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INNER METRIC ANALYSIS

Andreas Nestke — Thomas Noll

ABSTRACT. This article deals with a kind of metric analysis of pieces of music whose principal feature is the study of so-called “inner local meters”, their arithmetic as well as combinatorial relations and a variety of structures naturally arising thereof.

1. Introduction

In order to distinguish several domains of analytical subtasks we propose to consider the following two fundamental dichotomies:

inner vs. **outer** and **metric** vs. **rhythmic**.

An *inner* analysis takes into account exclusively the given set of musical entities; in our case, the set of onsets. On the contrary, an *outer* analysis tries to relate this set to exterior syntactic schemes, e.g., bar structures, time signatures, grouping structures, etc.

Metric analysis is directed towards the investigation of regular repetitions of onsets. *Inner* metric analysis aims at finding and studying such repetitions within the piece itself, whereas *outer* metric analysis tries to fit the onsets of the piece into an exterior frame of regular repetitions (possibly a whole hierarchy of frames like those described by Lerdahl and Jackendoff, cp. [4]).

Rhythmic analysis, on the other hand, concentrates on the investigation of groupings of onsets. The *outer* grouping structures are typically motivated by layers of musical description beyond that of pure onsets (harmony, melody, syntactic forms, ...). The rules governing the well-formedness of such grouping structures are exterior to the given set of onsets. The mathematical modeling of onsets by rational numbers immediately suggests the paradigm for the *inner* grouping by what *rhythmic symmetry patches* will be denoted (cp. [10]).

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Some authors (e.g., E p s t e i n in [1]) appear to assume a close concordance between metric and outer as well as rhythmic and inner structures of a musical piece. From the point of view of an analysis on neutral level in the sense of N a t t i e z (cp. [9] or [8], p. 90), however, it is necessary to start by considering them as separate dichotomies.

As mentioned in the beginning, this article explores the inner metric realm proposing a framework of concepts and related quantitative instruments for probing into this field. This is not intended to imply any kind of superiority of this particular topic over all the other sides of the above dichotomies. The interplay of the inner metric structure with the remaining three domains will rather be the subject of forthcoming investigations, an intention announced as the 'MaMuTh-Metro Project' (see `www.mamuth.de`).

2. Free meters

We will assume throughout that musical onsets are modeled by rational numbers $x \in \mathbb{Q}$ understood as a $\mathbb{Z}$-module. Distances between onsets will be called *periods*. Periods are described by positive rational numbers. First, we will introduce several notions related to the arithmetics of metric structure.

Each period δ generates a unique submodule $\langle\delta\rangle$ in $\mathbb{Q}$. The quotient module $\mathbb{Q}/\langle\delta\rangle$ is called the *meter space* for period δ. Its elements are called the *free meters* of period δ. The smallest non-negative element ϕ of a free meter Φ is called its *phase*. Given a period δ and a phase $0 \leq \phi < \delta$, the corresponding free meter is $\Phi(\delta, \phi) = \phi + \langle\delta\rangle$. The free meter $0 + \langle 1\rangle$ is sometimes called the *free tactus meter*, the period 1 the *tactus*.

Remark 1. If one is about to represent an empirically given set of musical onsets (e.g., from a musical score) in terms of a set $X \subset \mathbb{Q}$ of rational numbers, there are at least two degrees of freedom in order to calibrate it: a scalar factor $s \in \mathbb{Q}$ and a translation $t \in \mathbb{Q}$. X and $s \cdot X + t$ both meet the demands, as long as there is no additional motivation to load the arithmetically distinguished rational numbers 0 and 1 with special music-theoretical meaning. Concerning the choice of s and t we recommend two strategies.

A prima vista representation prefers a calibration such that the prescribed tactus (the quarter-note period in $\frac{4}{4}$, the eighth-note period in $\frac{6}{8}, \dots$) has period 1 and the first structurally strong barline has onset 0. This is appropriate for a metric analysis combining inner with outer aspects.

In contrast to this, a blind representation uses only integers and starts with 0 (i.e., 0 is the smallest element of X). In that case, the fundamental period (see below) is supposed to be equal to 1. This is more appropriate for a purely inner analysis.

Lemma 1. (Intersection of two free meters) *Consider two free integral meters $\Phi_1 = \Phi(\delta_1, \phi_1)$ and $\Phi_2 = \Phi(\delta_2, \phi_2)$, i.e., $\delta_1, \delta_2, \phi_1, \phi_2 \in \mathbb{Z}$, with different periods, $\delta_1 \neq \delta_2$.*

Φ_1 and Φ_2 have non-empty intersection if and only if their phase-difference $\phi_1 - \phi_2$ satisfies one of the following conditions:

1. *$\phi_1 - \phi_2 = 0$: In this case the intersection is equal to $\Phi\big(\mathrm{LCM}(\delta_1, \delta_2), \phi_1\big)$.*
2. *$\mathrm{GCD}(\delta_1, \delta_2) | (\phi_1 - \phi_2)$ with $\phi_1 - \phi_2 \neq 0$: In that case the intersection is equal to $\Phi\big(\mathrm{LCM}(\delta_1, \delta_2), \phi\big)$, where ϕ is the smallest non-negative integer representing the residue class*

$$\frac{\phi_1 k_2 \delta_2 + \phi_2 k_1 \delta_1}{\mathrm{GCD}(\delta_1, \delta_2)} \quad \mathrm{mod}\ \mathrm{LCM}(\delta_1, \delta_2),$$

k_1 and k_2 are any integers for which the relation $\mathrm{GCD}(\delta_1, \delta_2) = k_1 \delta_1 + k_2 \delta_2$ holds.

$\mathrm{GCD}(m, n)$ and $\mathrm{LCM}(m, n)$ denote the greatest common divisor and the least common multiple of the integers m and n, respectively.

Proof. Let x be the smallest non-negative element of the intersection $\Phi_1 \cap \Phi_2$. Hence we have

$$x = \phi_1 + t_1 \delta_1 = \phi_2 + t_2 \delta_2$$

for suitable integers t_1 and t_2. Equivalently,

$$\phi_1 - \phi_2 = t_2 \delta_2 - t_1 \delta_1 .$$

For $t_1 = t_2 = 0$, this condition is equivalent to $\phi_1 - \phi_2 = 0$, which is case 1. Obviously, the period of the intersection is $\mathrm{LCM}(\delta_1, \delta_2)$, and the corresponding phase coincides with $\phi_1 = \phi_2$.

