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# UNBIASED ESTIMATION OF LINEAR FUNCTION OF VARIANCE COMPONENTS IN BALANCED TWO-WAY RANDOM EFFECTS ANOVA MODEL

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*Dedicated to Lubomír Kubáček on the occasion of his 70th birthday*

ABSTRACT. In this paper a balanced two-way random effects ANOVA model is investigated. There can be a correlation into and between classes of observations. This model is treated as a linear model with unknown variance components. Sometimes the standard quadratic forms from fixed effects ANOVA are used as estimators of linear functions of unknown variance components, but except for a few special situations these quadratic forms are biased estimators. The aim of the paper is to give an overview of all possibles situations (models) taking into consideration a form of the criterion matrix for an unbiased estimation and then to investigate the class of unbiased linear quadratic estimators of linear functions of variance components for all possible situations. In the end, there is given a method which in case of existence of independent replications of the model enables to unbiasedly estimate each variance component in any situation.

## 1. Introduction

Let us consider a balanced two-way ANOVA model with random effects, where the factor $A$ has $q$ levels, the factor $B$ has $p$ levels, and we have $c$ observations in each group.

Denoting the random levels of the factor $A$ as $A_1, \dots, A_q$ and the random levels of the factor $B$ as $B_1, \dots, B_p$, we can arrange the observations $Y_{ijk}$, $i = 1, 2, \dots, q$, $j = 1, 2, \dots, p$, $k = 1, 2, \dots, c$ in the following table:

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| B / A | $B_1$ | $B_2$ | $\dots$ | $B_3$ |
|---|---|---|---|---|
| $A_1$ | $Y_{111}, Y_{112}, \dots, Y_{11c}$ | $Y_{121}, Y_{122}, \dots, Y_{12c}$ | $\dots$ | $Y_{1p1}, Y_{1p2}, \dots, Y_{1pc}$ |
| $A_2$ | $Y_{211}, Y_{212}, \dots, Y_{21c}$ | $Y_{221}, Y_{222}, \dots, Y_{22c}$ | $\dots$ | $Y_{2p1}, Y_{2p2}, \dots, Y_{2pc}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $A_q$ | $Y_{q11}, Y_{q12}, \dots, Y_{q1c}$ | $Y_{q21}, Y_{q22}, \dots, Y_{q2c}$ | $\dots$ | $Y_{qp1}, Y_{qp2}, \dots, Y_{qpc}$ |

Let us assume that

$$\begin{aligned} \mathrm{E}(Y_{ijk}) &= \mu_{ij}\,, \\ \mathrm{var}(Y_{ijk}) &= \sigma^2\,, \\ \mathrm{cov}(Y_{ijk}, Y_{ijm}) &= \rho_0\sigma^2\,, \quad k \neq m\,, \\ \mathrm{cov}(Y_{ijk}, Y_{imr}) &= \rho_1\sigma^2, \quad j \neq m\,, \\ \mathrm{cov}(Y_{ijk}, Y_{mjr}) &= \rho_2\sigma^2, \quad i \neq m\,, \\ \mathrm{cov}(Y_{ijk}, Y_{msr}) &= \rho_3\sigma^2, \quad i \neq m\,,\ j \neq s\,. \end{aligned}$$

Let

$$Y_{ijk} = h_{ij} + \varepsilon_{ijk}\,,$$

where $h_{ij}$ is the random effect of the combination of levels $A_i$ and $B_j$, and $\varepsilon_{ijk}$ are experimental errors which occur during observations of the random variable $h_{ij}$.

Denoting

$$\begin{aligned} \mathbf{Y}_{ij} &= (Y_{ij1}, Y_{ij2}, \dots, Y_{ijc})'\,, \qquad i = 1, \dots, q\,, \quad j = 1, \dots, p\,, \\ \mathbf{Y} &= (\mathbf{Y}'_{11}, \mathbf{Y}'_{12}, \dots, \mathbf{Y}'_{1p}, \mathbf{Y}'_{21}, \mathbf{Y}'_{22}, \dots, \mathbf{Y}'_{2p}, \dots, \mathbf{Y}'_{q1}, \mathbf{Y}'_{q2}, \dots, \mathbf{Y}'_{qp})', \\ \mathbf{h} &= (h_{11}\mathbf{1}'_c, \dots, h_{1p}\mathbf{1}'_c, h_{21}\mathbf{1}'_c, \dots, h_{2p}\mathbf{1}'_c, \dots, h_{q1}\mathbf{1}'_c, \dots, h_{qp}\mathbf{1}'_c)', \\ \boldsymbol{\varepsilon} &= (\varepsilon_{111}, \dots, \varepsilon_{11c}, \varepsilon_{121}, \dots, \varepsilon_{12c}, \dots, \varepsilon_{qp1}, \dots, \varepsilon_{qpc})', \end{aligned}$$

the situation described above can be expressed as the model

$$\mathbf{Y} = \mathbf{h} + \varepsilon \tag{1}$$

or the model

$$\left(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{5} \vartheta_i \mathbf{V}_i\right), \tag{2}$$

where

$$\mathrm{E}(\mathbf{Y}) = \mathrm{E}(\mathbf{h}) = \mathbf{X}\boldsymbol{\beta}\,, \tag{3}$$

$$\mathrm{E}(\varepsilon) = \mathbf{0}\,, \tag{4}$$

$$\mathrm{Var}(\mathbf{Y}) = \mathrm{Var}(\mathbf{h} + \varepsilon) = \mathbf{\Sigma} = \sum_{i=1}^{5} \vartheta_i \mathbf{V}_i\,, \tag{5}$$

$$\begin{aligned}
\vartheta_1 &= \sigma^2,\\
\vartheta_2 &= \rho_0 \sigma^2,\\
\vartheta_3 &= \rho_1 \sigma^2,\\
\vartheta_4 &= \rho_2 \sigma^2,\\
\vartheta_5 &= \rho_3 \sigma^2,\\
\mathbf{V}_1 &= \mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes \mathbf{I}_{cc}\,,\\
\mathbf{V}_2 &= \mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes (\mathbf{1}_c \mathbf{1}_c' - \mathbf{I}_{cc})\,,\\
\mathbf{V}_3 &= \mathbf{I}_{qq} \otimes (\mathbf{1}_p \mathbf{1}_p' - \mathbf{I}_{pp}) \otimes \mathbf{1}_c \mathbf{1}_c'\,,\\
\mathbf{V}_4 &= (\mathbf{1}_q \mathbf{1}_q' - \mathbf{I}_{qq}) \otimes \mathbf{I}_{pp} \otimes \mathbf{1}_c \mathbf{1}_c'\,,\\
\mathbf{V}_5 &= (\mathbf{1}_q \mathbf{1}_q' - \mathbf{I}_{qq}) \otimes (\mathbf{1}_p \mathbf{1}_p' - \mathbf{I}_{pp}) \otimes \mathbf{1}_c \mathbf{1}_c'\,.
\end{aligned}$$

