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ALGEBRAIC VARIETIES OF MUSICAL PERFORMANCES

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ABSTRACT. In this article first, we give an overview, without any pretension of being exhaustive, of some theoretical tool of the modern theory of musical performance. Then, we present a mathematical model of performance which is based on the principles of mathematical music theory. We prove a structure theorem describing the algebraic variety of performance system parameters which induce a determined performance. This algebro-geometric model has inspired a statistical model that concretely answers a typical question about the way a player performs. Accordingly, a claim like "Horowitz has a more local way than Cortot to play Schumann's 'Träumerei'" has a strong statistical evidence.

Montez tous à la fois, océans irrités!
Astres, détachez-vous des cieux épouvantes!
Et vous, formes de l'être à jamais disparues,
Gigantesques débris que heurtaient les charrues,
Pressez-vous sous la terre et dans vos lit poudreux.
Faites-nous une place, ô frères monstrueux!

Les Fossiles, Louis Bouilhet.

1. Performance grammars

The idea that performance should be executed along certain regular patterns comes from the fact that music is viewed as a vehicle of contents, and that these contents are, by the very nature of musical semantics, hidden, difficult and ambiguous. In other words, the way they are expressed is an essential condition for their communication. At present, only very elementary grammatical patterns are known. We distinguish *rule based* approaches from *rule learning* procedures.

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For an exhaustive, comprehensive overview of performance research we refer to [6].

Essentially, three directions in rule based grammars are known: the KTH school, Neil Todd's approach and Guerino Mazzola's work. The terminology of Johan Sundberg and his school at KTH recovers the idea of basing performance on rules in analogy with the linguistic grammar. For the first attempts toward this goal see Mathis Lussy [7]. He proposed that musical expression is not something added to a work through performers' actions. Instead, it is inherent in the structure of music. Against this background, the KTH school has developed a system dominated by local rules which are based on low-level structural analysis of the text. Semantics of this analysis are rational and — to less degree — gestural. They are found by the characteristic empirical 'analysis by synthesis' method (see [6]).

Todd's approach (see [15], [16]) is backed by a systematic formalism of performance as a function of structure and specific grammatical arguments. It relates simple structural data, such as grouping boundaries, to the expression by means of physically oriented transformation rules.

The method of Mazzola and his collaborators is centered around the concept of *weight*. These are numerical functions that encode analyses and serve as an input to the core of grammatical instances: trees and operators. Accordingly, a performance generated by the tree, a genealogical stemma of nodes representing local performance scores of successively refines upon performance quality. The generation of such a node from antecedent ones, involves performance operators. The latter are charged with weights and realize grammatical rules of different flavours. The nature of these rules is not further specified, and may include any of the system proposed by other approaches as long as they are based upon weights (in particular the KTH and Todd proposals). We will develop these aspects in the next section.

Under the term "learning grammars" we understand grammatical patterns generated by machine-based algorithmic and statistical learning from empirical performance data. A group around Giovanni de Poli has investigated various means for modeling interaction effects. They combined parameters used by KTH rules with a neural network and made it learn from performances. Further, they considered another alternative involving fuzzy algorithms (see [4], [5]). Several other learning grammars have also been built to study music performance. An interesting approach using machine learning was used by Gerhard Widmer [17]. He lets an artificial intelligence system infer performance rules from measured performances. We refer to [18] for a discussion of new results in this areas. However we should note that these methods have severe limits since much of a performer's behaviour cannot be explained at the note level ([18], § 4.5), on which these methods are focused. That is why in the model presented in the next section, we will consider not only the "notes" by themselves, but their relations to the other parts of the piece. As we will see, even in our simple linear model, it is possible to answer coherently to important musicological questions about the comparison of several performances.

2. Analytical stemmas

In western classical music, a composition is usually analyzed in at least three ways, namely with respect to its melodic, rhythmic, and harmonic contents. The idea of quantitative analysis is therefore to find rhythmic, melodic and harmonic structures and to quantify them in weight functions in the sense that each note event x is given a weight which quantifies its "importance" with respect to the melodic, rhythmic, and harmonic structure of the composition, respectively. Clearly, methods for finding musically meaningful analytical structures must incorporate knowledge of music theory and practice. To illustrate this, we refer to the weight functions described in Mazzola et al. ([12], [10]). These weights are obtained by aggregating local and global features at different levels of details.

