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SURVEY OF SOME TOPICS OF HARMONIC ANALYSIS

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ABSTRACT. There is given a historical survey of some topics of harmonic analysis.

An important topic in classical physics is the analysis of periodic phenomena. Applied to the study of sound produced by vibrating strings, columns of air, a piece of metal or a stretched surface and the like, this study is very properly called harmonic analysis a sound being composed of a number of pure sounds or harmonics.

In acoustics, and in the physico-mathematical studies of Eighteenth and Nineteenth Centuries, one was concerned with determining these components (harmonic analysis) and also with reconstructing a musical tone from its components (harmonic synthesis). A similar program can be proposed for many physical phenomena that either are periodic or contain "hidden" periods: celestial orbits, electromagnetic radiation, terrestrial magnetism and so on.

In the hands of a great many analysts, this branch of analysis has developed over the past 200 years into the theory of Fourier series and integrals, a mathematical doctrine having an existence quite apart from its origins in physics, but still bearing the name of harmonic analysis.

A harmonic analysis was originally the study of Fourier series and Fourier integrals directed to functions of real variable the problem being for a given function to find the harmonics which constitute it (harmonic analysis) and to recover it starting from these harmonics (the harmonic synthesis). This aspect of the question did not stop to develop till our days.

The harmonic analysis was generalized under impulse of A. Well to the case of arbitrary locally compact commutative groups; the most remarkable is the theorem of duality of Pontriagin. The majority of theorems proved for

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the series and integrals of Fourier were extended to the case of arbitrary locally compact commutative group G; the functions $x \mapsto e^{inx}$ (n integer) of the case of Fourier series and $x \mapsto e^{iax}$ (a real) of the case of Fourier integrals are here the *characters* of G, or continuous morphisms of G into the multiplicative group of complex numbers of the module 1, and the Fourier transformation associates to a function f (submitted to certain restrictions) on G the function $\hat{f}$ on the set of characters:

$$\hat{f}(\chi) = \int \overline{\langle \chi, s \rangle} \cdot f(s)\, ds$$

(ds being a Haar measure).

Gelfand and his school had related harmonic analysis to the theory of commutative Banach algebras introduced by them for this aim: let $A = L^1(G)$ be the commutative Banach algebra formed by the integrable functions on G with, as multiplication, the convolution product; the characters of G are identified with the characters of A or morphisms of A into the algebra $\mathbb{C}$ of complex numbers, or also maximal regular ideals of A; and Fourier transformation is nothing else as the Gelfand transformation associating to every element f of A the function $\hat{f}$ on the set of characters of A: $\hat{f}(\chi) = \chi(f)$. This point of view makes possible to use the notions and the results purely algebraic.

Take, for example, the classic Wiener theorem according to which if a function f on the torus $\mathcal{T}$ is sum of Fourier series absolutely convergent and does not null, then $\frac{1}{f}$ is also sum of Fourier series absolutely convergent; the hypothesis is translated into the language of commutative Banach algebras saying that f belongs to a certain commutative Banach algebra A with a unit and does not belong to any maximal ideal of A; it follows (Krull theorem) that f does not belong to any ideal of A and therefore is invertible in A. In the same order of ideas the problem of the harmonic synthesis is related to the problem algebraic as follows: is given ideal of an algebra or of a ring intersection of maximal ideals? Let us mention some generalizations of the present theory, and mainly harmonic analysis non commutative, field extremely active of mathematics: as for applications of the theory mentioned here often non commutative — in quantum physics and arithmetics.

The very important in harmonic analysis is spectrum. The following text is based on [8]. In twentieth century mathematics the word "spectrum" occurs in many places and is used to describe a wide variety of apparently unrelated ideas. The evolution of the meaning of this word begins with the rainbow of light created by a prism and passes by way of integral equations and quantum mechanics to the space of prime ideals of a ring. Along the way one glimpses much of modern harmonic analysis and operator theory.

The spectrum of colours produced by a prism or diffraction grating can be regarded as the Fourier transform of the incoming waveform. In the 1860's

Maxwell showed that light was but a small part of a vast electromagnetic spectrum. The discovery of radio waves by Hertz and the use of X-ray diffraction in crystallography confirmed this and so the notion of spectrum was extended to this wider domain. Eventually the word spectrum was applied to the plot of amplitude against frequency obtained by taking the Fourier transform of any function for which it was defined.

An atom, when suitably excited, emits radiation whose spectrum consists of a number of "sharp lines" characteristic of the atom in question. In 1917 Bohr found that this phenomenon could be explained in terms of the quantum ideas introduced by Planck. In 1925 Heisenberg and Schrödinger independently incorporated Bohr's rules into a coherent quantum theory. Four years later von Neumann showed these theories to be equivalent. In von Neumann's formulation each physical observable is represented by a self-adjoint operator on a Hilbert space and it is the set of eigenvalues of the energy operator which determines Bohr's "stationary states" of the atom; the frequency spectrum of the atom is obtained by taking differences of eigenvalues. The spectrum of an operator T consists of those complex numbers λ such that $T - \lambda I$ does not possess an inverse. The word "spectrum" was already in use in this context in a 1926 paper on quantum mechanics by Born, Heisenberg and Jordan. The work of Hilbert on integral equations and quadratic forms in infinitely many variables allowed to treat line spectra and continuous spectra on an equal footing. Hilbert had defined the point spectrum and the continuous spectrum of a quadratic form in 1906 but later expressed surprise that his mathematical spectrum coincided with the physical spectrum.