The symbol $\otimes$ denotes Kronecker (tensor) product of matrices, $\mathbf{I}_{nn}$ is an $n \times n$ identity matrix and $\mathbf{1}_n$ is an $n$-dimensional vector with all components equal to 1.

**Remark 1.1.** The form of the design matrix $\mathbf{X}$ and of the vector $\boldsymbol{\beta}$ depends on a priori information about mean values of the random effects of the factors $A$ a $B$.

(1) If the mean values $\mu_{ij}$ of the random effects $h_{ij}$ are different numbers, then

$$\begin{aligned}
\mathbf{X} &= [\mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes \mathbf{1}_c]\,,\\
\boldsymbol{\beta} &= (\mu_{11}, \mu_{12}, \dots, \mu_{1p}, \mu_{21}, \dots, \mu_{2p}, \dots, \mu_{q1}, \dots, \mu_{qp})' \in \mathbb{R}^{q \cdot p}\,.
\end{aligned}$$

(2) If $\mu_{ij} = \mu_{\bullet j}$, $i = 1, \dots, q$, $j = 1, \dots, p$ (i.e., the mean of $h_{ij}$ depends only on index $j$), then

$$\begin{aligned}
\mathbf{X} &= [\mathbf{1}_q \otimes \mathbf{I}_{pp} \otimes \mathbf{1}_c]\,,\\
\boldsymbol{\beta} &= (\mu_{\bullet 1}, \mu_{\bullet 2}, \dots, \mu_{\bullet p})' \in \mathbb{R}^{p}\,.
\end{aligned}$$

(3) If $\mu_{ij} = \mu_{i\bullet}$, $i = 1, \dots, q$, $j = 1, \dots, p$, (i.e., the mean of $h_{ij}$ depends only on index $i$), then

$$\begin{aligned}
\mathbf{X} &= [\mathbf{I}_{qq} \otimes \mathbf{1}_p \otimes \mathbf{1}_c]\,,\\
\boldsymbol{\beta} &= (\mu_{1\bullet}, \mu_{2\bullet}, \dots, \mu_{q\bullet})' \in \mathbb{R}^{q}\,.
\end{aligned}$$

(4) If $\mu_{ij} = \mu$, $i = 1, \dots, q$, $j = 1, \dots, p$, then

$$\mathbf{X} = [\mathbf{1}_q \otimes \mathbf{1}_p \otimes \mathbf{1}_c],$$
$$\boldsymbol{\beta} = \mu \in \mathbb{R}.$$

## 2. Notations, definitions and auxiliary statements

**Definition 2.1.** Let $\mathbf{Y}$ be an $n$-dimensional random vector. The model

$$\left(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{s} \vartheta_i \mathbf{V}_i\right), \tag{6}$$

where $\boldsymbol{\beta} \in \mathbb{R}^k$, $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_s)' \in \underline{\boldsymbol{\vartheta}} \subset \mathbb{R}^s$, is said to be the *model with variance components*, if it has following properties:

(1) the vector parameter $\boldsymbol{\theta}$ of the model is divided into two subvectors $\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$. The parametric space $\underline{\boldsymbol{\Theta}}$ is given as the Cartesian product $\mathbb{R}^k \times \underline{\boldsymbol{\vartheta}}$, where $\underline{\boldsymbol{\vartheta}} \subset \mathbb{R}^s$ contains an open sphere in $\mathbb{R}^s$,

(2)
$$\mathrm{E}_{\boldsymbol{\beta}}(\mathbf{Y}) = \int_{\mathbb{R}^n} \mathbf{y}\, dF(\mathbf{y}, \boldsymbol{\beta}, \boldsymbol{\vartheta}) = \mathbf{X}\boldsymbol{\beta}, \qquad \forall \boldsymbol{\beta} \in \mathbb{R}^k, \tag{7}$$

where $\mathbf{X}$ is a given $n \times k$ design matrix,

(3)
$$\mathrm{Var}_{\boldsymbol{\vartheta}}(\mathbf{Y}) = \int_{\mathbb{R}^s} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\, dF(\mathbf{y}, \boldsymbol{\beta}, \boldsymbol{\vartheta}) = \sum_{i=1}^{s} \vartheta_i \mathbf{V}_i, \tag{8}$$

for $\forall \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, where $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_s$ are given $n \times n$ symmetric matrices. The covariance matrix $\mathrm{Var}_{\boldsymbol{\vartheta}}(\mathbf{Y})$ is not a function of the vector $\boldsymbol{\beta}$.

**Definition 2.2.** Let $\mathbf{Y}$ be an $n$-dimensional random vector. The class of the statistics

$$\mathcal{A} = \left\{\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y} \colon \mathbf{a} \in \mathbb{R}^n,\ \mathbf{A} \text{ is an } n \times n \text{ symmetric matrix}\right\} \tag{9}$$

is said to be the *class of linear quadratic estimators.*

**Theorem 2.3.** *The linear function $f(\boldsymbol{\vartheta}) = g'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, is unbiasedly estimable in the class $\mathcal{A}$ of the linear quadratic estimators if and only if $g$ is from the column space $\mathcal{M}(\mathbf{C})$ of the matrix $\mathbf{C}$, where the entries of the matrix $\mathbf{C}$ are given as*

$$\{\mathbf{C}\}_{ij} = \mathrm{Tr}(\mathbf{V}_i \mathbf{V}_j - \mathbf{P}_{\mathbf{X}} \mathbf{V}_i \mathbf{P}_{\mathbf{X}} \mathbf{V}_j), \qquad i, j = 1, \dots, s, \tag{10}$$

*and* $\mathbf{P_X} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ *denotes the projection matrix onto the column space* $\mathcal{M}(\mathbf{X})$ *of the matrix* $\mathbf{X}$.

