ is a finitely continuous Darboux function, then $f(\mathbb{R}) \setminus f\big(\operatorname{int}(C_f)\big)$ is a nowhere dense set.*

Proof. Let us denote $U_f = \operatorname{int}(C_f)$.

Of course, if $f$ is constant, then $f(\mathbb{R}) \setminus f(U_f) = \emptyset$. So, in what follows we shall assume that $f$ has at least two values.

Since $f$ is a Darboux function, the image $f(\mathbb{R}) = J$ is a nondegenerate interval. Let $G \neq \emptyset$ be an arbitrary open set. We shall show that there exists a nonempty open set $U$ such that

$$U \subset (G \setminus J) \cup \big(G \cap f(U_f)\big).$$

In the case when $G \cap J = \emptyset$ it is sufficient to put $U = G$. Now, we suppose that $G \cap J \neq \emptyset$, and we shall prove that

$$\text{there exists a nonempty open set } U \subset G \cap f(U_f). \tag{1}$$

Let $(a, b)$ be a nonempty interval contained in $G \cap J$.

Consider a function $h \in \mathbb{R}^{\mathbb{R}}$ defined by the formula:

$$h(x) = \min\big(b, \max(f(x), a)\big), \qquad \text{for } x \in \mathbb{R}.$$

First, we shall show that

$$\begin{gathered}\text{there exists a component } K \text{ of the set } U_h = \operatorname{int}(C_h) \text{ such that} \\ \text{the restriction } h{\restriction}K \text{ is a nonconstant function.}\end{gathered} \tag{2}$$

Suppose, to the contrary, that the restriction $h$ to each component of the set $U_h$ is a constant function. Let us observe that $h$ is discontinuous (in the opposite case $h$ is constant, which contradicts the fact that $(a, b) \subset h(\mathbb{R})$).

According to the Proposition 1 from [6], $h$ is finitely continuous. So, let $\mathcal{A}$ be a finite cover of $\mathbb{R}$ such that $h$ is $\mathcal{A}$-continuous. Let $n = \operatorname{card} \mathcal{A} > 1$.

Now, we shall show that there exist sequences $(A_i)_{i=1}^n$, $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$, $(\delta_i)_{i=1}^n$, $(\varepsilon_i)_{i=1}^n$ fulfilling the following conditions (for $i \in \{1, \dots, n\}$):

(i) $A_i \in \mathcal{A}$, $A_i \neq A_j$ (for $i, j \in \{1, \dots, n\}$, $i \neq j$);

(ii) $x_i \in D_h \cap A_i$, $y_i \in \Big(\operatorname{int}\big(L^+(h, x_i)\big) \cup \operatorname{int}\big(L^-(h, x_i)\big)\Big) \setminus \big(h(U_h) \cup \{h(x_i)\}\big)$;

(iii) $h\big((x_i - \delta_i,\, x_i + \delta_i) \cap A_i\big) \subset \big(h(x_i) - \varepsilon_i,\, h(x_i) + \varepsilon_i\big)$;

(iv) $(y_i - \varepsilon_i,\, y_i + \varepsilon_i) \cap \bigcup\limits_{l=1}^{i} \big(h(x_l) - \varepsilon_l,\, h(x_l) + \varepsilon_l\big) = \emptyset$;

(v) $(x_i - \delta_i,\, x_i + \delta_i) \subset (x_{i-1} - \delta_{i-1},\, x_{i-1} + \delta_{i-1})$ (for $i \in \{2, \dots, n\}$);

(vi) $h(x_i) \in (y_{i-1} - \varepsilon_{i-1},\, y_{i-1} + \varepsilon_{i-1}) \supset (y_i - \varepsilon_i,\, y_i + \varepsilon_i)$ (for $i \in \{2, \dots, n\}$).

In fact, let $x_1 \in D_h$. Since $h$ possesses Darboux property, the cluster sets $L^+(h, x_1)$, $L^-(h, x_1)$ are intervals and $h(x_1) \in L^+(h, x_1) \cap L^-(h, x_1)$. It is easy to see that $\operatorname{int}\big(L^+(h, x_1)\big) \cup \operatorname{int}\big(L^-(h, x_1)\big) \neq \emptyset$ and $h(U_h)$ is a countable set. Then we can choose an element $y_1 \in \Big(\operatorname{int}\big(L^+(h, x_1)\big) \cup \operatorname{int}\big(L^-(h, x_1)\big)\Big) \setminus \big(\{h(x_1)\} \cup h(U_h)\big)$.