If not both, t_1 and t_2, are equal to zero, then, since $\mathrm{GCD}(\delta_1, \delta_2)$ divides the right hand side of the last equation, for some integer n

$$\phi_2 - \phi_1 = n\, \mathrm{GCD}(\delta_1, \delta_2) = n(k_1 \delta_1 + k_2 \delta_2).$$

We now consider the two projections

$$p_1 \colon \mathbb{Z}/\big\langle \mathrm{LCM}(\delta_1, \delta_2)\big\rangle \to \mathbb{Z}/\langle \delta_1 \rangle \qquad \text{and} \qquad p_2 \colon \mathbb{Z}/\big\langle \mathrm{LCM}(\delta_1, \delta_2)\big\rangle \to \mathbb{Z}/\langle \delta_2 \rangle .$$

We are going to show that the residue class of

$$\phi = \frac{\phi_1 k_2 \delta_2 + \phi_2 k_1 \delta_1}{\mathrm{GCD}(\delta_1, \delta_2)} \quad \mathrm{mod}\ \ \mathrm{LCM}(\delta_1, \delta_2)$$

is mapped to the class of ϕ_1 and ϕ_2 by p_1 and p_2, respectively. Write $\bar{\delta}_1 = \frac{\delta_1}{\mathrm{GCD}(\delta_1, \delta_2)}$ and $\bar{\delta}_2 = \frac{\delta_2}{\mathrm{GCD}(\delta_1, \delta_2)}$. Then we have $1 = k_1 \bar{\delta}_1 + k_2 \bar{\delta}_2$, and ϕ can be written as

$$\begin{aligned}\phi &= \phi_1 k_2 \bar{\delta}_2 + \phi_2 k_1 \bar{\delta}_1 \\ &= \phi_1(1 - k_1\bar{\delta}_1) + \phi_2(k_1\bar{\delta}_1) \\ &= (\phi_2 - \phi_1)k_1\bar{\delta}_1 + \phi_1 \\ &= nk_1\delta_1 + \phi_1 .\end{aligned}$$

Hence, under the projection p_1 ϕ is mapped to ϕ_1. A similar calculation shows that ϕ is mapped to ϕ_2 under the projection p_2. So, in case 2, the intersection of Φ_1 and Φ_2 is $\Phi\big(\mathrm{LCM}(\delta_1, \delta_2), \bar{\phi}\big)$, where $\bar{\phi}$ is the smallest non-negative integer in the residue class $\phi_1 k_2 \bar{\delta}_2 + \phi_2 k_1 \bar{\delta}_1 \mod \mathrm{LCM}(\delta_1, \delta_2)$.

Note that this class does not depend on the actual choice of k_1 and k_2. □

Corollary 1. *If the total intersection of a finite family of free integral meters is non-empty, then this intersection is again a free integral meter whose period is the least common multiple of the periods of the original meters.*

Remark 2. Lemma 1 and Corollary 1 dealt with integral meters. In the following discussion of intersections of finitely many free meters this restriction can be lifted as they can be made integral without changing their incidences.

Lemma 2. *Consider three free meters,* $\Phi_1 = \Phi(\delta_1, \phi_1)$, $\Phi_2 = \Phi(\delta_2, \phi_2)$ *and* $\Phi_3 = \Phi(\delta_3, \phi_3)$. *If all three pairwise intersections* $\Phi_1 \cap \Phi_2$, $\Phi_1 \cap \Phi_3$ *and* $\Phi_2 \cap \Phi_3$ *are non-empty, then the three-fold intersection* $\Phi_1 \cap \Phi_2 \cap \Phi_3$ *is nonempty, too.*

Proof. Choose integers $x \in \Phi_1 \cap \Phi_2$, $y \in \Phi_2 \cap \Phi_3$ and $z \in \Phi_3 \cap \Phi_1$. Since both, x and z, belong to Φ_1, we have $x - z = r_1\delta_1$ for some integer r_1. Analogously, there are numbers r_2 and r_3 for which $x - y = r_2\delta_2$ and $z - y = r_3\delta_3$. We are done, if two of the three elements x, y and z coincide. Hence, let us suppose that r_1, r_2 and r_3 are all different from zero.

Then we have

$$r_2\delta_2 = x - y = (x - z) + (z - y) = r_1\delta_1 + r_3\delta_3 .$$

Defining $\delta_0 = \mathrm{GCD}(\delta_1, \delta_2, \delta_3)$ the expression on the left-hand side can be expressed as $r_2\delta_2 = r_2\delta_0\frac{\delta_2}{\delta_0}$, while the right-hand side is equal to $r_4\,\mathrm{GCD}(\delta_1, \delta_3)$ for some integer r_4. By comparison we conclude that $\frac{\delta_2}{\delta_0}$ divides r_4. Since $\frac{\delta_2}{\delta_0}\,\mathrm{GCD}(\delta_1, \delta_3) = \mathrm{GCD}\big(\mathrm{LCM}(\delta_1, \delta_2), \mathrm{LCM}(\delta_2, \delta_3)\big)$, we can write

$$\begin{aligned}x - y &= r_5\frac{\delta_2}{\delta_0}\,\mathrm{GCD}(\delta_1, \delta_3) \\ &= r_5\,\mathrm{GCD}\big(\mathrm{LCM}(\delta_1, \delta_2), \mathrm{LCM}(\delta_2, \delta_3)\big) \\ &= r_6\,\mathrm{LCM}(\delta_1, \delta_2) + r_7\,\mathrm{LCM}(\delta_2, \delta_3) .\end{aligned}$$

Hence, there is an integer t such that

$$t = x - r_6 \operatorname{LCM}(\delta_1, \delta_2) = y + r_7 \operatorname{LCM}(\delta_2, \delta_3).$$

t belongs to $\Phi_1 \cap \Phi_2$, since $t = x - r_6 \operatorname{LCM}(\delta_1, \delta_2)$, as well as to $\Phi_2 \cap \Phi_3$ due to $t = y + r_7 \operatorname{LCM}(\delta_2, \delta_3)$. □

The preceding lemma can be generalized to the case of more than three meters. We prepare this by proving

Lemma 3. *Let $x_1 = k_1\delta_1, \dots, x_n = k_n\delta_n$ belong to the free integral meter Φ. Then some multiple of $\operatorname{LCM}(\delta_1, \dots, \delta_n)$ is also contained in Φ.*

Proof. We use induction on the number n of points. For $n = 1$ this is trivial, and for $n = 2$ it follows immediately from Lemma 2 and Remark 2 by considering the case $\Phi_1 = \Phi(\delta_1, 0)$, $\Phi_2 = \Phi(\delta_2, 0)$ and $\Phi_3 = \Phi$. Suppose that the lemma holds true for $n-1$ points. Given n points, we conclude that Φ contains some multiple $x = k \operatorname{LCM}(\delta_1, \dots, \delta_{n-1})$. But this additional point x together with the nth point $x_n = k_n\delta_n$ brings us back into the two-point situation above. □

Proposition 1. *Consider a finite family M of free meters with the property that all pairwise intersections are non-empty. Then the total intersection of M is non-empty, too.*

Proof. We use induction on the number n of meters. For $n = 3$ the proposition holds true by Lemma 2. Now suppose that every family of $n-1$ free meters with non-empty pairwise intersections has a nontrivial total intersection. Given n meters, we can pick any one of them, say Φ_n, and then conclude that the remaining $n-1$ meters, $\Phi_1, \dots, \Phi_{n-1}$, have a common intersection Φ_0 which is a free meter due to Corollary 1 and Remark 2.