From the point of view of a composer or a performer this is a natural approach: Composition and performance are results of shaping processes at different levels of refinement. For instance, a performer may rehearse the general tempo of a piece as a whole and then refine the performance at increasingly detailed levels. Evidently, the actual sequence in which the various levels are emphasized during the rehearsing process varies individually. The ultimate result is an aggregation of all levels in one performance.

More precisely, the music score has a natural hierarchical structure. Namely, it is commonly decomposed into periods, each period into bars, each bar into chords and notes. If necessary one can even divide up periods into smaller intermediate pieces; the same holds for bars. In any case one can naturally produce a tree from each given score. The main idea of Mazzola's approach to a performance grammar (see [9]) is that this tree should canonically simulate the learning method of the interpreter: one first analyzes a period, then one refines the knowledge of the score through an analysis of its bars, and so on. At the end of this learning process the musician should produce the performance that the public listens to. Here we present a mathematical model of this process. It is based on a family of linear representations of a special type of quivers (directed multigraphs) which we call the *stemma quiver*.

Definition 1. The *stemma quiver* is a finite directed graph $T = (V, A)$ with the vertex set V and the arrow set A, including multiple arrows and loops. The construction is as follows. We start with a directed tree $T_0 = (V, A_0)$ with a root r, i.e., each vertex can be reached by a unique path starting from the root. If

$x \to y$ is an arrow of T_0, we say that x is the *mother* of y and y is a *daughter* of x (in combinatorics they are known as 'father' and 'son', respectively, here we try to be politically correct). If we have a path $x \to y \to z$, then z is a *granddaughter* of x, while x is the *grandmother* of z, and so on. For a vertex x of T_0, the set of vertices which are daughters of x is denoted by $D_x(T_0)$. The vertices x which are not mothers, i.e., $D_x(T_0) = \emptyset$, are called *the final* (the 'leaves' in graph theory). Similarly for each vertex $x \in V(T)$ we define $M_x(T_0) \subseteq V(T_0)$ as the set of vertices lying on the unique path from x to the root (x included).

In order to define the stemma quiver T the directed tree T_0 is enriched by the following set of arrows (no vertices added): First, each vertex x is given a loop $x \circlearrowleft$, and for any couple of sisters x_1, x_2, i.e., of daughters of a common mother y, we add an arrow $x_1 \to x_2$. The resulting quiver $T = (V, A)$ is called a stemma quiver of the tree T_0, which is uniquely determined by the stemma quiver and is called the stemma tree. We therefore may define $D_x(T) = D_x(T_0)$ and $M_x(T) = M_x(T_0)$.

Definition 2. With these graph-theoretical data, a $\mathbb{C}$-*stemma* is a family of $\mathbb{C}$-linear representations of a stemma quiver T which is defined by the following data:

(1) For each vertex $x \in V(T)$ let A_x, B_x be two vector spaces, B_x of finite dimension s_x.

(2) Each mother-daughter arrow $x \to y$ is represented by a surjective linear map $r_{x,y} \colon A_x \to A_y$.

(3) For all $x \in V(T)$ let $\varphi_x \colon B_x \to \mathrm{End}(A_x)$ be an affine map, i.e., a linear map φ_x^0, followed by a displacement by $\varphi_x^t \in \mathrm{End}(A_x)$. So each loop $x \circlearrowleft$ is represented by a family of endomorphisms $(\varphi_x(b) \colon A_x \to A_x)_{b \in B_x}$ parametrized by the *parameter space* B_x.

(4) For all pairs of sisters $x, y \in D_m(T)$, the sister arrow $x \to y$ is represented by an isomorphism $i_{x,y} \colon A_x \to A_y$ with $i_{y,x} = i_{x,y}^{-1}$.

Example 1. In stemma theory it is shown that a number of parameterization of theoretical interest [12] and of practical use in the RUBATO® software [13] can be subsumed under the method of Lie operators in the following sense.