Proof. Cf. [2] p. 21. □

In case the observation vector $\mathbf{Y}$ is normally distributed, a stronger theorem holds.

**THEOREM 2.4.** *Let the vector* $\mathbf{Y}$ *be normally distributed, i.e.,* $\mathbf{Y} \sim N_n\left(\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{s} \vartheta_i \mathbf{V}_i\right)$, $\boldsymbol{\beta} \in \mathbb{R}^k$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$. *The linear function* $f(\boldsymbol{\vartheta}) = g'\boldsymbol{\vartheta}$ *is unbiasedly estimable if and only if* $g \in \mathcal{M}(\mathbf{C})$.

Proof. Cf. [2] p. 23. □

## 3. Estimation of variance components

In this section we will search for the class of the functions $f(\boldsymbol{\vartheta}) = g'\boldsymbol{\vartheta}$ unbiasedly estimable in the class $\mathcal{A}$ of linear quadratic estimators.

According Theorem 2.3, the linear function $g'\boldsymbol{\vartheta}$ is unbiasedly estimable in the class $\mathcal{A}$ if and only if $g$ is from a column space of the matrix $\mathbf{C}$, which is determined by its entries (10). For investigating types of functions $g$ from a column space $\mathcal{M}(\mathbf{C})$ we must determine facts which influence the form of the matrix $\mathbf{C}$.

It is seen from (10) that the form and the dimension of the matrix $\mathbf{C}$ are influenced by

(1) the form of the design matrix $\mathbf{X}$ (because $\mathbf{P_X} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, all the possible forms of $\mathbf{X}$ are given in Remark 1.1),

(2) the fact that some of variance components $\vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5$ could be zero (if $\vartheta_i = 0$, $i \in \{2,3,4,5\}$, then the covariance matrix $\boldsymbol{\Sigma}$ is changed in the model (2) — the matrix $\mathbf{V}_i$ is omitted — and the dimension of the matrix $\mathbf{C}$ is decreased).

Combinations of cases (1) a (2) determine possible forms of the matrix $\mathbf{C}$ and they are arranged in the following table:

In the next we will investigate a base (generator) of the column space of the matrix $\mathbf{C}$ for all situations given in the table. This base will determine the class of linear functions of variance components unbiasedly estimable in the class $\mathcal{A}$.

The method of analysis is universal for all models of given situations, only the design matrix $\mathbf{X}$ and the covariance matrix $\boldsymbol{\Sigma}$ with variance components and variance matrices $\mathbf{V}_i$, $i \in \{1,2,3,4,5\}$, have to be changed.

| Var. comp. / $\mathbf{X}$ | $\vartheta_i \neq 0$, $i = 1,2,\dots,5$ | $\vartheta_2 = 0$, $\vartheta_i \neq 0$, $i = 1,3,4,5$ | $\vartheta_3 = 0$, $\vartheta_i \neq 0$, $i = 1,2,4,5$ | $\vartheta_4 = 0$, $\vartheta_i \neq 0$, $i = 1,2,3,5$ | $\vartheta_5 = 0$, $\vartheta_i \neq 0$, $i = 1,2,3,4$ |
|---|---|---|---|---|---|
| $\mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes \mathbf{1}_c$ | I.A | I.B | I.C | I.D | I.E |
| $\mathbf{1}_q \otimes \mathbf{I}_{pp} \otimes \mathbf{1}_c$ | II.A | II.B | II.C | II.D | II.E |
| $\mathbf{I}_{qq} \otimes \mathbf{1}_p \otimes \mathbf{1}_c$ | III.A | III.B | III.C | III.D | III.E |
| $\mathbf{1}_q \otimes \mathbf{1}_p \otimes \mathbf{1}_c$ | IV.A | IV.B | IV.C | IV.D | IV.E |

**Theorem 3.1.** *The column space $\mathcal{M}(\mathbf{C})$ of the criterion matrix $\mathbf{C}$*

(1) *is* $\{g \in \mathbb{R}^5 \colon g = (a,-a,0,0,0)',\ a \in \mathbb{R}\}$ *in the situation I. A,*

(2) *is* $\{g \in \mathbb{R}^4 \colon g = (a,0,0,0)',\ a \in \mathbb{R}\}$ *in the situation I.B,*

(3) *is* $\{g \in \mathbb{R}^4 \colon g = (a,-a,0,0)',\ a \in \mathbb{R}\}$ *in the situations I.C, I.D and I.E,*

(4) *is* $\{g \in \mathbb{R}^5 \colon g = (a,b,d,-(a+b),-d)',\ a,b,d \in \mathbb{R}\}$ *in the situation II.A,*

(5) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,d,-b)',\ a,b,d \in \mathbb{R}\}$ *in the situation II.B,*

(6) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,-(a+b),d)',\ a,b,d \in \mathbb{R}\}$ *in the situation II.C,*

(7) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,d,-d)',\ a,b,d \in \mathbb{R}\}$ *in the situation II.D,*

(8) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,d,-(a+b))',\ a,b,d \in \mathbb{R}\}$ *in the situation II.E,*

(9) *is* $\{g \in \mathbb{R}^5 \colon g = (a,b,-(a+b),d,-d)',\ a,b,d \in \mathbb{R}\}$ *in the situation III.A,*

(10) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,d,-d)',\ a,b,d \in \mathbb{R}\}$ *in the situation III.B,*

(11) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,d,-d)',\ a,b,d \in \mathbb{R}\}$ *in the situation III.C,*

(12) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,-(a+b),d)',\ a,b,d \in \mathbb{R}\}$ *in the situation III.D,*

(13) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,-(a+b),d)',\ a,b,d \in \mathbb{R}\}$ *in the situation III.E,*

(14) *is* $\{g \in \mathbb{R}^5 \colon g = (a,b,d,e,-(a+b+d+e))',\ a,b,d,e \in \mathbb{R}\}$ *in the situation IV.A,*

(15) *is* $\{g \in \mathbb{R}^4 \colon g = (a,b,d,e)',\ a,b,d,e \in \mathbb{R}\}$ *in the situations IV.B, IV.C, IV.D and IV.E.*

Proof.