Without loss of generality we may also suppose that all meters are integral and that their total intersection contains the point zero. Hence, each of the meters $\Phi_1, \dots, \Phi_{n-1}$ has phase zero:

$\Phi_0 = \Phi\big(\operatorname{LCM}(\delta_1, \dots, \delta_{n-1})\big)$, where $\delta_1, \dots, \delta_{n-1}$ denote the periods of $\Phi_1, \dots, \Phi_{n-1}$, respectively. Since $\Phi_j \cap \Phi_n \neq \emptyset$ for $j = 1, \dots, n-1$, Φ_n contains points $k_1\delta_1, \dots, k_{n-1}\delta_{n-1}$. From Lemma 3 we conclude that $\Phi_0 \cap \Phi_n \neq \emptyset$. □

Remark 3. We will refer to this proposition in the last section concerning the combinatorial analysis of global meters.

We conclude this section by some remarks concerning the periods determined by a piece of music.

Whenever $\langle\delta_1\rangle$ is contained in $\langle\delta_2\rangle$, i.e., if there is a natural number k with $k\delta_2 = \delta_1$, we say that the period δ_1 *dominates* the period δ_1. In this case, there is a natural projection from the meter spaces $\mathbb{Q}/\langle\delta_1\rangle$ for δ_1 onto the meter space $\mathbb{Q}/\langle\delta_2\rangle$ for δ_2.

We will benefit from the ambiguity of the word "piece" which may refer to a piece of music as well as its pieces (parts). Let X be a finite subset of $\mathbb{Q}$ describing the set of onsets of a given piece. By $\Delta(X)$ we denote the set of all periods defined by elements of X:

$$\Delta(X) = \{\delta\colon \text{ there exist two elements } x \leq y \in X \text{ with } \delta = y - x\}.$$

A first quantitative overall view of the periods occurring in a given set of musical onsets is provided by their weights.

Definition 1. Let $X \subset \mathbb{Q}$ be a finite set of onsets. For a given period $\delta \in \Delta(X)$ the corresponding *period weight* is defined as the number of all pairs $x < y$, $x, y \in X$, with period δ,

$$w_{\text{period}}(\delta) = \#\big\{(x,y) \in X \times X\colon x < y \text{ and } y - x = \delta\big\}.$$

The following three figures display the period weights for three pieces each consisting of forty notes.

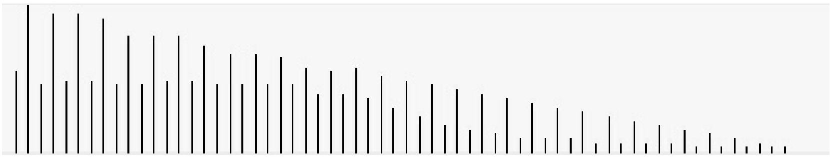

Figure 1. Period Weight of Fuge C Minor from J. S. Bach's *Wohltemperiertes Klavier Part I.*

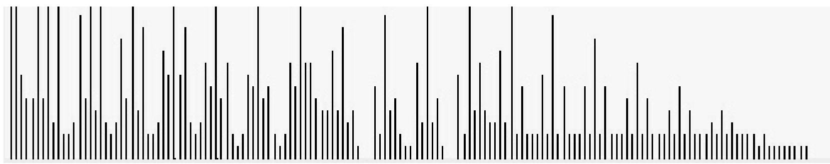

Figure 2. Period Weight of Fuge D Major from J. S. Bach's *Wohltemperiertes Klavier Part I.*

Figure 3. Period Weight of a random set of Onsets.

By $\langle\Delta(X)\rangle$ we denote the submodule in $\mathbb{Q}$ generated by all periods of X. Since X is finite, there exists a unique positive rational number $d(X)$ satisfying $\langle\Delta(X)\rangle = \langle\delta(X)\rangle$. This number is called the *principal period* of the set X.

All elements $x \in X$ belong to the same free meter $\Phi(X) \in \mathbb{Q}/\delta(X)$ which is called the principal free meter of X. Its phase $\phi(X)$ is called the *principal phase.* For every period δ of X we have the canonical projection

$$\pi_{[\delta,X]} : \mathbb{Q}/\langle\delta\rangle \to \mathbb{Q}/\langle\delta(X)\rangle .$$

3. Local meters

We will now investigate *bounded meters*, i.e., restrictions of free meters with respect to *timespans*, as well as *local meters*, i.e., the resulting segments of free meters.

Definition 2. Each pair of rational numbers a, b such that $a \leq b$ defines a (closed) *timespan* $[a, b] \subset \mathbb{Q}$ of *size* $b - a$. A pair $(\Phi, [a, b])$ consisting of a free meter Φ and a timespan $[a, b]$ is called a *bounded meter.*

We say that two bounded meters *intersect weakly*, if both their defining free meters and their defining timespans intersect.

Non-empty intersections of timespans with free meters are called *local meters.* The leftmost onset of a local meter is called its *beginning.* For a meter m, its *length*, $l(m)$, is the number of all onsets following the beginning. For a given profile $p \in \mathbb{R}$ we define the *p-weight* of the meter m by $w^p(m) = l^p$.

Remark 4. Two bounded meters are said to *intersect strongly* if the corresponding local meters have non-trivial intersection. Hence, strong intersection implies weak intersection.

Although the defining timespan of a local meter is not unique, we can always choose the minimal one determined by the first and the onset of the meter. Moreover, note that the definition does not exclude local meters with just a single onset. This is convenient for several purposes, but has to be taken into account, e.g., when talking about periods. In fact, only for meters of positive length there is a well-defined period inherited from the unique free meter defining it.

Now we will introduce a numerical characteristic for each meter with respect to a given piece X, i.e., a set of onsets, and apply it to distinguish among three types one of which will be of particular interest to us in the sequel.

Definition 3. Consider a meter m of length l and a piece X. The quotient

$$\frac{\#(X \cap m)}{l + 1}$$

is called the *density* of m with respect to X. *Inner* meters of X are those of density 1, while *complementary* meters are defined by having density 0. Meters which have positive density < 1 with respect to X are called *outer meters.*

Remark 5. Typically, outer meters of large density play a prominent role in the usual study of the bar structure of a piece. Each kind of analysis focussing on (suitably chosen) outer meters is, in this sense, an *outer metric analysis.* If, on the other hand, one wanted to analyse a piece leaving aside the bar lines as well as the time signature in the score, then one would, preferably, start by considering inner meters. For this approach we will use the term *inner metric analysis.*

The next defined notions allow a systematic inventory of the variety of inner meters of a given piece.

Definition 4. Let $X \subset \mathbb{Q}$ be a piece. For each $\delta \in \Delta(X)$ we associate with every $x \in X$ the (unique) maximal inner meter $m_X^\delta(x)$ of period δ in X containing x. For a fixed period $\delta \in \Delta(X)$ and any phase $\phi \in \mathbb{Q}$, $0 \leq \phi < \delta$, the corresponding *phase-context*, $\mathrm{phasecont}_X^\delta(\phi)$, is defined as the set of all meters $m_X^\delta(x)$ with $x \in \Phi(\delta, \phi)$.

For a given set of periods $D \subset \Delta(X)$, the *D-context* of $x \in X$, $\mathrm{cont}_X^D(x)$, is defined as the set of all $m_X^\delta(x)$ for $\delta \in D$. For a set of natural numbers $L \subset \mathbb{N}$ we also define the (D, L)- *context*, $\mathrm{cont}_X^{(D,L)}(x)$, as the set of all those $m_X^\delta(x)$ whose length belongs to L.