Let $\varepsilon_1 > 0$ be a real number such that

$$(y_1 - \varepsilon_1,\, y_1 + \varepsilon_1) \cap \big(h(x_1) - \varepsilon_1,\, h(x_1) + \varepsilon_1\big) = \emptyset .$$

Moreover, let $A_1 \in \mathcal{A}$ be a set such that $x_1 \in A_1$. From the continuity of the restrictions $f \restriction A_1$ we conclude that there exists $\delta_1 > 0$ such that $h\big((x_1 - \delta_1,\, x_1 + \delta_1) \cap A_1\big) \subset \big(h(x_1) - \varepsilon_1,\, h(x_1) + \varepsilon_1\big)$. Now we suppose that $1 < k \leq n$ and that the sequences $(A_i)_{i=1}^{k-1}$, $(x_i)_{i=1}^{k-1}$, $(y_i)_{i=1}^{k-1}$, $(\delta_i)_{i=1}^{k-1}$, $(\varepsilon_i)_{i=1}^{k-1}$ have been defined in such a way that the conditions (i), (ii), (iii), (iv) (for $i \in \{1, \dots, k-1\}$) and (v), (vi) (for $i \in \{2, \dots, k-1\}$) are satisfied.

First, we shall show that there exists a point $x_k$ such that

$$x_k \in (x_{k-1} - \delta_{k-1},\, x_{k-1} + \delta_{k-1}) \cap D_h \quad \text{and} \quad h(x_k) \in (y_{k-1} - \varepsilon_{k-1},\, y_{k-1} + \varepsilon_{k-1}) .$$

Our proof starts with the observation that $x_{k-1}$ is a Darboux point of $h$. Since $y_{k-1} \in \Big(\operatorname{int}\big(L^+(h, x_{k-1})\big) \cup \operatorname{int}\big(L^-(h, x_{k-1})\big)\Big) \setminus \{h(x_{k-1})\}$, then there exits an element $c_{k-1} \in (x_{k-1} - \delta_{k-1},\, x_{k-1} + \delta_{k-1})$ such that $h(c_{k-1}) = y_{k-1}$. Of course, $c_{k-1} \notin U_h$.

If $c_{k-1} \in D_h$, then it is sufficient to put $x_k = c_{k-1}$.

In the opposite case $c_{k-1} \in C_h$ and we can choose $\tau_{k-1} > 0$ such that

$$(c_{k-1} - \tau_{k-1},\, c_{k-1} + \tau_{k-1}) \subset (x_{k-1} - \delta_{k-1},\, x_{k-1} + \delta_{k-1})$$

and

$$h\big((c_{k-1} - \tau_{k-1},\, c_{k-1} + \tau_{k-1})\big) \subset (y_{k-1} - \varepsilon_{k-1},\, y_{k-1} + \varepsilon_{k-1}) .$$

Since $c_{k-1} \notin U_h$ we have $c_{k-1} \in \operatorname{der}(D_h)$. Therefore we can choose $x_k \in (c_{k-1} - \tau_{k-1},\, c_{k-1} + \tau_{k-1}) \cap D_h$. Of course, $h(x_k) \in (y_{k-1} - \varepsilon_{k-1},\, y_{k-1} + \varepsilon_{k-1})$.

Since $x_k \in D_h$, $h(x_k) \in (y_{k-1} - \varepsilon_{k-1}, y_{k-1} + \varepsilon_{k-1})$ and $h$ is Darboux function, then $\Big(\mathrm{int}\big(L^+(h, x_k)\big) \cup \mathrm{int}\big(L^-(h, x_k)\big)\Big) \cap (y_{k-1} - \varepsilon_{k-1}, y_{k-1} + \varepsilon_{k-1}) \neq \emptyset$. Observe that $h(U_h)$ is a countable set, and so, there exists a point $y_k$ such that $y_k \in \Big[\Big(\mathrm{int}\big(L^+(h, x_k)\big) \cup \mathrm{int}\big(L^-(h, x_k)\big)\Big) \cap (y_{k-1} - \varepsilon_{k-1}, y_{k-1} + \varepsilon_{k-1})\Big] \setminus \big[\{h(x_k)\} \cup h(U_h)\big]$.

Let $\varepsilon_k > 0$ be a real number such that $(y_k - \varepsilon_k, y_k + \varepsilon_k) \subset (y_{k-1} - \varepsilon_{k-1}, y_{k-1} + \varepsilon_{k-1})$ and $\big(h(x_k) - \varepsilon_k, h(x_k) + \varepsilon_k\big) \cap (y_k - \varepsilon_k, y_k + \varepsilon_k) = \emptyset$.

Let $A_k \in \mathcal{A}$ be a set such that $x_k \in A_k$ and let $\delta_k > 0$ be a real number such that:

$$(x_k - \delta_k, x_k + \delta_k) \subset (x_{k-1} - \delta_{k-1}, x_{k-1} + \delta_{k-1})$$

and

$$h\big((x_k - \delta_k, x_k + \delta_k) \cap A_k\big) \subset \big(h(x_k) - \varepsilon_k, h(x_k) + \varepsilon_k\big).$$

It is not difficult to verify that for the sequences $(A_i)_{i=1}^n$, $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$, $(\delta_i)_{i=1}^n$, $(\varepsilon_i)_{i=1}^n$ constructed above the conditions (i), (ii), (iii), (iv), (v) and (vi) take place.