A major result in Hilbert's work on integral equations is his theorem expressing a completely continuous quadratic form as an integral over the spectrum. This generalized the "Principal Axes Theorem" on the reduction of a finite quadratic form to a sum of squares and developed into the "Spectral Theorem" of von Neumann, Stone and others. One introduces a projection valued measure Π (which depends on the operator T), defined on the Borel sets of the spectrum and then expresses T as an integral $\int \lambda \, d\Pi$. The measure Π is an isomorphism between the Boolean lattice of Borel sets of the spectrum and a lattice of orthogonal projections on the Hilbert space. In this way the notion of spectrum can be transferred to measures (simply meaning the support of the measure) and, more significantly, to Boolean lattices.

That is what led Stone (in 1935) to consider Boolean lattices in general and to study their representations as lattices of subsets of a set. Stone observed that Boolean lattices are equivalent to Boolean rings (commutative rings in which every element is idempotent). The connection between the ring and lattice operations is given by $xy = x \wedge y$ and $x + y = (x \wedge y') \vee (x' \wedge y)$. In the case of the measure Π, the points of the spectrum correspond to the maximal ideals of the Boolean ring; hence the name "spectrum" for the set of maximal ideals of a general Boolean ring. Stone obtained his representation theorem for Boolean lattices by first passing to the Boolean ring and then associating each element to the set of maximal ideals which contain it. In another paper Stone introduced a topology on the spectrum by taking for each ideal the set of maximal ideals which contain it and calling these the closed sets.

These ideas were soon extended to the theory of commutative Banach algebras by Gelfand and his school. The ring of all continuous complex valued functions defined on a compact Hausdorff space X is a commutative Banach algebra $C(X)$ and the set of maximal ideals of $C(X)$, topologized as above, is homeomorphic to the original space X. Gelfand showed that, for any commutative Banach algebra B, this topology makes the set $\mathrm{Max}(B)$ of maximal ideals into a compact Hausdorff space and there is a natural homomorphism from B into $C(\mathrm{Max}(B))$. In general this map is neither one-to-one nor onto.

If T is a bounded self-adjoint operator on a Hilbert space, then T generates a commutative Banach algebra B. The space $X = \mathrm{Max}(B)$ is homeomorphic to the spectrum of T and B is isometrically *-isomorphic to $C(X)$. This X can be considered as a compact subspace of the real line and the element of $C(X)$ corresponding to T is simply the function $f(x) = x$. All these considerations extend to the case of a normal operator.

Around 1957 it was realized by Grothendieck and others that the passage from a commutative ring to its space of maximal ideals does not define a functor and that instead of maximal ideals one should study the prime ideals. The topology is defined as before. To each ideal of the ring one associates the set of all prime ideals which contain it: these are the closed sets. This is now called the Zariski topology and the set of all prime ideals of the ring A furnished with this topology is called the *spectrum*, $\mathrm{Spec}(A)$, of A.

One can go further and associate to each (basic) open set of $\mathrm{Spec}(A)$ a suitable localization of A in such a way that the resulting structure is a sheaf of rings. Ringed spaces of this form may be glued together to form "schemes". These are the objects of study in modern abstract algebraic geometry.

Commutative algebra generalizes both algebraic geometry and algebraic number theory and the theory of schemes is a further generalization. The study of schemes and sheaves led Grothendieck to the related notions of site (a category with a "topology") and topos (a special type of category generalizing a sheaf). It turns out that topoi represent a fusion of both geometry and logic and that the notion of spectrum can be generalized to this context. It can be thought of as the construction of a "free local ring" on a given ring object of the topos. In carrying out the construction one is forced to change to a new topos.

The development from the rainbow to topos theory covers most of the uses of "spectrum" within mathematics. There are of course other occurrences such as the "spectral sequences" and "spectra of spaces" of algebraic topology. These uses tend to conform to the common meaning of "a range of interrelated values, objects etc." and are not connected with eigenvalues of operators.

Methods of harmonic analysis can be applied in many situations, e.g., data transmission, multiplexing, filtering, image enhancement, pattern recognition etc. They are implemented on high speed computers and can be used with very little storage space.

Recall that listening to music, telephoning, watching of television transmission are such everyday actions that require a treatment of information by our sensory organs and our brain. These transformations of signals are mathematically analyzable: scientists which want to save, transmit, manipulate the signal decompose it to simple elements. Fourier analysis transforms it to sinusoidal waves, Walsh-Fourier analysis uses Walsh series. These days certain signals are transformed in wavelet. To obtain these results mathematicians developed theories whose aesthetics is comparable with results of other parts of mathematics.

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