Note that the set $\{x\}$ is contained in the D-context of $x \in X$ iff there exists a $\delta \in D$ such that $m_X^\delta(x) = \{x\}$, i.e., $x - \delta$ and $x + \delta$ do not belong to X.

Having defined the contexts we will now quantify these sets introducing corresponding weights accumulating the weights of the single meters within them.

Definition 5. Let $X \subset \mathbb{Q}$ be a piece, $\delta \in \Delta(X)$ a period of X, $\phi \in \mathbb{Q}$ the phase of a free meter of period δ and $p \in \mathbb{R}$ a profile. Then, the *phase-weight*, $pw_X^p(\phi)$, of ϕ is a sum of the p-weights of all meters in the phase-context $\mathrm{phasecont}_X^\delta(\phi)$,

$$pw_X^p(\phi) = \sum_{m \in \ \mathrm{phasecont}_X^\delta(\phi)} w^p(m)\,.$$

The following two figures show the phase weights of two musical examples: Each of the concentric circles corresponds to one period and its gray level exemplifies the associated period weight. Along a single circle, there are, regularly placed, as many small disks as there are phases for this period (modulo principal period). The phase weights for each period are normalized in a way that the highest occurring value is depicted by a black disk, and lighter ones correspond

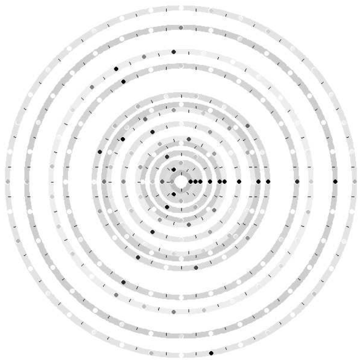

FIGURE 4. Phase Weight of the 2nd voice of the C-Minor Fugue (see Figure 1)

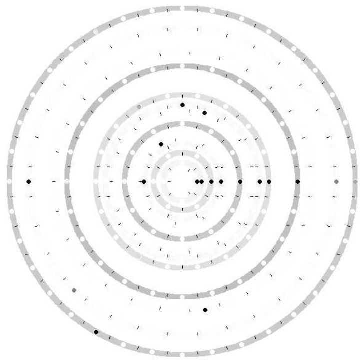

FIGURE 5. Phase Weight of the first eight bars of Chopin's Polonaise Op. 40, No. 1.

to smaller values. The periods chosen here are the integers not bigger than 36 having prime factors 2 and 3 only. Note that in the Bach Example the principal period is present in the piece, while this is not the case for the Chopin Example. This fact is partly responsible for the qualitative difference between both phase weights.

There is a refined strategy in the metrical analysis of a given piece which uses variable phase weight calculations of restricted timespans. The global variation of the resulting phase weight (e.g., while moving a timespan of fixed size through the piece) provides useful information about the degree of phase homogeneity with respect to a chosen period. The following figure shows for the Chopin Example how explicitly the phases of a fixed period are occupied by local meters of that period:

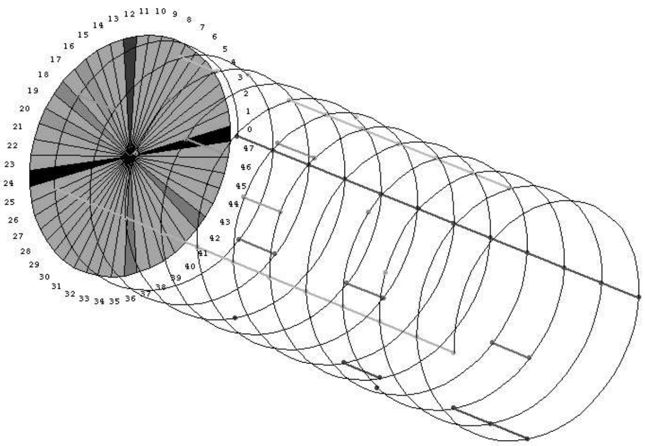

FIGURE 6. All onsets of the Chopin Example are distributed along a spiral (onsets from left to right, and phases with respect to period 48 on a circle). Lines from left to right stand for local meters. Gray values on the disk to the very left express phase weights.

Definition 6. For a given set of periods $D \subset \Delta(X)$, and a set of natural numbers $L \subset \mathbb{N}$, the *inner metric weight* $mw_X^{D,L,p}(x)$, of the onset $x \in X$ for the profile $p \in \mathbb{R}$ is defined as the sum of the p-weights of all meters in the $(D, L) - context$ of x,

$$mw_X^{D,L,p}(x) = \sum_{m \in \mathrm{cont}_X^{(D,L)}(x)} w^p(m).$$

The way the metric weight is defined here differs from the following one due to Mazzola (cp. [6]) who uses the same formula, but takes into account only those meters which are not contained in any other meter, regardless of their periods. If m is a maximal inner meter of a piece X with respect to the period δ of m, we denote by $\overline{m}$ the set of all those inner meters n of X containing m, that are themselves not contained in any other inner meter of X. We call these meters *totally maximal*.

The meters n within $\overline{m}$ have periods dividing δ and their associated timespans contain the timespan $[a, b]$ associated with m, but they are all contained in the open timespan $(a - \delta, b + \delta)$. Totally maximal meters m are characterized by $\overline{m} = \{m\}$.

Definition 7. With the same ingrediences as in Definition 6, the *total metric weight* of an onset $x \in X$ is the sum of the p-weights of all those meters in the $(D, L) - context$ of x satisfying $\overline{m} = \{m\}$,

$$\overline{mw}_X^{D,L,p}(x) = \sum_{\substack{\overline{m}=\{m\} \\ m \in \mathrm{cont}_X^{(D,L)}(x),}} w^p(m).$$

We will see at the end of this section how the two weights can be related to one another in a natural way.

As a last ingredient for now, we recall a convenient notion of distance which is quite useful for metric analysis, the Hausdorff distance for sets.

Definition 8. Consider $\mathbb{Q}$ as a metric space with respect to the euclidean distance. For two non-empty subsets A, $B \subset \mathbb{Q}$, the *asymmetric set distance*, $d_A(B)$, of the set A to the set B is defined by

$$d_A(B) = \sup_{a \in A} d_a(B) = \sup_{a \in A} \inf_{b \in B} \{|a - b|\}.$$

Moreover, the larger of the two asymmetric set distances $d_A(B)$ and $d_B(A)$ is called the *Hausdorff distance* of the sets A and B,

$$d_H(A, B) = \max\{d_A(B), d_B(A)\}.$$

The most important property of the Hausdorff distance is the following fact which, of course, is just the very special case, referring to the situation discussed here, of a general result, cp. ([3]).

Proposition 2. *The Hausdorff distance is a metric on the set of all finite non-empty subsets of* $\mathbb{Q}$.

As we are exclusively dealing with finite sets, the suprema and infima in the above definition may, of course, be replaced by maxima and minima, respectively. To indicate the immediate usefulness of the Hausdorff distance for our purposes, we note the following. We note the following obvious fact.