Observe that from the condition (i) we deduce that $\mathcal{A} = \{A_1, \dots A_n\}$. According to (iii) and (v) we infer that

$$\begin{aligned} h\big((x_n - \delta_n, x_n + \delta_n)\big) &= \bigcup_{i=1}^{n} \big(h\big((x_i - \delta_i, x_i + \delta_i)\big) \cap A_i\big) \\ &\subset \bigcup_{i=1}^{n} \big(h(x_i) - \varepsilon_i, h(x_i) + \varepsilon_i\big). \end{aligned}$$

Applying (iv) we obtain

$$h\big((x_n - \delta_n, x_n + \delta_n)\big) \cap (y_n - \varepsilon_n, y_n - \varepsilon_n) = \emptyset. \tag{3}$$

Observe that $y_n \in \Big(\mathrm{int}\big(L^+(h, x_n)\big) \cup \mathrm{int}\big(L^-(h, x_n)\big)\Big) \setminus \{h(x_n)\}$. Since $x_n$ is a Darboux point of $h$, we have $h^{-1}(y_n) \in (x_n - \delta_n, x_n + \delta_n) \neq \emptyset$, which contradicts (3). This contradiction ends the proof of (2).

Let $(r, s)$ be a component of the set $U_h$ such that $h$ is a nonconstant on $(r, s)$. It is not difficult to verify that:

*there exists $t_0 \in (r, s)$ such that $h(t_0) \in (a, b)$ and $h$ is a nonconstant function in any neighbourhood of the point $t_0$.*

Of course, there exists a real number $\delta_0 > 0$ such that $(t_0 - \delta_0, t_0 + \delta_0) \subset (r, s)$ and $h\big((t_0 - \delta_0, t_0 + \delta_0)\big) \subset (a, b)$. According to the definition of $h$ we conclude that if $x \in \mathbb{R}$ and $h(x) \in (a, b)$, then $h(x) = f(x)$. From this we infer that $f \restriction (t_0 - \delta_0, t_0 + \delta_0) = h \restriction (t_0 - \delta_0, t_0 + \delta_0)$ and, consequently, $(t_0 - \delta_0, t_0 + \delta_0) \subset U_f$

and $f$ is nonconstant on $(t_0-\delta_0, t_0+\delta_0)$, which gives that $f\big((t_0-\delta_0, t_0+\delta_0)\big)$ is a nondegenerate interval. Put $U = \operatorname{int}\big(f\big((t_0-\delta_0, t_0+\delta_0)\big)\big)$. Then we have $\emptyset \neq U \subset (a,b) \cap f(U_f) \subset G \cap f(U_f)$. In this way we prove that the condition (1) holds, which ends the proof of this theorem. □

As a simple consequence we can observe that under assumptions of this theorem the graph of $f\restriction \operatorname{int}(C_f)$ is a dense set in the graph of $f$, which means that $f$ is quasi-continuous and possesses the Denjoy–Clarkson property[2].

## 2. Path–continuity

The second direction of our investigations concerning the general topic *restrictions of functions* is connect with the theory of path. We restrict our considerations to the real line.

The starting point of our considerations will be the following definitions [2], [1]:

**Definition 1.** Let $x \in \mathbb{R}$. A *path* (more precisely, a *bilateral path*) leading to $x$ is a set $E$ such that $x \in E$ and $x$ is a point of bilateral accumulation of $E$.

A *system of paths at* $x \in \mathbb{R}$ is a nonempty family $\mathcal{F}$ of subsets of $\mathbb{R}$ such that $E$ is a path leading to $x$, for each $E \in \mathcal{F}$.

A *system of paths on* $\mathbb{R}$ is an arbitrary mapping $\varepsilon\colon \mathbb{R} \to 2^{2^{\mathbb{R}}}$ such that $\varepsilon(x)$ is a system of paths at $x$, for any $x \in \mathbb{R}$.

**Definition 2.** Let $\varepsilon$ be a system of paths in $\mathbb{R}$ and let $f \in \mathbb{R}^{\mathbb{R}}$. We say that $f$ is an *$\varepsilon$-continuous function at $x$* (or $x$ is an *$\varepsilon$-continuity point* of $f$), if there exists a path $E \in \varepsilon(x)$ such that $f\restriction E$ is a continuous function at $x$.

The set of all $\varepsilon$-continuity points of $f$ will be denoted by $C_f^{\varepsilon}$.

**Definition 3.** Let $\varepsilon$, $\varepsilon'$ be two systems of paths on $\mathbb{R}$. We say that the system $\varepsilon$ is a *refinement* of the system $\varepsilon'$ if for each $x \in \mathbb{R}$ and $E \in \varepsilon(x)$ there exists a path $E' \in \varepsilon'(x)$ fulfilling the following conditions:

(a) $E \subset E'$;

(b) $C \cap E \neq \emptyset$, for each component $C$ of $E'$.

The following system of paths connected with a fixed open and dense set on the real line will be very useful for our considerations:

For a fixed open and dense set $D$ of $\mathbb{R}$, let us denote by symbol $\varepsilon_D$ the system of paths on $\mathbb{R}$ such that: $\varepsilon_D(x) = \big\{D \cup \{x\}\big\}$, for $x \in \mathbb{R}$.

[2] This corollary resulted from the discussion with prof. T. Natkaniec and I. Recław.