(1) Let us consider the model corresponding to the situation I.A. Then the projection matrix $\mathbf{P}_{\mathbf{X}}$ has the form

$$\begin{aligned}\mathbf{P}_{\mathbf{X}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' &= \left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}_c\right]\times\\ &\quad\times\left\{\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}'_c\right]\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}_c\right]\right\}^{-1}\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}'_c\right]\\ &= \left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}_c\right]\left\{\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes c\right]\right\}^{-1}\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}'_c\right]\\ &= \left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}_c\right]\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\frac{1}{c}\right]\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{1}'_c\right]\\ &= \left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\frac{\mathbf{1}_c\mathbf{1}'_c}{c}\right].\end{aligned}$$

Using Theorem 2.3 we obtain the element $\{\mathbf{C}\}_{11}$ of the matrix $\mathbf{C}$

$$\begin{aligned}\{\mathbf{C}\}_{11} &= \mathrm{Tr}\{\mathbf{V}_1\mathbf{V}_1 - \mathbf{P}_{\mathbf{X}}\mathbf{V}_1\mathbf{P}_{\mathbf{X}}\mathbf{V}_1\}\\ &= \mathrm{Tr}\left\{\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{I}_{cc}\right]\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{I}_{cc}\right]\right\}\\ &\quad - \mathrm{Tr}\left\{\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\frac{\mathbf{1}_c\mathbf{1}'_c}{c}\right]\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{I}_{cc}\right]\right.\\ &\qquad\qquad\left.\times\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\frac{\mathbf{1}_c\mathbf{1}'_c}{c}\right]\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{I}_{cc}\right]\right\}\\ &= \mathrm{Tr}\left\{\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\mathbf{I}_{cc}\right]\right\} - \mathrm{Tr}\left\{\left[\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes\frac{\mathbf{1}_c\mathbf{1}'_c}{c}\right]\right\} = qp(c-1)\,.\end{aligned}$$

The remaining elements of the matrix $\mathbf{C}$ can be obtained analogously, thus we have

$$\mathbf{C} = \begin{pmatrix} qp(c-1) & -qp(c-1) & 0 & 0 & 0\\ -qp(c-1) & qp(c-1) & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}. \tag{11}$$

It is evident that the column space of the matrix $\mathbf{C}$ is

$$\mathcal{M}(\mathbf{C}) = \left\{g\in\mathbb{R}^5\colon\, g = (a,-a,0,0,0)',\ a\in\mathbb{R}\right\}.$$

(2) In the situation II.B the type of the matrix $\mathbf{C}$ is $4\times 4$. By Theorem 2.3 we obtain

$$\mathbf{C} = \begin{pmatrix} qp(c-1) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{12}$$

It is evident again that

$$\mathcal{M}(\mathbf{C}) = \left\{g\in\mathbb{R}^4\colon\, g = (a,0,0,0)',\ a\in\mathbb{R}\right\}.$$

(3) The matrix $\mathbf{C}$ corresponding to the situations I.C, I.D and I.E has the same form

$$\mathbf{C} = \begin{pmatrix} qp(c-1) & -qp(c-1) & 0 & 0 \\ -qp(c-1) & qp(c-1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

The column space of this matrix is

$$\mathcal{M}(\mathbf{C}) = \{g \in \mathbb{R}^4 \colon g = (a, -a, 0, 0)',\ a \in \mathbb{R}\}.$$

(4) In the situation II.A the matrix $\mathbf{C}$ has the form

$$\begin{pmatrix} (qc-1)p & -p(c-1) & 0 & -q(c-1) & 0 \\ -p(c-1) & (qc-c+1)p(c-1) & 0 & -(q-1)p(c-1)c & 0 \\ 0 & 0 & (q-1)(p-1)pc^2 & 0 & -(q-1)(p-1)pc^2 \\ -(q-1)pc & -(q-1)p(c-1)c & 0 & (q-1)pc^2 & 0 \\ 0 & 0 & -(q-1)(p-1)pc^2 & 0 & (q-1)(p-1)pc^2 \end{pmatrix}.$$

Because only the column space of the matrix $\mathbf{C}$ is in interest, it will be more advantageous to work with the matrix $\widetilde{\mathbf{C}}$, which can be created from the matrix $\mathbf{C}$ by the transformation $\mathbf{T}$, i.e., $\widetilde{\mathbf{C}} = \mathbf{C}T$, where

$$\mathbf{T} = \begin{pmatrix} \frac{1}{p} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{p(c-1)} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(q-1)(p-1)pc^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{(q-1)pc} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{(q-1)(p-1)pc^2} \end{pmatrix}. \tag{14}$$

The matrix $\widetilde{\mathbf{C}}$ has the same column space as the matrix $\mathbf{C}$ ($\mathcal{M}(\widetilde{\mathbf{C}}) = \mathcal{M}(\mathbf{C})$), but it has a more simple form

$$\widetilde{\mathbf{C}} = \begin{pmatrix} qc-1 & -1 & 0 & -1 & 0 \\ -(c-1) & qc-c+1 & 0 & -(c-1) & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -(q-1)c & -(q-1)c & 0 & c & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}. \tag{15}$$

Denoting the $i$th column of the matrix $\widetilde{\mathbf{C}}$ as $\mathbf{s}^i$, $i = 1, 2, \dots, 5$, it follows from (15) that the column space generated by the columns $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^4$ is vertical on the column space generated by the columns $\mathbf{s}^3$ and $\mathbf{s}^5$, i.e.,

$$\mathcal{M}(\mathbf{s}^1, \mathbf{s}^2, \mathbf{s}^4) \perp \mathcal{M}(\mathbf{s}^3, \mathbf{s}^5).$$

Then

$$\mathcal{M}(\widetilde{\mathbf{C}}) = \mathcal{M}(\mathbf{s}^1, \mathbf{s}^2, \mathbf{s}^4) \oplus \mathcal{M}(\mathbf{s}^3, \mathbf{s}^5)$$

holds, the symbol $\oplus$ is the direct sum of spaces.