Lemma 4. *For every pair of non-empty sets* $A, B \subset \mathbb{Q}$ *and any* $\varepsilon > 0$ *the inequality* $d_A(B) < \varepsilon$ *holds iff for each* $a \in A$ *there exists an element* $b \in B$ *such that* $|a - b| < \varepsilon$.

In other words, $d_A(B) < \varepsilon$ if and only if the set A is contained in the ε-neighbourhood of the set B.

As in the case of inner rhythmic as well as motivic analysis (cp. [10]), where measures of closeness are quite important, one can use both the asymmetric and the Hausdorff distance in order to measure the relative position of rhythms and meters on the one hand, and of meters among themselves, on the other.

Coming back now to the comparison of the two metric weights, we observe that Mazzola neglects exactly those inner meters whose asymmetric set distance to another inner meter is zero. The difference between both viewpoints is reflected by the opposite asymmetric set distance, i.e., the Hausdorff distance. Hence, a generalized version of metric weight has to include a second profile weighting this new contribution to the formula. As a preparation we define a function d_X measuring the mean Hausdorff distance of an inner meter to all totally maximal inner meters, containing m: Given a piece X and an inner meter m (being maximal with respect to its own period δ) we define

$$d_X(m) := \frac{1}{\#(\overline{m})} \sum_{n \in \overline{m}} d_H(m, n).$$

Definition 9. For a given set of periods $D \subset \Delta(X)$ and a set of natural numbers $L \subset \mathbb{N}$, the *extended inner metric weight* $emw_X^p(x)$, of the onset $x \in X$ for the profiles p, $q \in \mathbb{R}$ is defined as a sum of the p-weights of all meters m in the (D, L) – *context* of x each multiplied by a new factor that equals the mean Hausdorff distance to the totally maximal meters in $\overline{m}$ plus 1, taken to the power $-q$,

$$emw_X^{D,L,p,q}(x) = \sum_{m \in \mathrm{cont}_X^{(D,L)}(x)} w^p(m) \left(1 + d_X(m)\right)^{-q}.$$

If δ_1 and δ_2 are the periods of m and $n \in \overline{m}$, respectively, with $\delta_2 = k \cdot \delta_1$, then one has $d_H(m, n) = k/2 \times \delta_1$ for k even, and $d_H(m, n) = (k-1)/2 \times \delta_1$ for k odd.

Lemma 5. *In the situation of the definition we have*

$$emw_X^{D,L,p,0} \equiv mw_X^{D,L,p}\,,$$
$$emw_X^{D,L,p,q}(x) \to \overline{mw}_X^{D,L,p}(x) \qquad \text{for } q \to \infty\,.$$

In other words, for $q = 0$ we obtain exactly the same formula as in Definition 6. In addition, the total metrical weight (Definition 7) is recovered by taking the limit of $emw_X^{D,L,p,q}(x)$ for $q \to \infty$.

4. Combinatorial structures

Starting from a given set of meters $\mathcal{F}$ contained in a piece $X \subset \mathbb{Q}$ we will now describe a series of structures arising from it and discuss several aspects of their interrelations.

Definition 10. Let $X \subset \mathbb{Q}$ be a piece and $\mathcal{F}$ a (finite) set of meters in X which are supposed to be maximal in the sense that none of them is contained in any of the others.

The *support* of $\mathcal{F}$ is the union of all the sets in $\mathcal{F}$,

$$\operatorname{supp}(\mathcal{F}) = \bigcup_{M \in F} M \subset X\,.$$

Moreover, the *$\mathcal{F}$-topology*, $\mathcal{T}_{\mathcal{F}}$, is the uniquely defined topology on $\operatorname{supp}(\mathcal{F})$ with the set $\mathcal{F}$ as its subbasis.

Finally, the components of the topological space $(\operatorname{supp}(\mathcal{F}), \mathcal{T}_{\mathcal{F}})$ are called the *$\mathcal{F}$-components* of X.

All that is to follow will take place within or be related to the elements in the support $\operatorname{supp}(\mathcal{F})$. In general, this is a proper subset of the piece X, whereas, for a sufficiently rich family $\mathcal{F}$, the support will, as a rule, coincide with the whole set X. Expressed differently, a set $U \subset \operatorname{supp}(\mathcal{F})$ belongs to $\mathcal{T}_{\mathcal{F}}$ ("is open in this topology") if and only if U is a union of finite intersections of sets from $\mathcal{F}$.

Remark 6. As the intersection of two meters is a meter itself, the open sets in this topology are all unions of meters within the support of $\mathcal{F}$. On the other hand, these intersections may well be degenerate in various ways.

The extreme of these situations is that the intersection consists of a single point only. Next, two meters may have two points in common. If, in this case, the periods coincide, then both meters combine to a larger one with this same period containing both. If the periods are different, then one of them may divide the other, but, in general, both are just divisors of the distance between the common points, since there has to be at least one more point of either meter in between. Among these an extreme situation occurs, if both meters are bounded by the common points. Besides this, almost everything is possible, e.g., the meters can be nested, one meter can extend to any side of the other or even to both sides. Hence, in general, the $\mathcal{F}$-topology typically has a discrete part as well as rather large minimal open sets.

Note that the $\mathcal{F}$-components do not, in general, form any kind of segmentation of the piece X. By definition, a component comprises all those points in the support of $\mathcal{F}$ which can be joined by segments of the meters in $\mathcal{F}$. However, finding a way to *mathematically* define a segmentation of X, possibly by choosing particular meter families and perhaps changing them suitably, that serves the *musicological* tasks and corresponds to prominent proposals in the literature, is definite one of the aims of the current investigations.

Remark 7. In [5], Guerino Mazzola discusses certain aspects of the topology defined by the family of maximal meters in a given set of onsets. In particular, he describes the irreducible closed sets of this topology and constructs the associated sober space.

Next we are going to define a combinatorial structure encoding the incidence relations among the sets in a given meter family $\mathcal{F}$ for a set $X \subset \mathbb{Q}$.

Definition 11. Let $X \subset \mathbb{Q}$ be a piece and $\mathcal{F}$ a family of meters in X. The sets of the family $\mathcal{F}$ form, by definition, a covering of the support $\mathrm{supp}(\mathcal{F}) \subset X$. Hence, the *nerve*, $\mathrm{N}_{\mathcal{F}}$, is a well-defined (abstract) simplicial complex whose vertices correspond to the sets in $\mathcal{F}$. A set of these vertices form a simplex of the nerve $\mathrm{N}_{\mathcal{F}}$ iff the intersection of the corresponding meters from $\mathcal{F}$ is nonempty.

As any subset of a simplex obviously is a simplex itself, the nerve is a simplicial complex.

Example 1. Consider the family of all meters with one out of two given periods, d and D, one of which divides the other, $d|D$. Then, first, the set of vertices of the nerve is partitioned into the subsets corresponding to the different phases and, second, the nerve is just 1-dimensional with each edge describing a pair of meters, one for either period, having at least one point in common. Similarly reduced, but metrically nevertheless important, situations arise, whether one starts from specially chosen "period diagrams" which in turn lead to rather few possibilities for non-trivial intersections of meters.