**Theorem 2.** *Let $D$ be an arbitrary open and dense set in $\mathbb{R}$ and let $\varepsilon$ be a system of paths which is a refinement of $\varepsilon_D$. Then if $f\lceil D$ is a Darboux function and $\mathbb{R} \setminus D \subset C_f^{\varepsilon}$, then $f$ is Darboux as well.*

P r o o f. Assume that $f\lceil D$ is Darboux and $\mathbb{R} \setminus D \subset C_f^{\varepsilon}$ and suppose, on the contrary, that $f$ is not a Darboux function.

First, we shall show that there exist real numbers $a, b, \alpha$ such that:

$$a, b \in \mathbb{R} \setminus D,\ a < b,\ f(a) \neq f(b),\ \alpha \in I_{(f(a),\ f(b))} \quad \text{and} \quad f^{-1}(\alpha) \cap (a, b) = \emptyset. \tag{4}$$

Of course, our hypothesis that $f$ does not posses Darboux property allows one to suppose that there exist real numbers $a' < b'$ and a number $\alpha$ such that $f(a') \neq f(b')$, $\alpha \in I_{(f(a'),f(b'))}$ and $f^{-1}(\alpha) \cap (a', b') = \emptyset$. Without loss of generality we may assume that $f(a') < f(b')$.

If $a' \notin D$, then we put $a = a'$. In the opposite case, let $U_{a'}$ be a component of $D$ containing $a'$. Observe that $b' \notin U_{a'}$. Let $a$ be the right-hand endpoint of $U_{a'}$, then $a \leq b'$. Since $f \in \mathbb{R}^D$ is a Darboux function, $f(a') < \alpha$ and $\alpha \notin f\big([a', a)\big)$, then $f\big([a', a)\big) \subset (-\infty, \alpha)$. Let $E \in \varepsilon(a)$ be a path such that $f\lceil E$ is continuous at $a$. Then there exists a sequence $(t_n)_{n=1}^{\infty}$ such that $t_n \in E \cap (a', a)$, for $n = 1, 2, \dots$ and $t_n \nearrow a$. It is easy to see that $\lim\limits_{n\to\infty} f(t_n) = f(a)$ and $f(t_n) \in (-\infty, \alpha)$ (for $n = 1, 2, \dots$), which means that $f(a) \leq \alpha$. Of course, $f(b') > \alpha$ and, consequently, $a < b'$ and $f(a) < \alpha$.

If $b' \notin D$, then it is sufficient to put $b = b'$. In the opposite case, let $b$ be the left-hand endpoint of the component $U_{b'}$ of the set $D$ containing the point $b'$. Similarly as above, we can prove that $a < b < b'$ and $f(b) > \alpha$, thus the condition (4) is proved.

There is no loss of generality in assuming that $f(a) < f(b)$.

Now we shall denote:

$$\mathcal{S} = \big\{I \in \ell(D) \colon I \subset (a, b) \wedge f(I) \subset (-\infty, \alpha)\big\}$$

and

$$\mathcal{S}' = \big\{I \in \ell(D) \colon I \subset (a, b) \wedge f(I) \subset (\alpha, \infty)\big\}\,.$$

Observe that

$$\text{if} \quad I \in \ell(D) \quad \text{and} \quad I \cap (a, b) \neq \emptyset, \qquad \text{then} \quad I \in \mathcal{S} \cup \mathcal{S}'. \tag{5}$$

If $\mathcal{S} = \emptyset$, then put $c = a$. In the opposite case, let $c = \sup\big\{\sup(I) \colon I \in \mathcal{S}\big\}$. It is easy to check that $\emptyset \neq \big\{\sup(I) \colon I \in \mathcal{S}\big\} \subset [a, b] \setminus D$ and so $c \in [a, b] \setminus D$.

Let $E \in \varepsilon(c)$ be a path such that $f\restriction E$ is continuous at $c$. We shall show that

$$f(c) \leq \alpha\,.$$

It is sufficient to consider the case $c > a$ (in the case $c = a$ we have $f(c) < \alpha$). Observe that

$$\text{there exists a sequence } (x_n)_{n=1}^{\infty} \text{ such that } x_n \nearrow c,\ \lim_{n\to\infty} f(x_n) = f(c) \tag{6}$$

and

$$f(x_n) \in (-\infty, \alpha)\,, \qquad \text{for} \quad n = 1, 2, \dots$$

In fact, if $c = \sup(I_0)$ (for some $I_0 \in \mathcal{S}$) then there exists a sequence $(x_n)_{n=1}^{\infty} \subset I_0 \cap E$ such that $x_n \nearrow c$. Then $\lim\limits_{n\to\infty} f(x_n) = f(c)$ and $f(x_n) \in (-\infty, \alpha)$ (for $n = 1, 2, \dots$). In the opposite case there exists a sequence $(I_n)_{n=1}^{\infty} \subset \mathcal{S}$ such that $\sup(I_n) \nearrow c$. Then $I_n \cap E \neq \emptyset$. Let $x_n \in I_n \cap E$ ($n = 1, 2, \dots$). Thus $x_n \nearrow c$, $\lim\limits_{n\to\infty} f(x_n) = f(c)$ and $f(x_n) \in (-\infty, \alpha)$. This ends the proof of the fact that the condition (6) is true.