To begin with, fix a number n of musical parameters. Each vertex x of the stemma tree T_0 is associated with a closed rectangle $R_x = [l_{x_1}, u_{x_1}] \times \cdots [l_{x_n}, u_{x_n}]$ where the music events of the score are placed. For each mother m and daughter x, we suppose that $R_x \subset R_m$, and for each couple of sisters x_1, x_2, $R_{x_1} \cap R_{x_2} = \emptyset$. This corresponds to a restriction of a larger portion of a musical score to a disjoint grouping of smaller portions. Here the above system ((1)–(4)) quiver representations is used:

Consider the vector space F_x of C^∞ functions on R_x. Then we set $A_x = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Der}(F_x)$, the space of complexified derivations, i.e., the C^∞ vector fields on R_x. In performance theory, such "performance fields" represent the deformations of the score data under a determined performance. They are the adequate generalization of tempo curves, the one-dimensional performance fields in time, see also [8]. The surjective maps $r_{x,y}$ are defined as the complexified restrictions of vector fields on the mother's rectangle R_x to the daughter's rectangle R_y. To define the representations for a sister arrow $x_1 \to x_2$, consider the unique affine morphism $a_{x_2,x_1}\colon R_{x_2} \to R_{x_1}$ on the sisters' rectangles such that the respective vertices are mapped onto each other. Then the sister arrow representations are the complexified isomorphisms $i_{x_1,x_2}\colon A_{x_1} \to A_{x_2}$ induced by the transport of a vector field Z on R_{x_1} to $Z \cdot a_{x_2,x_1}$ on R_{x_2}.

To define the operation of parameter family, first take a vector field $Z \in \mathrm{Der}(F_x)$, and a function $\Lambda \in F_x$. Denote by $\mathrm{Aff}(U,V)$ the vector space of affine morphisms from a vector space U to a vector space V. For $\mathrm{Dir} \in \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^n)$, consider the corresponding vector field $\Gamma_{\mathrm{Dir}}(t) = (t, \mathrm{Dir}(t))$ on R_x. Then we have a new vector field $Z - L_Z\Lambda \cdot \Gamma_{\mathrm{Dir}}$, where L is the Lie derivative. This is a $\mathbb{R}$-linear operator on Z, and the 'deformation' part $L_Z\Lambda \cdot \Gamma_{\mathrm{Dir}}$ is $\mathbb{R}$-bilinear in the function and the affine endomorphism.

Now we take a finite dimensional subspace W_x of F_x which in performance theory represents the weight functions issued from analyses of metrical, motivic, and harmonic structures of the given score. We now set $B_x = \mathbb{C} \otimes_{\mathbb{R}} W_x \otimes_{\mathbb{R}} \mathrm{Aff}(\mathbb{R}^n, \mathbb{R}^n)$, and we obtain a $\mathbb{C}$-linear map $\varphi_x^0\colon B_x \to \mathrm{End}(A_x)$ defined by $\varphi_x^0(\Lambda \otimes \mathrm{Dir})(Z) = -L_Z\Lambda \cdot \Gamma_{\mathrm{Dir}}$. Setting $\varphi_x^t = \mathrm{Id}_{A_x}$, we have defined the required affine map $\varphi_x = \varphi_x^t + \varphi_x^0\colon B_x \to \mathrm{End}(A_x)$ with

$$\varphi_x(\Lambda \otimes \mathrm{Dir})(Z) = Z - L_Z\Lambda \cdot \Gamma_{\mathrm{Dir}}.$$

This is a typical example to be memorized in the following abstract discussion of a stemma.

Let $m \in V(T)$ be a mother. Then for each daughter $x \in D_m(T)$, we define a triaffine (affine in each argument) map

$$f_x\colon A_m \times \mathbb{C}^{\# D_m(T)} \times \prod_{y \in D_m(T)} B_y \to A_x, \tag{1}$$

by

$$\begin{aligned}\big(a_m, (c_{y,x}^m)_{y\in D_m(T)}, (b_y)_{y \in D_m(T)}\big) &\mapsto \sum_{y \in D_m(T)} c_{y,x}^m i_{y,x}\Big(\varphi_y(b_y)\big(r_{m,y}(a_m)\big)\Big) \\ &= \sum_{y \in D_m(T)} c_{y,x}^m i_{y,x}\Big(\varphi_y^0(b_y)\big(r_{m,y}(a_m)\big)\Big) + \sum_{y \in D_m(T)} c_{y,x}^m i_{y,x}\Big(\varphi_y^t\big(r_{m,y}(a_m)\big)\Big).\end{aligned}$$