The combinatorial structure just introduced is the origin of a framework as well as strategy for studying the mutual relations of the meters in the family $\mathcal{F}$ from the point of view of combinatorial topology. Each of the notions, constructions and computations based on simplicial complexes should first be questioned as to the particular properties they exhibit, whether the complex arises from a family of meters. Then, there is a further requirement to look for those consequences allowing an interpretation in terms of the original subject, i.e., music.

The data possibly of particular relevance and discussed here are the facets (maximal simplices with respect to inclusion) of the complex and their dimensions (one less than the number of vertices), the star of each vertex (the subcomplex of all simplices containing it) together with its cardinality and the occurring simplex dimensions, and, of course, the homology groups, Betti numbers (their ranks) and torsion coefficients. Both simplicial complexes, the nerve and the $\mathcal{F}$-complex to be defined next, play an important role in the classification theory of global compositions (cf. [5]).

Definition 12. Let $X \subset \mathbb{Q}$ be a piece and $\mathcal{F}$ a family of meters in X. $\mathcal{F}$ defines a mapping $\mathrm{cont}_{\mathcal{F}}\colon \mathcal{P}\big(\mathrm{supp}(\mathcal{F})\big) \to \mathcal{P}(\mathcal{F})$, the *context* defined by the family $\mathcal{F}$, associating with each subset $Y \subset X$ of the set $\mathrm{cont}_{\mathcal{F}}(Y)$ of all those meters in $\mathcal{F}$ which contain the whole set Y.

Moreover, reducing the information represented by the image of $\mathrm{cont}_{\mathcal{F}}$ to the cardinality of that set, we obtain the *$\mathcal{F}$-weight*,

$$\begin{aligned} &w_{\mathcal{F}}\colon \mathcal{P}\big(\mathrm{supp}(\mathcal{F})\big) \to \mathbb{N}\,,\\ &w_{\mathcal{F}}(Y) = \#\big(\mathrm{cont}_{\mathcal{F}}(Y)\big)\,. \end{aligned}$$

Finally, the *$\mathcal{F}$-complex*, $\mathrm{C}_{\mathcal{F}}$, is defined to be the support of $\mathrm{cont}_{\mathcal{F}}$ (or, equivalently, the support of $\mathrm{w}_{\mathcal{F}}$), i.e., the family of subsets Y of the set X satisfying $\mathrm{cont}_{\mathcal{F}}(Y) \neq \emptyset$ ($\Leftrightarrow \mathrm{w}_{\mathcal{F}}(Y) \neq 0$).

In other words, the $\mathcal{F}$-complex is the simplicial complex defined as having the sets in $\mathcal{F}$ as its facets. Thus, the simplices are all subsets of the sets in $\mathcal{F}$. We will call them the (metric) $\mathcal{F}$-patches. The following lemma is straightforward.

Lemma 6. *For a piece $X \subset \mathbb{Q}$ and a family of meters $\mathcal{F}$ in X, the context mapping is monotonous, i.e., for $Z \subset Y \subset \mathrm{supp}(\mathcal{F})$ we have $\mathrm{cont}_{\mathcal{F}}(Z) \supset \mathrm{cont}_{\mathcal{F}}(Y)$. Moreover, the $\mathcal{F}$-weight is non-increasing, i.e., $Z \subset Y \subset \mathrm{supp}(\mathcal{F})$ implies $w_{\mathcal{F}}(Z) \geq w_{\mathcal{F}}(Y)$.*

To illustrate how to make use of these structures we mention a (partial) relation among onsets in the piece to be analysed discussed in relation with its musical interpretation and consequences in [2]. The different kinds of weights, and this relation in particular, are (first) steps towards extracting pieces of information from the results of a metric analysis directly referring to the onsets of the piece. This is decisive for the purpose of generating performances expressing analytical insights.

Example 2. Let $X \subset \mathbb{Q}$ be a piece, $\mathcal{F}$ a family of meters in X, and let

$$\mathrm{cont}_{\mathcal{F}}\colon \mathcal{P}\big(\mathrm{supp}(\mathcal{F})\big) \to \mathcal{P}(\mathcal{F})$$

be the corresponding context mapping. The point $x \in \mathrm{supp}(\mathcal{F}) \subset X$ is then said to *dominate* the point $y \in \mathrm{supp}(\mathcal{F}) \subset X$ with respect to $\mathcal{F}$ iff $\mathrm{cont}_{\mathcal{F}}\big(\{y\}\big) \subset \mathrm{cont}_{\mathcal{F}}\big(\{x\}\big)$, i.e., every meter in $\mathcal{F}$ containing y also contains x.

The dominance relation provides a convenient method to describe hierarchical structures in a given set of onsets with respect to a given family of meters. Unfortunately, unless the set is rather small, there will be no pairs of points in which one dominates the other or each one dominates the other (metric equivalence). Hence, a reasonable application of this relation requires to choose (sequences of) sets with, say, 10 to 15 tones.

Starting from the $\mathcal{F}$-weight, instances of the dominance relation can be easily computed by considering its values both on individual onsets and pairs of onsets, i.e., restricting $w_{\mathcal{F}}$ to the 1-skeleton of the meter complex $\mathrm{C}_{\mathcal{F}}$. The corresponding data can be neatly arranged in the form of a graph whose vertices are the onsets two of which are joined by an edge, whenever there is a meter in $\mathcal{F}$ containing both. Labelling, in addition, the vertices as well as the edges by their weights allows the straightforward determination of dominances. The single observation to be made here is that one onset dominates a second iff the label of the latter, i.e., its metric weight, coincides with the label of the edge joining both onsets. If, moreover, all three labels agree, then both onsets are metrically equivalent. For a thorough discussion and examples we refer to [2].

In a way, the metric weight assigns a (numerical) metric importance (with respect to the given meter family) to every onset in the support of the family by just counting the number of meters to which the onset belongs. A somewhat more refined "weight", could be defined by taking the number of $(k-1)$-element sets within $\mathrm{cont}_{\mathcal{F}}\big(\{x\}\big)$ as the kth coefficient of a polynomial in an undetermined t. This resulted in a polynomial weight for every point in the support of the family $\mathcal{F}$ which, in addition, included information about the length of the participating meters. A further refinement could take into account the period as well. (Compare also [5] for the notion of *simplicial metric weight.*)

For the family of all maximal meters, $\mathcal{F}_{\max}(X)$ within a given set X a considerable amount of empirical and explorative investigations into their musical relevance has already been carried out, most of which using the software RUBATO (cp. [7]).

In the concluding part of this note we want to address two closely related topics.

1. **Realization:** Which (finite) simplicial complexes can be realized as nerves of meter families?

2. **Homology:** Are there any general homological properties common to the (meter and nerve) complexes associated with meter families?