According to (6) we infer that $f(c) \leq \alpha$, which means that $c < b$. From the definition of the point $c$ and (5) we conclude that $f\big((c, b) \cap D\big) \subset (\alpha, \infty)$. Let $(y_n)_{n=1}^{\infty} \subset E \cap (c, b)$ be a sequence such that $y_n \searrow c$. Then $\lim\limits_{n\to\infty} f(y_n) = f(c)$ and $y_n \in D \cap (c, b)$ (for $n = 1, 2, \dots$) and so $f(y_n) \in (\alpha, \infty)$ and, consequently, $f(c) \geq \alpha$.

The above consideration proves that $c \in f^{-1}(\alpha) \cap (a, b)$, which contradicts (4). This contradiction ends the proof of Theorem 2. $\square$

The above theorem lead us to the question whether an analogous theorem for almost continuous functions (in the Stallings sense) is also true. We can give a positive answer, but our assumptions are a little stronger than the ones in the theorem above.

**Definition 4.** Let $\varepsilon$, $\varepsilon'$ be two systems of paths on $\mathbb{R}$. We say that the system $\varepsilon$ is a *bilateral refinement* of the system $\varepsilon'$ if for each $x \in \mathbb{R}$ and $E \in \varepsilon(x)$ there exists a path $E' \in \varepsilon'(x)$ fulfilling the conditions (a) and (b) of the Definition 3, and, moreover, the systems $\varepsilon$, $\varepsilon'$ fulfill the following condition: (c) if $\inf(C) > -\infty$, then $\inf(C)$ is a right-hand accumulation point of $E$ and if $\sup(C) < \infty$, then $\sup(C)$ is a left-hand accumulation point of $E$, for each nondegenerate component $C$ of $E'$.

Before stating the theorem connected with the almost continuity we shall present two lemmas:

**Lemma 1.** ([9]) *If $X_0$ is a closed subspace of the metric space $X$ and $f \in Y^X$ is an almost continuous function, then $f\restriction X_0$ is an almost continuous function.*

**Lemma 2.** ([7]) *If $f \in \mathbb{R}^{[a,b]}$ is an almost continuous function and $G \subset [a,b] \times \mathbb{R}$ is an open neighbourhood of $f$, then there exists a continuous function $g \in \mathbb{R}^{[a,b]}$ such that $g \subset G$, $g(a) = f(a)$ and $g(b) = f(b)$.*

**Theorem 3.** *Let $D$ be an arbitrary open and dense set in $\mathbb{R}$ and let $\varepsilon$ be a system of paths bilateral refinement of $\varepsilon_D$. Then if $f \restriction D$ is an almost continuous function and $\mathbb{R} \setminus D \subset C_f^{\varepsilon}$, then $f$ is an almost continuous function.*

P r o o f. Let $G \subset \mathbb{R}^2$ be an open set such that $f \subset G$. Let $(p_z)_{z \in \mathbb{Z}}$ be a sequence of elements of $D$ such that $p_z < p_{z+1}$ (for $z \in \mathbb{Z}$) and $\lim_{z \to \infty} p_z = \infty$, $\lim_{z \to -\infty} p_z = -\infty$.

Fix $z \in \mathbb{Z}$. We shall show:

$$\begin{gathered}\text{there exists a continuous function } g_z \in \mathbb{R}^{[p_z, p_{z+1}]} \text{ such that} \\ g_z \subset G,\ g(p_z) = f(p_z) \quad \text{and} \quad g(p_{z+1}) = f(p_{z+1}).\end{gathered} \tag{7}$$

Let $C = [p_z, p_{z+1}] \setminus D$. If $C = \emptyset$, then from Lemmas 1 and 2 we infer that there exists a function $g_z$ fulfilling the above condition.

So, in the further considerations we shall assume that $C \neq \emptyset$. Let $\varepsilon_x$ be a positive number such that $G_x = (x - \varepsilon_x, x + \varepsilon_x) \times (f(x) - \varepsilon_x, f(x) + \varepsilon_x) \subset G$, for $x \in C$.

Now we shall show that there exists a family $\mathcal{J} = \{(t_i, J_i, E_i) \colon i = 1, \dots k\}$ fulfilling the following conditions (for $i, j \in \{1, \dots k\}$):

(i) $J_i$ is an open interval, $t_i \in C \cap J_i$, $E_i \in \mathcal{E}(t_i)$,

(ii) $J_j \not\subset \bigcup_{\substack{i=1 \\ i \neq j}}^{k} J_i$,

(iii) $C \subset \bigcup_{i=1}^{k} J_i$,

(iv) $\mathrm{bd}(J_i) \subset E_i$, $f \restriction \big(\mathrm{cl}(J_i) \cap E_i\big) \subset G_{t_i}$,

(v) either $\mathrm{cl}(J_i) \cap \mathrm{cl}(J_j) = \emptyset$ or $J_i \cap J_j \neq \emptyset$,

(vi) if $J_i \cap J_j \neq \emptyset$ and $\inf(J_i) < \inf(J_j)$, then $t_i > t_j$.