Referring to the above example, this formula describes the following: In order to determine the performance field on a frame F_x, we use the field of its mother m, first we restrict that field to any of its daughters y and we get the fields $r_{m,y}(a_m)$. These sister fields to daughter x are then deformed under the endomorphisms $\varphi_y(b_y) = \varphi_y^t + \varphi_y^0(b_y)$ induced by the system parameters b_y. These deformed fields are then transported to x and weighted by factors $c_{y,x}^m$. These sister fields influence the final value of the field at daughter x. Musically, this means that the performance at x is influenced by surrounding sister fields, which are typically the fields of past and future times (past or future periods, bars, etc.).

Technically speaking, we proceed as follows: For each final vertex $z \in V(T)$ let $m_{z0} = r$, m_{z1}, $m_{z2}, \dots, m_{zn_z} = z$, be the ordered sequence of elements of $M_z(T)$. For the root r and for each final vertex $z \in V(T)$ let us fix arbitrary $e_r \in A_r$, $e_z \in A_z$, respectively. The main task of *inverse performance theory* [9] is to study the set of solutions of the system of equalities

$$e_z = f_{m_{zn_z}}\Big(f_{m_{zn_z-1}}(\dots f_{m_{z1}}(e_r,(c^r_{y_1,m_{z1}}),(b_{y_1})),\dots),(c^{m_{zn_z-1}}_{y_{n_z},m_{zn_z}}),(b_{y_{n_z}})\Big), \tag{2}$$

where for $k = 1, \dots, n_z$, the vertices y_k lie in the set $D_{m_{zk-1}}(T)$.

For the explicit calculation, we select a basis $v_{x1}, \dots, v_{xs_x}$ of the vector space B_x. This means that for every vertex $x \in V(T)$ we may consider the linear operators $O_{xi} := \varphi_x^0(v_{xi}) \in \mathrm{End}(A_x)$, $i = 1, \dots, s_x$, as a part of our data. Whence, if we identify B_x with $\mathbb{C}^{s_x}$ through the basis $v_{x1}, \dots, v_{xs_x}$, we can define the homomorphism (1) as a triaffine map

$$f_x \colon A_m \times \mathbb{C}^{\# D_m(T)} \times \prod_{y \in D_m(T)} \mathbb{C}_y^{s_y} \to A_x$$

by associating the sum

$$\sum_{y\in D_m(T)} \sum_{j=1}^{s_y} c^m_{y,x} b_{yj}\, i_{y,x}\Big(O_{yj}(r_{m,y}(a_m))\Big) + \sum_{y\in D_m(T)} c^m_{y,x}\, i_{y,x}\Big(\varphi_y^t(r_{m,y}(a_m))\Big)$$

to $\big(a_m, (c^m_{y,x})_{y\in D_m(T)}, (b_{yj})_{y\in D_m(T),\, j=1,\dots,s_y}\big)$.

The equations (2) become:

$$\begin{aligned} e_z = &\sum c^{m_{zn_z-1}}_{y_{n_z},m_{zn_z}} b_{y_{n_z}j_{n_z}} \cdots c^r_{m_{y_1},z1} b_{y_1j_1} m_{z,y_1,\dots,y_{n_z},j_1,\dots,j_{n_z}} \\ &+ \dots + \sum c^{m_{zn_z-1}}_{y_{n_z},m_{zn_z}} \cdots c^r_{y_1,m_{z1}} m_{z,y_1,\dots,y_{n_z}}, \end{aligned} \tag{3}$$

where the sum is over all $y_1 \in D_r(T)$, $y_2 \in D_{m_{z1}}(T), \dots,$ $y_{n_z} \in D_{m_{zn_z-1}}(T)$ and all $j_k = 1, \dots, s_{y_k}$, for $k = 1, \dots, n_z$. The leading vector summand of this linear combination equals

$$\begin{aligned} &m_{z,y_1,\dots,y_{n_z},j_1,\dots,j_{n_z}} \\ &\quad = i_{y_{n_z},m_{zn_z}}\Big(O_{y_{n_z}j_{n_z}}(r_{m_{n_z-1},y_{n_z}}(\dots i_{y_1,m_{z1}}(O_{y_1j_1}(r_{r,y_1}(e_r)))\dots))\Big). \end{aligned} \tag{4}$$