That is why we can investigate the spaces $\mathcal{M}(\mathbf{s}^1,\mathbf{s}^2,\mathbf{s}^4)$ and $\mathcal{M}(\mathbf{s}^3,\mathbf{s}^5)$ separately and the column space of the matrix $\mathbf{C}$ will be the span of these two spaces.

First we will investigate the space $\mathcal{M}(\mathbf{s}^1,\mathbf{s}^2,\mathbf{s}^4)$. It holds

$$\begin{aligned}\mathcal{M}(\mathbf{s}^1,\mathbf{s}^2,\mathbf{s}^4) = \big\{g \in \mathbb{R}^5\colon\ & g=(a,b,0,d,0)',\ \text{where } a,b,d \text{ are real numbers}\\ & \text{such that numbers } u,v,z\in\mathbb{R} \text{ possessing the property}\\ & g = u\mathbf{s}^1 + v\mathbf{s}^2 + z\mathbf{s}^4 \text{ exist}\big\}.\end{aligned}$$

Now we must determine the numbers $a,b,d$ such that the system

$$u\cdot\mathbf{s}^1 + v\cdot\mathbf{s}^2 + z\cdot\mathbf{s}^4 = \begin{pmatrix} a\\ b\\ 0\\ d\\ 0\end{pmatrix} \tag{16}$$

has a solution. We can express (16) as

$$\begin{aligned} (qc-1)u - v \qquad\qquad\quad & - z &= a\\ -(c-1)u + (qc-c+1)v & -(c-1)z &= b\\ -(q-1)cu - (q-1)cv \quad & + cz &= d\end{aligned}$$

The solution of this system will exist if

$$d = -(a+b).$$

holds. Then we have

$$\mathcal{M}(\mathbf{s}^1,\mathbf{s}^2,\mathbf{s}^4) = \big\{g\in\mathbb{R}^5\colon\ g = \big(a,b,0,-(a+b),0\big)',\ a,b\in\mathbb{R}\big\}.$$

Further it is evident that the space generated by the vectors $\mathbf{s}^3$ and $\mathbf{s}^5$ is

$$\mathcal{M}(\mathbf{s}^3,\mathbf{s}^5) = \big\{g\in\mathbb{R}^5\colon\ g=(0,0,d,0,-d)',\ d\in\mathbb{R}\big\}.$$

The column space of the matrix $\mathbf{C}$ is

$$\begin{aligned}\mathcal{M}(\mathbf{C}) = \mathcal{M}(\widetilde{\mathbf{C}}) &= \mathcal{M}(\mathbf{s}^1,\mathbf{s}^2,\mathbf{s}^4)\oplus\mathcal{M}(\mathbf{s}^3,\mathbf{s}^5)\\ &= \big\{g\in\mathbb{R}^5\colon\ g = g_1+g_2,\ g_1\in\mathcal{M}(\mathbf{s}^1,\mathbf{s}^2,\mathbf{s}^4),\ g_2\in\mathcal{M}(\mathbf{s}^3,\mathbf{s}^5)\big\}\\ &= \big\{g\in\mathbb{R}^5\colon\ g=\big(a,b,d,-(a+b),-d\big)',\ a,b,d\in\mathbb{R}\big\}.\end{aligned}$$

(5) In the situation II.B the matrix $\mathbf{C}$ is

$$\begin{pmatrix} (qc-1)p & 0 & -(q-1)c & 0\\ 0 & (q-1)(p-1)pc^2 & 0 & -(q-1)(p-1)pc^2\\ -(q-1)pc & 0 & (q-1)pc^2 & 0\\ 0 & -(q-1)(p-1)pc^2 & 0 & (q-1)(p-1)pc^2\end{pmatrix}.$$

This matrix can be transformed similarly as in (4.) into the matrix $\widetilde{\mathbf{C}}$ with the same column space and a more simple form.

$$\widetilde{\mathbf{C}} = \begin{pmatrix} qc-1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -(q-1)c & 0 & c & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{17}$$

Denoting the $i$th column of the matrix $\widetilde{\mathbf{C}}$ as $\mathbf{s}^i$, $i = 1,2,3,4$, we can write

$$\mathcal{M}(\widetilde{\mathbf{C}}) = \mathcal{M}(\mathbf{s}^1, \mathbf{s}^3) \oplus \mathcal{M}(\mathbf{s}^2, \mathbf{s}^4).$$

Further

$$\begin{aligned} \mathcal{M}(\mathbf{s}^1, \mathbf{s}^3) = \big\{ g \in \mathbb{R}^4 \colon g = (a,0,b,0)', \ &\text{where } a, b \text{ are real numbers} \\ &\text{such that the numbers } u, v \in \mathbb{R} \text{ possessing the property} \\ &g = u\mathbf{s}^1 + v\mathbf{s}^3 \text{ exist} \big\}. \end{aligned}$$

We will investigate again when the system

$$u \cdot \mathbf{s}^1 + v \cdot \mathbf{s}^3 = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \tag{18}$$

has a solution for numbers $a, b$. The system of equations

$$\begin{aligned} (qc-1)u - v \ &= a, \\ -(q-1)cu - cv &= b, \end{aligned}$$

has a solution for arbitrary numbers $a, b$. Then

$$\mathcal{M}(\mathbf{s}^1, \mathbf{s}^3) = \big\{ g \in \mathbb{R}^4 \colon g = (a,0,b,0)', \ a, b \in \mathbb{R} \big\}.$$

Further

$$\mathcal{M}(\mathbf{s}^2, \mathbf{s}^4) = \big\{ g \in \mathbb{R}^4 \colon g = (0,d,0,-d)', \ d \in \mathbb{R} \big\}.$$