We will show it in three steps.

**The first step.** Let (for $x \in C$) $F_x$ be a set and $\delta_x$ be a real number such that $\delta_x > 0$, $F_x \in \mathcal{E}(x)$, $f\big(F_x \cap (x - \delta_x, x + \delta_x)\big) \subset \big(f(x) - \varepsilon_x, f(x) + \varepsilon_x\big)$. Moreover, let $\tau_x = \min\{\varepsilon_x, \delta_x\}$. Observe that $x$ is a point of bilateral accumulation of $F_x$. So, we can choose elements $l_x \in (x - \tau_x, x) \cap (p_z, p_{z+1}) \cap F_x$ and $r_x \in (x, x + \tau_x) \cap (p_z, p_{z+1}) \cap F_x$. Let $I_x = (l_x, r_x)$. Then $x \in I_x$, $l_x, r_x \in D$, $f \restriction \big(\mathrm{cl}(I_x) \cap F_x\big) \subset G_x$.

The family $\{I_x\colon x \in C\}$ is an open cover of the compact set $C$. So, let $\{I_{x_i}\colon i = 1,\dots,k_0\}$ be a finite subcover of $\{I_x\colon x \in C\}$. We can assume that $I_{x_j} \not\subset \bigcup\limits_{\substack{i=1\\ i\neq j}}^{k_0} I_{x_i}$, for $i,j = 1,\dots,k_0$. Let $\mathcal{I} = \{(x_i, I_{x_i}, F_{x_i})\colon i = 1,\dots,k_0\}$. Observe that the family $\mathcal{I}$ fulfils conditions (i)–(iv).

**The second step.** If the family $\mathcal{I}$ satisfies the condition (v), we put $\mathcal{I}'' = \mathcal{I}$. In the opposite case, there are intervals $I_{x_{i_0}}$, $I_{x_{i_1}}$ (where $i_0, i_1 \in \{1,\dots,k_0\}$) such that $\mathrm{cl}(I_{x_{i_0}}) \cap \mathrm{cl}(I_{x_{i_1}}) \neq \emptyset$ and $I_{x_{i_0}} \cap I_{x_{i_1}} = \emptyset$. Let $p \in \mathrm{cl}(I_{x_{i_0}}) \cap \mathrm{cl}(I_{x_{i_1}})$.

According to (ii), $\inf(I_{x_{i_0}}) \neq \inf(I_{x_{i_1}})$. Without loss of generality we may assume that $\inf(I_{x_{i_0}}) < \inf(I_{x_{i_1}})$.

We have $I_{x_{i_0}} = (l_{x_{i_0}}, r_{x_{i_0}})$, $I_{x_{i_1}} = (l_{x_{i_1}}, r_{x_{i_1}})$ and $l_{x_{i_0}} < r_{x_{i_0}} = p = l_{x_{i_1}} < r_{x_{i_1}}$. Observe that $p = r_{x_{i_0}} \in D$.

Let $(a_p, b_p)$ be a component of $D$ containing $p$. Then $x_{i_0} \le a_p < p < b_p \le x_{i_1}$. Since $\mathcal{E}$ is a bilateral refinement of the system $\mathcal{E}_D$, $a_p$ is a right-hand accumulation point of $F_{x_{i_0}}$ and $b_p$ is a left-hand accumulation point of $F_{x_{i_1}}$. Thus there exist $r'_{x_{i_0}} \in (a_p, p) \cap F_{x_{i_0}}$ and $l'_{x_{i_1}} \in (p, b_p) \cap F_{x_{i_1}}$. Let $l'_{x_{i_0}} = l_{x_{i_0}}$, $r'_{x_{i_1}} = r_{x_{i_1}}$ and $I'_{x_{i_0}} = (l'_{x_{i_0}}, r'_{x_{i_0}})$, $I'_{x_{i_1}} = (l'_{x_{i_1}}, r'_{x_{i_1}})$. Then $x_{i_0} < r'_{x_{i_0}}$, $l'_{x_{i_1}} < x_{i_1}$, thus $x_{i_0} \in I'_{x_{i_0}}$ and $x_{i_1} \in I'_{x_{i_1}}$. If $i \notin \{i_0, i_1\}$ then we put $I'_{x_i} = I_{x_i}$. Let us denote by $T$ the set $\Big\{i \in \{i_0, i_1\}\colon I'_{x_i} \subset \bigcup\limits_{\substack{l=1\\ l\neq i}}^{k_0} I'_{x_l}\Big\}$. Now, let $\mathcal{I}' = \{(x_i, I'_{x_i}, F_{x_i})\colon i \in \{1,\dots,k_0\} \setminus T\}$. The family $\mathcal{I}'$ satisfies the conditions (i)–(iv) and $\mathrm{cl}(I'_{x_{i_0}}) \cap \mathrm{cl}(I'_{x_{i_1}}) = \emptyset$. If $\mathcal{I}'$ does not satisfy (v), we can repeat the second step.