The general vector summand refers to a choice of endomorphisms $X_{\ldots} = O_{y_{n_z} j_{n_z}}$ or $X_{\ldots} = \varphi^t_{y_{n_z}}$, etc. and is equal to

$$i_{y_{n_z}, m_{z n_z}} \Big(X_{\ldots} (r_{m_{n_z - 1}, y_{n_z}} (\ldots i_{y_1, m_{z1}} (X_{\ldots} (r_{r, y_1} (e_r))) \ldots))\Big) . \tag{5}$$

In order to describe the solution variety we can interpret (3) as a system of linear equations with parameters,

$$e_z = \sum_{y \in D_{m_{z, n_z - 1}}(T)} c^{m_{n_z - 1}}_{y,z} l_{yz} . \tag{6}$$

Observe that e_z, l_{yz} are vectors in A_z, so the solutions $c^{m_{n_z-1}}_{y,z}$ of equations (6) result from simultaneous solutions in the vectors' coordinates if we are given a basis of each A_z. The coefficients l_{yz} equal linear combinations of the vector summands (4) and (5) with coefficients which are monomials in the remaining variables $c^{\cdot}_{\ldots}$ and the $b_{\ldots}$.

Now if we assume that all variables $c^{m_{n_z-1}}_{zy}$ are independent, we see that for any two different $z \in D_{m_{n_z - 1}}(T)$, the solutions of the equations (6) do not interact. This means that the solution of the system (6) for $z \in D_{m_{n_z - 1}}(T)$ is either empty (if the value e_z is not in the image of the linear map (6)) or equals

$$\prod_{z \in D_{m_{n_z-1}}(T)} L_z \subseteq \prod_{z \in D_{m_{n_z-1}}(T)} \mathbb{C}^{\# D_{m_{n_z-1}}(T)},$$

where L_z are linear subspaces of $\mathbb{C}^{\# D_{m_{n_z-1}}(T)}$. Their codimension can be read form the matrix of the coefficients. Since there is no other condition on the other variables $c^{m_{zk}}_{y,x}$, we obtain altogether the following result:

Theorem 1. *The solution space of* (3) *is a linear fibration over some appropriate affine space.*

Corollary 1. *The dimension of the non-empty fibers of this fibration is generically minimal, and it increases along some finite union of proper algebraic subvarieties* (*the loci where the minors of the coefficients' matrices vanish*)*. This means that the non-empty fibers* $\mathrm{Fib}(e_{\cdot})$, *i.e., the solution spaces defined by* (2) *over a given output set* $e_{\cdot} = (e_z)_z$, *are all generically isomorphic, i.e., isomorphic when restricted to appropriate open subschemes. However, the configuration of the specialization subschemes is not evident and depends on the particular vector summands. Also, the condition for non-empty fibers is not evident in general.*

How can we interpret this result in musicological terms? A quantitative measurement of a performance can be done by recording the values of the parameters that characterize a note, i.e., loudness, duration, and so on. In the last paragraph we have denoted these data by the vectors e_z. Equation (3) explains that in order to produce a given values of the parameters from the weight functions, one has to find suitable values for all $c_{y,x}^{m_{zk}}$. It would be nice if the choice of this coefficients could be small, because this would mean that in order to produce a given performance one does not have a lot of freedom in the choice of system parameters. However, this in not the case from this model. In fact, Corollary 1 tells us that for a fixed performance one has either no one or, else, infinitely many possibilities to choose system parameters which produce the given performance. Moreover, we learn that all performances with non-empty fibers $\mathrm{Fib}(e_.)$ have open dense subsets $U(e_.) \subset \mathrm{Fib}(e_.)$ which are all isomorphic with each other, so these generic subsets $U(e_.)$ are qualitatively equivalent. *In other words, the non-empty fibers only differ from on their special loci apart from generic open subschemes $U(e_.)$.*