The column space $\mathcal{M}(\mathbf{C})$ has the form

$$\begin{aligned} \mathcal{M}(\mathbf{C}) = \mathcal{M}(\widetilde{\mathbf{C}}) &= \mathcal{M}(\mathbf{s}^1, \mathbf{s}^3) \oplus \mathcal{M}(\mathbf{s}^2, \mathbf{s}^4) \\ &= \big\{ g \in \mathbb{R}^4 \colon g = g_1 + g_2, \ g_1 \in \mathcal{M}(\mathbf{s}^1, \mathbf{s}^3), \ g_2 \in \mathcal{M}(\mathbf{s}^2, \mathbf{s}^4) \big\} \\ &= \big\{ g \in \mathbb{R}^4 \colon g = (a,b,d,-b)', \ a, b, d \in \mathbb{R} \big\}. \end{aligned}$$

The statements (6)–(15) can be proved analogously as the statements (1)–(5). □

**Remark 3.2.** The class of linear functions of variance components unbiasedly estimable in the class of linear quadratic estimators is

$$\big\{ f(\boldsymbol{\vartheta}) = g'\boldsymbol{\vartheta}, \ g \in \mathcal{M}(\mathbf{C}) \big\},$$

$\mathcal{M}(\mathbf{C})$ is given in Theorem 3.1.

## 4. Replicated model

In the preceding sections we showed that except for the situations IV.B–IV.E we cannot unbiasedly estimate all variance components in the class $\mathcal{A}$ and therefore neither an arbitrary linear function of these variance components. In this section the theory will be given which enables us to unbiasedly estimate the unknown variance components in all situations under certain circumstances.

Let us consider that $m$ independent replications $\mathbf{Y}^1, \dots, \mathbf{Y}^m$ of the linear model (2) exist. The model

$$\left(\underline{\mathbf{Y}},\ \mathbf{1}_m \otimes \mathbf{X}\boldsymbol{\beta}, \mathbf{I}_{mm} \otimes \boldsymbol{\Sigma}\right), \tag{19}$$

where

$$\underline{\mathbf{Y}} = \left(\mathbf{Y}^{1\prime}, \mathbf{Y}^{2\prime}, \dots, \mathbf{Y}^{m\prime}\right)',$$

is said to be an *$m$-replicated linear model.*

Let us denote

$$\overline{\mathbf{Y}} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{Y}^i = \left[\frac{\mathbf{1}'_m}{m} \otimes \mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes \mathbf{I}_{cc}\right]\underline{\mathbf{Y}}\,,$$

$$\mathbf{S} = \frac{1}{m-1}\sum_{i=1}^{m}(\mathbf{Y}^i - \overline{\mathbf{Y}})(\mathbf{Y}^i - \overline{\mathbf{Y}})'.$$

**Theorem 4.1.** *Within replicated linear model* (19) *a linear function* $f(\boldsymbol{\vartheta}) = g'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, *is unbiasedly estimable in the class* $\mathcal{A}$ *if and only if*

$$g \in \mathcal{M}\left[(m-1)\mathbf{C}_0 + \mathbf{C}\right],$$

*where*

$$\{\mathbf{C}_0\}_{ij} = \operatorname{Tr}\{\mathbf{V}_i\mathbf{V}_j\}\,, \qquad i,j = 1,2,\dots,s\,, \tag{20}$$

*and the matrix* $\mathbf{C}$ *is defined by* (10).

Proof. C. f. [2] p. 270. ☐

**Theorem 4.2.** *Let the function* $f(\boldsymbol{\vartheta}) = g'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, *within replicated linear model* (19) *satisfy the condition* $g \in \mathcal{M}\left[(m-1)\mathbf{C}_0 + \mathbf{C}\right]$. *Then for every unbiased estimator of the form* $\underline{\mathbf{Y}}'\mathbf{T}_{mn,mn}\underline{\mathbf{Y}}$, $\mathbf{T} = \mathbf{T}'$, *there exists an unbiased estimator*

$$\underline{\mathbf{Y}}'\left[(\mathbf{M}_m \otimes \mathbf{Q}) + (\mathbf{P}_m \otimes \mathbf{R})\right]\underline{\mathbf{Y}} = (m-1)\operatorname{Tr}\{\mathbf{S}Q\} + m\underline{\mathbf{Y}}'\mathbf{R}\underline{\mathbf{Y}}$$

*possessing the property*

$$\operatorname{Var}(\underline{\mathbf{Y}}'\mathbf{T}\underline{\mathbf{Y}}) \geq \operatorname{Var}\left[(m-1)\operatorname{Tr}\{\mathbf{S}Q\} + m\underline{\mathbf{Y}}'\mathbf{R}\underline{\mathbf{Y}}\right] \tag{21}$$

$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\}$ *and* $\forall\{\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}\}$.

Proof. Cf. [2] p. 274. ☐

## 5. Estimation of variance components in the replicated linear model

In this section we will look for unbiased estimators of variance components in the replicated linear model of the situation I.A. The method can be applied to every situations mentioned above where variance components cannot be unbiasedly estimated without the replications. The only difference is that we must use the covariance matrix $\mathbf{\Sigma}$ and the vector $\boldsymbol{\vartheta}$ corresponding to the considered situation.

First we have to investigate which variance components are unbiasedly estimable in the class of linear quadratic estimators in the replicated model. We will use Theorem 4.1. The matrix $\mathbf{C}_0$ given by (20) has the form

$$\mathbf{C}_0 = qpc \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (c-1) & 0 & 0 & 0 \\ 0 & 0 & (p-1)c & 0 & 0 \\ 0 & 0 & 0 & (q-1)c & 0 \\ 0 & 0 & 0 & 0 & (q-1)(p-1)c \end{pmatrix}. \tag{22}$$

Since the matrix $\mathbf{C}_0$ is regular, the matrix $(m-1)\mathbf{C}_0 + \mathbf{C}$ is regular and there holds $\mathbf{e}_i \in \mathcal{M}\big[(m-1)\mathbf{C}_0 + \mathbf{C}\big]$ for $i = 1, 2, \dots, 5$, where $\mathbf{e}_i$ denotes the vector from $\mathbb{R}^5$, which has the $i$th component equal to $1$ and the other components equal to $0$. Then by Theorem 4.1 every variance component $\vartheta_i = \mathbf{e}_i{}'\boldsymbol{\vartheta}$, $i = 1, 2, \dots, 5$, (and an arbitrary linear function $g'\boldsymbol{\vartheta}$) is unbiasedly estimable in the class $\mathcal{A}$.