EXAMPLE 3. (Triangles) To start with, let us have a closer look at two very simple examples. Consider the "pieces" $X_1 = \{0, 2, 3, 4, 7, 10\}$, $X_2 = \{0, 1, 2, 4, 7\}$ and corresponding meter families

$$\mathcal{F}_1 = \big\{\{2, 3, 4\}, \{0, 2, 4\}, \{4, 7, 10\}\big\}, \quad \mathcal{F}_2 = \big\{\{0, 1, 2\}, \{0, 2, 4\}, \{1, 4, 7\}\big\}.$$

Though apparently similar, the complexes and nerves of these meter families are, combinatorially, quite different. In particular, the meter complex $\mathrm{C}_{\mathcal{F}_2}$ as well as the nerve $\mathrm{N}_{\mathcal{F}_2}$ contain a non-trivial cycle, while $\mathrm{C}_{\mathcal{F}_1}$ and $\mathrm{N}_{\mathcal{F}_1}$ are acyclic.

EXAMPLE 4. (Combinatorial Möbius strip) With respect to the second of the above problems this already shows that, in general, non-trivial homology has to be expected. In particular, it is not difficult to construct meter families with large Betti numbers even in higher dimensions. What is still open and being studied by computations is the question, whether meter families may give rise to simplicial complexes with non-trivial torsion. As a first indication of what can be realized we include a simple example of a family of five meters whose nerve turns out to be a Möbius strip.

FIGURE 7. Rhythm X_1 composed of 3 local meters.

FIGURE 8. Rhythm X_2 composed of 3 local meters.

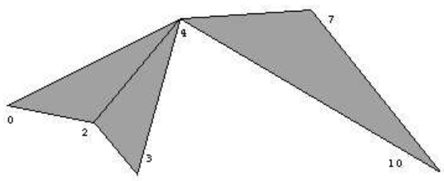

Figure 9. Meter Complex $C_{\mathcal{F}_1}$.

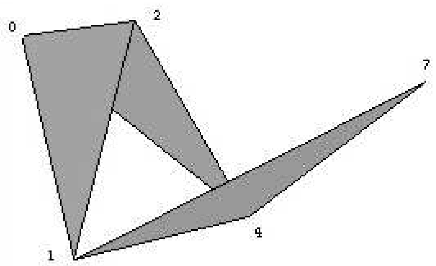

Figure 10. Meter Complex $C_{\mathcal{F}_2}$.

Lemma 2 implies that five *free* meters with non-trivial pairwise intersections always have a 4-simplex as their nerve. Hence, there is no combination of five free meters such that the corresponding nerve yields a Moebius strip (formed by 5 triangles). The same is obviously true for any combination of five timespans.

However, the following example shows that 4 free meters may have a non-trivial cycle in their nerve.

EXAMPLE 5. (Non-trivial 1-cycle of free meters) Consider $\Phi_1 = \Phi(3,0)$, $\Phi_2 = \Phi(2,1)$, $\Phi_3 = \Phi(3,1)$ and $\Phi_4 = \Phi(2,0)$.

Then there are exactly 4 non-empty intersections of these four meters:

$$\begin{aligned} \Phi_1 \cap \Phi_2 &= \Phi(6,3)\,, \\ \Phi_2 \cap \Phi_3 &= \Phi(6,1)\,, \\ \Phi_3 \cap \Phi_4 &= \Phi(6,4)\,, \\ \Phi_4 \cap \Phi_1 &= \Phi(6,0)\,. \end{aligned}$$

All this shows that to study the combinatorial structure of meter families is not the same as studying the combinatorial structures of the associated families of timespans and free meters. Hence, for music-theoretical investigations it may be useful to take into account the notions of strong and weak intersection of bounded meters.

FIGURE 11. Score of five local meters with period 4, 4, 10, 2, and 1.

Of course, the Möbius strip has homology groups isomorphic to $\mathbb{Z}$ in dimension zero and one, and trivial homology elsewhere. Nevertheless, its first

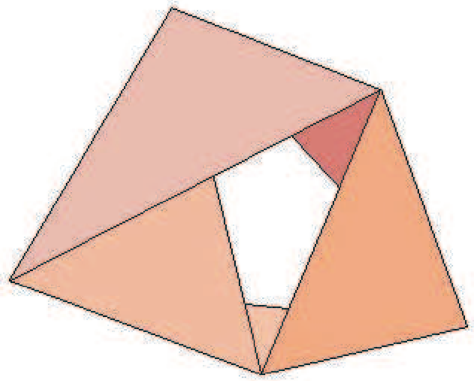

FIGURE 12. Visualization of the nerve formed by the 5 meters in Figure 11.

homology as a two-dimensional pseudo-manifold relative to the boundary is isomorphic to $\mathbb{Z}_2$. The natural candidate to be studied next is the projective plane. In particular, we are looking for a realization of the triangulation of this space by ten triangles, fifteen edges and six points as the nerve of a meter family. While this writing, we were neither able to produce an example nor to show that there is no such meter family. In general, it appears to be completely open, whether there are any conditions a simplicial complex has to satisfy in order to be realizable as the nerve of a meter family. The "typical" complexes arising from a meter family are, of course, neither pseudo-manifolds nor even homogeneous.

Finally, we will discuss the relationship between the two simplicial complexes which were defined for any meter family $\mathcal{F}$, the $\mathcal{F}$-complex $\mathrm{C}_{\mathcal{F}}$ and the nerve $\mathrm{N}_{\mathcal{F}}$. As already mentioned, both encode slightly different aspects of the combinatorial situation represented by the given family of meters. The nerve has vertices corresponding to the elements of $\mathcal{F}$ and expresses the incidence among them as an alternative. Either these meters have a common non-empty intersection, and thus define a simplex in the nerve, or their intersection is empty, and they do not contribute to the nerve. In contrast to this, the vertices of the $\mathcal{F}$-complex are the onsets in the support of $\mathcal{F}$, a subset of which constitutes a simplex, if they all belong to (at least) one set in $\mathcal{F}$. The apparent duality between these simplicial complexes is made a little more explicit by the following two mappings between the sets $\mathrm{N}_{\mathcal{F}}$ and $\mathrm{C}_{\mathcal{F}}$.

DEFINITION 13. Let X be a piece and $\mathcal{F}$ a family of meters in X. The mappings $S\colon \mathrm{C}_{\mathcal{F}} \to \mathrm{N}_{\mathcal{F}}$ and $T\colon \mathrm{N}_{\mathcal{F}} \to \mathrm{C}_{\mathcal{F}}$ are defined by the equations

$$S(Y) = \mathrm{cont}(Y)\,,$$

$$T\big(\{m_1, \ldots, m_k\}\big) = \bigcap_{j=1}^{k} m_j\,.$$

In the first equation, the context, which is understood as a simplex in the nerve, is non-empty, since Y belongs to $\mathrm{C}_{\mathcal{F}}$ and, hence, there is a meter in $\mathcal{F}$ containing this set. In the second, the meters $m_1, \ldots, m_k \in \mathcal{F}$ are supposed to form a simplex in the nerve, i.e., the intersection on the right-hand side is non-empty and, moreover, a simplex in the $\mathcal{F}$-complex.

It has to be stressed that both these mappings are not simplicial, although the image of a simplex in the first complex is a simplex in the second, in both cases. However, they do not map vertices to vertices, and, moreover, each one obviously reverses the face relation.