In this way, (after a finite number of steps) we obtain the family $\mathcal{I}'' = \{(w_s, I''_{w_s}, F_{w_s})\colon s = 1,\dots,k_1\}$ (where $k_1 \in \{1,\dots k_0\}$, $w_s \in \{x_i\colon i = 1,\dots,k_0\}$) fulfilling conditions (i)–(v).

**Final step.** If $\mathcal{I}''$ satisfies condition (vi), we put $\mathcal{J} = \mathcal{I}''$. In the opposite case, there are $s_0, s_1 \in \{1,\dots,k_1\}$ such that $I''_{w_{s_0}} \cap I''_{w_{s_1}} \neq \emptyset$, $\inf(I''_{w_{s_0}}) < \inf(I''_{w_{s_1}})$ and $w_{s_0} < w_{s_1}$. Then we have $(w_{s_0}, w_{s_1}) \cap I''_{w_{s_0}} \cap I''_{w_{s_1}} \neq \emptyset$. Since $D$ is a dense set, there exists $p' \in (w_{s_0}, w_{s_1}) \cap I''_{w_{s_0}} \cap I''_{w_{s_1}} \cap D$. Let $(a_{p'}, b_{p'})$ be a component of $D$ containing $p'$. Then $w_{s_0} \le a_{p'} < p' < b_{p'} \le w_{s_1}$. Let us observe that $a_{p'}$ is a right-side accumulation point of $F_{w_{s_0}}$ and $b_{p'}$ is a left-side accumulation point of $F_{w_{s_1}}$.

So, let $r'''_{w_{s_0}} \in (a_{p'}, p') \cap F_{w_{s_0}}$, $l'''_{w_{s_1}} \in (p', b_{p'}) \cap F_{w_{s_1}}$, $l'''_{w_{s_0}} = l''_{w_{s_0}}$, $r'''_{w_{s_1}} = r''_{w_{s_1}}$. Moreover, let $I'''_{w_{s_0}} = \big(l'''_{w_{s_0}}, r'''_{w_{s_0}}\big)$, $I'''_{w_{s_1}} = \big(l'''_{w_{s_1}}, r'''_{w_{s_1}}\big)$.

Observe that $w_{s_0} \in I'''_{w_{s_0}}$, $w_{s_1} \in I'''_{w_{s_1}}$. If $s \notin \{s_0, s_1\}$ then we put $I'''_{w_s} = I''_{w_s}$. Let us denote $S = \Big\{ s \in \{s_0, s_1\} \colon I'''_{x_s} \subset \bigcup\limits_{\substack{l=1\\ l\neq s}}^{k_1} I'''_{x_l} \Big\}$. Now, let $\mathcal{I}''' = \big\{(w_s, I'''_{w_s}, F_{w_s}) \colon s \in \{1, \dots k_1\} \setminus S\big\}$. The family $\mathcal{I}'''$ satisfies conditions (i)–(iv), (v) and $\operatorname{cl}\big(I'''_{w_{s_0}}\big) \cap \operatorname{cl}\big(I'''_{w_{s_1}}\big) = \emptyset$. If $\mathcal{I}'''$ does not satisfy condition (vi), we can repeat this step. In this way (after a finite number of steps) we obtain the family $\mathcal{J} = \big\{(t_i, J_i, E_i) \colon i \le k\big\}$, (where $l \in \{1, \dots, k_1\}$, $t_i \in \{w_s \colon s = 1, \dots, k_1\}$) fulfilling conditions (i)–(vi).

Now, let $\big\{(c_m, d_m) \colon m = 1, \dots, k_2\big\}$ be the family of all components of $\bigcup\limits_{i=1}^{k} J_i$. We may assume that $c_m < c_n$ (for $1 \le m < n \le k_2$). Observe that from (v) we conclude that $d_m < c_{m+1}$, for $m = 1, \dots, k_2$. Let $m \in \{1, \dots, k_2\}$. We shall show that there exists a continuous function $\eta_m \in \mathbb{R}^{[c_m, d_m]}$ such that

$$\eta_m(c_m) = f(c_m), \quad \eta_m(d_m) = f(d_m), \qquad \eta_m \subset G. \tag{8}$$

If there exists $i \in \{1, \dots, k\}$ such that $(c_m, d_m) = J_i$ then condition (iv) implies that $\operatorname{bd}(J_i) \subset E_i$ and $f\restriction\big(\operatorname{cl}(J_i) \cap E_i\big) \subset G_{t_i}$. Let $\eta_m \in \mathbb{R}^{[c_m, d_m]}$ be a linear function such that $\eta_m(c_m) = f(c_m)$, $\eta_m(d_m) = f(d_m)$. Then $\eta_m$ is a continuous function and $\eta_m \subset G_{t_i} \subset G$.

In the opposite case, there are $i_0, i_1 \in \{1, \dots, k\}$ such that $i_0 \ne i_1$, $J_{i_0} \cap J_{i_1} \ne \emptyset$ and $J_{i_0} \cup J_{i_1} \subset (c_m, d_m)$.