Let $\underline{\mathbf{Y}}'\mathbf{T}_i\underline{\mathbf{Y}}$ be an unbiased estimator of the variance component $\vartheta_i$, $i = 1, 2, \dots, 5$. Since by Theorem 4.2 there exists an unbiased estimator of the form

$$\underline{\mathbf{Y}}'\big[(\mathbf{M}_m \otimes \mathbf{Q}_i) + (\mathbf{P}_m \otimes \mathbf{R}_i)\big]\underline{\mathbf{Y}} = (m-1)\operatorname{Tr}(\mathbf{S}Q_i) + m\underline{\mathbf{Y}}'\mathbf{R}_i\underline{\mathbf{Y}} \tag{23}$$

possessing the property (21), we can investigate only the class of quadratic estimators of the form (23) and try to find an unbiased estimator of the variance component $\vartheta_i$, $i = 1, 2, \dots, 5$, in this class.

Let $\underline{\mathbf{Y}}'\big[(\mathbf{M}_m \otimes \mathbf{Q}_i) + (\mathbf{P}_m \otimes \mathbf{R}_i)\big]\underline{\mathbf{Y}}$ be an unbiased estimator of the variance component $\vartheta_i$, $i = 1, 2, \dots, 5$, then the following must hold:

$$\mathrm{E}\big\{\underline{\mathbf{Y}}'\big[(\mathbf{M}_m \otimes \mathbf{Q}_i) + (\mathbf{P}_m \otimes \mathbf{R}_i)\big]\underline{\mathbf{Y}}\big\} = \mathbf{e}_i'\boldsymbol{\vartheta} = \vartheta_i\,. \tag{24}$$

Since

$$
\begin{aligned}
&\mathrm{E}\{\underline{\mathbf{Y}}'\big[(\mathbf{M}_m \otimes \mathbf{Q}_i) + (\mathbf{P}_m \otimes \mathbf{R}_i)\big]\underline{\mathbf{Y}}\} \\
&= (\mathbf{1}_m \otimes \mathbf{X}\boldsymbol{\beta})'\big[(\mathbf{M}_m \otimes \mathbf{Q}_i) + (\mathbf{P}_m \otimes \mathbf{R}_i)\big](\mathbf{1}_m \otimes \mathbf{X}\boldsymbol{\beta}) \\
&\quad + \mathrm{Tr}\big\{\big[(\mathbf{M}_m \otimes \mathbf{Q}_i) + (\mathbf{P}_m \otimes \mathbf{R}_i)\big](\mathbf{I}_{mm} \otimes \boldsymbol{\Sigma})\big\} \\
&= (\mathbf{1}_m \otimes \mathbf{X}\boldsymbol{\beta})'(\mathbf{P}_m \otimes \mathbf{R}_i)(\mathbf{1}_m \otimes \mathbf{X}\boldsymbol{\beta}) \\
&\quad + \mathrm{Tr}\big\{(\mathbf{M}_m \otimes \mathbf{Q}_i\boldsymbol{\Sigma}) + (\mathbf{P}_i \otimes \mathbf{R}_i\boldsymbol{\Sigma})\big\} \\
&= [m \otimes \boldsymbol{\beta}'\mathbf{X}'\mathbf{R}_i\mathbf{X}\boldsymbol{\beta}] + \sum_{j=1}^{5} \vartheta_j\big\{(m-1)\,\mathrm{Tr}(\mathbf{Q}_i\mathbf{V}_j) + \mathrm{Tr}(\mathbf{R}_i\mathbf{V}_j)\big\}\,,
\end{aligned}
$$

we obtain the conditions

$$\boldsymbol{\beta}'\mathbf{X}'\mathbf{R}_i\mathbf{X}\boldsymbol{\beta} = 0\,, \tag{25}$$

$$(m-1)\,\mathrm{Tr}(\mathbf{Q}_i\mathbf{V}_j) + \mathrm{Tr}(\mathbf{R}_i\mathbf{V}_j) = \delta_{ij}\,, \tag{26}$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

Let $\mathbf{R}_i = \mathbf{O}_{qpc,qpc}$, then the condition (25) holds and the condition (26) is

$$(m-1)\,\mathrm{Tr}(\mathbf{Q}_i\mathbf{V}_j) = \delta_{ij}\,. \tag{27}$$

Further, since

$$\mathrm{Tr}(\mathbf{V}_i\mathbf{V}_j) = \begin{cases} = 0 & \text{for } i \neq j\,, \\ \neq 0 & \text{for } i = j\,, \end{cases} \tag{28}$$

we can choose the matrix $\mathbf{Q}_i$, $i = 1, 2, \dots, 5$, to be

$$\mathbf{Q}_i = \frac{1}{(m-1)\,\mathrm{Tr}(\mathbf{V}_i\mathbf{V}_i)}\mathbf{V}_i\,, \quad i = 1, 2, \dots, 5\,. \tag{29}$$

Hence we have

$$
\begin{aligned}
\mathbf{Q}_1 &= \frac{1}{qpc(m-1)}\big[\mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes \mathbf{I}_{cc}\big]\,, \\
\mathbf{Q}_2 &= \frac{1}{qpc(c-1)(m-1)}\big[\mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes (\mathbf{1}_c\mathbf{1}_c' - \mathbf{I}_{cc})\big]\,, \\
\mathbf{Q}_3 &= \frac{1}{qp(p-1)c^2(m-1)}\big[\mathbf{I}_{qq} \otimes (\mathbf{1}_p\mathbf{1}_p' - \mathbf{I}_{pp}) \otimes \mathbf{1}_c\mathbf{1}_c'\big]\,, \\
\mathbf{Q}_4 &= \frac{1}{q(q-1)qc^2(m-1)}\big[(\mathbf{1}_q\mathbf{1}_q' - \mathbf{I}_{qq}) \otimes \mathbf{I}_{pp} \otimes \mathbf{1}_c\mathbf{1}_c'\big]\,, \\
\mathbf{Q}_5 &= \frac{1}{q(q-1)p(p-1)c^2(m-1)}\big[(\mathbf{1}_q\mathbf{1}_q' - \mathbf{I}_{qq}) \otimes (\mathbf{1}_p\mathbf{1}_p' - \mathbf{I}_{pp}) \otimes \mathbf{1}_c\mathbf{1}_c'\big]\,.
\end{aligned}
$$