Lemma 7. *For the compositions of the mappings S and T we have the following relations.*

$$(T \circ S)(Y) = \bigcap_{m \in \mathrm{cont}(Y)} m\,,$$

$$(S \circ T)\big(\{m_1, \ldots, m_k\}\big) = \mathrm{cont}\left(\bigcap_{j=1}^{k} m_j\right).$$

Each of the mappings $P = T \circ S\colon C_{\mathcal{F}} \to C_{\mathcal{F}}$ and $Q = S \circ T\colon N_{\mathcal{F}} \to N_{\mathcal{F}}$ is idempotent, i.e.,

$$Q^2 = Q \qquad \text{and} \qquad P^2 = P\,.$$

By the last statement, each of these compositions can be considered as a projection in the corresponding complex. The first one, P, associates with each subset of the support of $\mathcal{F}$ the smallest $\mathcal{F}$-open set containing this set. The points in this open set cannot be separated from each other in the $\mathcal{F}$-topology $\mathcal{T}_{\mathcal{F}}$. Hence, the image of P precisely contains the *minimal* open sets.

On the other hand, Q maps a given simplex in the nerve to the biggest simplex in this complex containing the given one for which the intersection of the meters forming its vertices coincides with the intersection of the meters in the original simplex. Therefore, the image of Q are the *maximal* simplices defining the open sets in $\mathcal{T}_{\mathcal{F}}$.

Although obviously closely related, both simplicial complexes are actually quite different in general. This was already illustrated by the very simple examples above. Moreover, a look at the number of vertices alone may suffice. For the $\mathcal{F}$-complex this is nothing but the number of points in the support of $\mathcal{F}$, whereas for the nerve, the number of vertices is equal to the cardinality of $\mathcal{F}$ itself. Thus, in general, for the purpose of analysis one could and, in fact, should try to benefit as much as possible from having both structures at hand.

One obvious way to make use of these structures for the purpose of analysis is to consider the homology groups, Betti numbers, torsion coefficients, Euler characteristic, Poincare polynomial... This appears to be an interesting field of its own which will be treated elsewhere. Here we just include the following result implying that, from the point of view of $\mathbb{Z}_2$-homology, there is no distinction between the $\mathcal{F}$-complex and the nerve of $\mathcal{F}$.

PROPOSITION 3. *Let X be a piece and $\mathcal{F}$ a family of meters in X. Then, the $\mathbb{Z}_2$-homologies of the nerve $N_{\mathcal{F}}$ and the $\mathcal{F}$-complex $C_{\mathcal{F}}$ are isomorphic.*

Proof. Working with $\mathbb{Z}_2$-coefficients has the advantage that each chain can naturally be considered as a subcomplex of the given complex. It is formed by all those simplices having non-trivial coefficients (together with all their sides).

The argument itself is now based on the following observations using essentially the mappings $S\colon \mathrm{C}_{\mathcal{F}} \to \mathrm{N}_{\mathcal{F}}$ and $T\colon \mathrm{N}_{\mathcal{F}} \to \mathrm{C}_{\mathcal{F}}$ defined above.

1. For each of the integers $k \in \{0, \dots, \dim(\mathrm{C}_{\mathcal{F}})\}$ there is a homomorphism $\mathrm{S}_k\colon \mathrm{H}_k\ (\mathrm{C}_{\mathcal{F}}; \mathbb{Z}_2) \to \mathrm{H}_k(\mathrm{N}_{\mathcal{F}}; \mathbb{Z}_2)$ defined by associating with each homology class $[\sigma_1 + \dots + \sigma_r] \in \mathrm{H}_k(\mathrm{C}_{\mathcal{F}}; \mathbb{Z}_2)$ the unique class in $\mathrm{H}_k(\mathrm{N}_{\mathcal{F}}; \mathbb{Z}_2)$ represented by the k-skeleton of the subcomplex of $\mathrm{N}_{\mathcal{F}}$ generated by the simplices $S(\sigma_1), \dots, S(\sigma_r)$. To see this, note that:
   - For every k-cycle σ in $\mathrm{C}_{\mathcal{F}}$ the image $\mathrm{S}_k(\sigma)$ is at least k-dimensional.
   - The image of a cycle in $\mathrm{C}_{\mathcal{F}}$ is a cycle in $\mathrm{N}_{\mathcal{F}}$.
   - The image cycle is independent of the cycle chosen to represent the class in $\mathrm{H}_k(\mathrm{C}_{\mathcal{F}}; \mathbb{Z}_2)$.
   - S_k is a homomorphism.
2. Analogously, we have for each integer $l \in \{0, \dots, \dim(\mathrm{N}_{\mathcal{F}})\}$ a homomorphism $\mathrm{T}_l\colon \mathrm{H}_l(\mathrm{N}_{\mathcal{F}}; \mathbb{Z}_2) \to \mathrm{H}_l(\mathrm{C}_{\mathcal{F}}; \mathbb{Z}_2)$ mapping the homology class $[\tau_1 + \dots + \tau_s] \in \mathrm{H}_l(\mathrm{N}_{\mathcal{F}}; \mathbb{Z}_2)$ to the unique class in $\mathrm{H}_l(\mathrm{C}_{\mathcal{F}}; \mathbb{Z}_2)$ represented by the k-skeleton of the subcomplex of $\mathrm{C}_{\mathcal{F}}$ generated by the simplices $T(\tau_1), \dots, T(\tau_s)$.
3. By Lemma 3, both compositions $\mathrm{T}_k \circ \mathrm{S}_k$ and $\mathrm{S}_k \circ \mathrm{T}_k$ induce the identity on $\mathrm{H}_k(\mathrm{C}_{\mathcal{F}}; \mathbb{Z}_2)$ and $\mathrm{H}_k(\mathrm{N}_{\mathcal{F}}; \mathbb{Z}_2)$, respectively.

□

REFERENCES

[1] EPSTEIN, D.: *Beyond Orpheus — Studies in Musical Structure*, Oxford University Press, Oxford, 1987.

[2] FLEISCHER, A.—MAZZOLA, G.—NOLL, T.: *Computergestützte Musiktheorie*, Musiktheorie **4** (2000), 314–325.

[3] HAUSDORFF, F.: *Grundzüge der Mengenlehre*, Leipzig, 1914.

[4] LERDAHL, F.—JACKENDOFF, R.: *A Generative Theory of Tonal Music*, MIT Press, Cambridge, 1983.

[5] MAZZOLA, G.: *The Topos of Music*, Birkhäuser, Basel, 2002 (to appear).

[6] MAZZOLA, G.—ZAHORKA, O.: *Geometry and Logic of Musical Performance I, II, III*, SNSF Research Reports, Universität Zürich, 1993–1995.

[7] MAZZOLA, G.—ZAHORKA, O.: `www.encyclospace.org`.

[8] MONELLE, R.: *Linguistics and Semiotics of Music*, Harwood, Chur, 1992.

[9] NATTIEZ, J. J.: *Fondements d'une Sémiologie de la Musique*, Paris, 1975.

[10] NESTKE, A.: *Paradigmatic Motivic Analysis*, Electronic Bulletin Mexican Mathematical Society, Mexico City, 2002 (to appear).

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Department of Computer Science
Technische Universität
Berlin
Germany

E-mail: nestke@cs.tu-berlin.de
noll@cs.tu-berlin.de