In this case we shall show that $J_{i_0} \cup J_{i_1} = (c_m, d_m)$. Without loss of generality we may assume that $\inf\big(J_{i_0}\big) < \inf\big(J_{i_1}\big)$. Then from (vi) we conclude that $t_{i_0} > t_{i_1}$ and also that $t_{i_0} \in J_{i_0} \cap J_{i_1}$. Let $i_2 \in \{1, \dots, k\} \setminus \{i_0, i_1\}$.

If $J_{i_2} \cap J_{i_0} \ne \emptyset$, then from (ii), we have $\inf\big(J_{i_2}\big) < \inf(J_{i_0})$. Then according to (vi), we have $t_{i_2} > t_{i_0}$ and also $t_{i_0} \in J_{i_0} \cap J_{i_1} \cap J_{i_2}$, but from (ii) we deduce that $J_{i_1} \cap J_{i_2} \cap J_{i_0} = \emptyset$. This contradiction proves that $J_{i_2} \cap J_{i_0} = \emptyset$. Similarly we obtain $J_{i_2} \cap J_{i_1} = \emptyset$. This means that $J_{i_0} \cup J_{i_1} = (c_m, d_m)$.

It is easy to see that $c_m = \hat{l}_0 < \hat{l}_1 < t_{i_1} < t_{i_0} < \hat{r}_0 < \hat{r}_1 = d_m$, where $J_{i_0} = \big(\hat{l}_0, \hat{r}_0\big)$, $J_{i_1} = \big(\hat{l}_1, \hat{r}_1\big)$. Let $u \in \big(t_{i_1}, t_{i_0}\big) \cap D$ and $(a_u, b_u)$ be a component of the set $D$ containing $u$. Since $a_u$ is a right-hand accumulation point of $E_{i_0}$ and $b_u$ is a left-hand accumulation point of $E_{i_1}$, there are $u_m \in (a_u, u) \cap E_{i_0}$ and $v_m \in (u, b_u) \cap E_{i_1}$. From (iv) we deduce that $\big(u_m, f(u_m)\big) \in G_{t_{i_0}}$, $\big(v_m, f(v_m)\big) \in G_{t_{i_1}}$. Let $h_0 \in \mathbb{R}^{[c_m, u_m]}$ be a linear function such that $h_0(c_m) = f(c_m)$, $h_0(u_m) = f(u_m)$. Then $h_0$ is a continuous function and $h_0 \subset G_{t_{i_0}} \subset G$. Similarly, let $h_1 \in \mathbb{R}^{[v_m,d_m]}$ be a linear function such that $h_1(v_m) = f(v_m)$, $h_1(d_m) = f(d_m)$. Then $h_1$ is a continuous function, $h_1 \subset G_{t_{i_1}} \subset G$. Since $[u_m, v_m] \subset D$ (according to Lemmas 1 and 2), there exists a continuous function $h_2 \in \mathbb{R}^{[u_m,v_m]}$ such that $h_2 \subset G$, $h_2(u_m) = f(u_m)$, $h_2(v_m) = f(v_m)$. Let $\eta_m = \bigcup_{i=1}^{3} h_i$. Then $\eta_m \in \mathbb{R}^{[c_m,d_m]}$, $\eta_m(c_m) = f(c_m)$, $\eta_m(d_m) = f(d_m)$, $\eta_m \subset G$. The proof of the condition (8) is finished.

The intervals $[p_z, c_1], [d_1, c_2], \dots, [d_{k_2}, p_{z+1}]$ are subsets of $D$. So, there exist functions $\xi_0 \in \mathbb{R}^{[p_z,c_1]}$, $\xi_{k_2} \in \mathbb{R}^{[d_{k_2},p_{z+1}]}$, $\xi_i \in \mathbb{R}^{[d_i,c_{i+1}]}$, $(0 < i < k_2)$ such that $\xi_i$ is a continuous function and $\xi_i \subset G$ $(i \in \{0, \dots, k_2\})$, and $\xi_0(p_z) = f(p_z)$, $\xi_0(c_1) = f(c_1)$, $\xi_{k_2}(d_{k_2}) = f(d_{k_2})$, $\xi_{k_2}(p_{z+1}) = f(p_{z+1})$, $\xi_i(d_i) = f(d_i)$, $\xi_i(c_{i+1}) = f(c_{i+1})$, (for $0 < i < k_2$). Let $\mathcal{F} = \{\eta_m : m = 1, \dots, k_2\} \cup \{\xi_i : i = 0, 1, \dots, k_2\}$ and $g_z = \bigcup \mathcal{F}$. Then $g_z \in \mathbb{R}^{[p_z,p_{z+1}]}$ is a continuous function and $g_z \subset G$, $g(p_z) = f(p_z)$, $g(p_{z+1}) = f(p_{z+1})$. The assertion (7) is proved.

Finally, we take $g = \bigcup_{z\in\mathbb{Z}} g_z$. Then $g \in \mathbb{R}^{\mathbb{R}}$ is a continuous function such that $g \subset G$. □

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Received December 14, 2000

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