Then

$$\widehat{\vartheta}_1 = \frac{1}{qpc(m-1)}\underline{\mathbf{Y}}'\big[\mathbf{M}_m \otimes \mathbf{I}_{qq} \otimes \mathbf{I}_{pp} \otimes \mathbf{I}_{cc}\big]\,\underline{\mathbf{Y}},$$

$$\begin{aligned}
\widehat{\vartheta}_2 &= \frac{1}{qpc(c-1)(m-1)}\,\underline{\mathbf{Y}}'\big[\mathbf{M}_m\otimes\mathbf{I}_{qq}\otimes\mathbf{I}_{pp}\otimes(\mathbf{1}_c\mathbf{1}_c'-\mathbf{I}_{cc})\big]\underline{\mathbf{Y}},\\
\widehat{\vartheta}_3 &= \frac{1}{qp(p-1)c^2(m-1)}\,\underline{\mathbf{Y}}'\big[\mathbf{M}_m\otimes\mathbf{I}_{qq}\otimes(\mathbf{1}_p\mathbf{1}_p'-\mathbf{I}_{pp})\otimes\mathbf{1}_c\mathbf{1}_c'\big]\underline{\mathbf{Y}},\\
\widehat{\vartheta}_4 &= \frac{1}{q(q-1)pc^2(m-1)}\,\underline{\mathbf{Y}}'\big[\mathbf{M}_m\otimes(\mathbf{1}_q\mathbf{1}_q'-\mathbf{I}_{qq})\otimes\mathbf{I}_{pp}\otimes\mathbf{1}_c\mathbf{1}_c'\big]\underline{\mathbf{Y}},\\
\widehat{\vartheta}_5 &= \frac{1}{q(q-1)p(p-1)c^2(m-1)}\\
&\quad\times\underline{\mathbf{Y}}'\big[\mathbf{M}_m\otimes(\mathbf{1}_q\mathbf{1}_q'-\mathbf{I}_{qq})\otimes(\mathbf{1}_p\mathbf{1}_p'-\mathbf{I}_{pp})\otimes\mathbf{1}_c\mathbf{1}_c'\big]\underline{\mathbf{Y}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\widehat{\vartheta}_1 &= \frac{1}{qpc(m-1)}\sum_{r=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{p}\sum_{k=1}^{c}\big(Y_{ijk}^{(r)}-\overline{Y}_{ijk}\big)^2,\\
\widehat{\vartheta}_2 &= \frac{1}{qpc(c-1)(m-1)}\sum_{r=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{p}\Bigg[\sum_{k=1}^{c}\sum_{l=1}^{c}\big(Y_{ijk}^{(r)}-\overline{Y}_{ijk}\big)\big(Y_{ijl}^{(r)}-\overline{Y}_{ijl}\big)\\
&\qquad\qquad -\sum_{k=1}^{c}\big(Y_{ijk}^{(r)}-\overline{Y}_{ijk}\big)^2\Bigg],\\
\widehat{\vartheta}_3 &= \frac{1}{qp(p-1)c^2(m-1)}\sum_{r=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{p}\Bigg[\sum_{s=1}^{p}\sum_{k=1}^{c}\sum_{l=1}^{c}\big(Y_{ijk}^{(r)}-\overline{Y}_{ijk}\big)\big(Y_{isl}^{(r)}-\overline{Y}_{isl}\big)\\
&\qquad\qquad -\sum_{k=1}^{c}\sum_{l=1}^{c}\big(Y_{ijk}^{(r)}-\overline{Y}_{ijk}\big)\big(Y_{ijl}^{(r)}-\overline{Y}_{ijl}\big)\Bigg],\\
\widehat{\vartheta}_4 &= \frac{1}{q(q-1)pc^2(m-1)}\sum_{r=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{p}\Bigg[\sum_{s=1}^{q}\sum_{k=1}^{c}\sum_{l=1}^{c}\big(Y_{ijk}^{(r)}-\overline{Y}_{ijk}\big)\big(Y_{sjl}^{(r)}-\overline{Y}_{sjl}\big)\\
&\qquad\qquad -\sum_{k=1}^{c}\sum_{l=1}^{c}\big(Y_{ijk}^{(r)}-\overline{Y}_{ijk}\big)\big(Y_{ijl}^{(r)}-\overline{Y}_{ijl}\big)\Bigg],\\
\widehat{\vartheta}_5 &= \frac{1}{q(q-1)p(p-1)c^2(m-1)}\\
&\quad\times\sum_{r=1}^{m}\sum_{i=1}^{q}\sum_{j=1}^{p}\Bigg[\sum_{s=1}^{q}\sum_{t=1}^{p}\sum_{k=1}^{c}\sum_{l=1}^{c}\big(Y_{stk}^{(r)}-\overline{Y}_{stk}\big)\big(Y_{ijl}^{(r)}-\overline{Y}_{ijl}\big)\\
&\qquad\qquad -\sum_{s=1}^{q}\sum_{k=1}^{c}\sum_{t=1}^{c}\big(Y_{sjk}^{(r)}-\overline{Y}_{sjk}\big)\big(Y_{ijl}^{(r)}-\overline{Y}_{ijl}\big)
\end{aligned}$$

$$- \sum_{t=1}^{p} \sum_{k=1}^{c} \sum_{l=1}^{c} \big(Y_{itk}^{(r)} - \overline{Y}_{itk}\big)\big(Y_{ijl}^{(r)} - \overline{Y}_{ijl}\big)$$
$$+ \sum_{k=1}^{c} \sum_{l=1}^{c} \big(Y_{ijk}^{(r)} - \overline{Y}_{ijk}\big)\big(Y_{ijl}^{(r)} - \overline{Y}_{ijl}\big)\Bigg] .$$

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