In musical terms, a fiber $\mathrm{Fib}(e_.)$ could be called a *critical fiber* because its points are the possible background parameters — in the present stemmatic model — which lead to the given performance output data $e_.$, such as local tempi, articulations, dynamics, detunings, etc. So the fiber really includes the possible ways of understanding why a performer is playing his actual performance. In fact, finding out which parameters the interpreter could have used is (or should be) the core activity of a music critic. Having generically isomorphic fibers means that in any two critical fibers of two given performances, there are "dominant" open sets of "criticisms" (i.e., points in the fiber!) which are isomorphic with each other. This does not mean that the criticisms are the same for the two given performances, but that their structural contexts my be identified. Again, this does not mean that the relevant criticisms may be identified — on the contrary: maybe these isomorphic contexts just describe the criticisms which are what everybody could say if no supplementary information about the specific performance culture of the interpreter is known.

So we should not discard this model as insignificant, instead we need to look at it in the right way. In fact, it is like pretending to explain the geometry of smooth plane curves, just using lines. The theory of lines can be useful for local questions, but not for global ones. The same thing happens here. The great flexibility of this model enables us to adapt it to the questions that one tries to answer, but they can not be too general. Indeed, one can select subschemes that are appropriate for the study of a particular problem. For example, if one is interested in comparing several performers under the point of view of the local/global way of playing, one could try to restrict the research just at the level of the daughters of the root, and to use some more of the structures of the vector space A_x (if any). An example of this procedure will be presented in the next section. Another question that could be asked is that of the final/causal way of playing. This requires to impose mutual dependence conditions upon the variables $c_{y,x}^{m_{zk}}$ for varying $z \in D_{m_{zk}}$ (see [12]).

3. Consequence for the choice of statistical models

An increasing number of numerical data is becoming available so a statistical approach to answering these questions is feasible, and also necessary in order to cope with the large amount of data. Two particular classes of statistical methods, which are particularly suitable for performance analysis, have been introduced in the papers [1], [2], and [3]. Let us describe them briefly:

Approach 1. Decompose the weight functions w into components w_i that correspond to different levels ($i = 1, 2, 3, \ldots$) of refinement. Fit a model of the form $y(s_i) = g(s_i, w_1, w_2, \ldots) + \varepsilon_i$, where ε_i is a suitable zero mean process (see [1], [2]).

Approach 2. Decompose not only w but also the tempo series y into subseries $y_1, y_2, \ldots$ that corresponds to different levels of "zooming-in". For each level, fit a model of the form $y_j(s_i) = g(s_i, w_j) + \varepsilon_{ij}$. Then the final model is $y_j(s_i) = \sum_j g(s_i, w_j) + \varepsilon_{ij}$ (see [3]).

The first approach has been encoded in the so-called HISMOOTH models ([1], [2]) and formalizes the idea that a musical performance combines local and global structures in a final as well as a causal way. This is done in a way that may be interpreted as complementary to the rooted tree introduced in Section 2. Instead of looking for deep and finer structures of the explanatory time series x, one decomposes it 'horizontally' as a sum of simple pieces. This however has the advantage to provide a method allowing the classification of performances. In fact, the estimated bandwidth values b_{rj} can be used for clustering ([1], § 5). Whence the claims like the fact that the μ-curve emphasizes features that may correspond to personal style, or the fact that Horowitz plays more 'locally' than Cortot (at least what concerns Schumann's "Träumerei") are now supported by strong statistical evidences ([1], § 6.3).

The second approach can be realized by studying an alternative to HISMOOTH models, called HIWAVE models (see [3]), obtained by using wavelet smoothing. The idea now is that one can use the information of hierarchical structure defined in 2, and assume that y is related to several "layers" of the weight function constructed by classifying the coefficients in the wavelet decomposition of the weight function into different bands of magnitude. It is interesting to note that the special shape of the rooted tree 2 should be useful for the choice of the family of wavelets that are more appropriate for the quantitative investigations. We plan to pursue this research in a subsequent work.

REFERENCES

[1] BERAN, J.—MAZZOLA, G.: *Visualizing the relationship between two time series by hierarchical smoothing models*, J. Comput. Graph. Statist. **8** (1999), 213–238.

[2] BERAN, J.—MAZZOLA, G.: *Analyzing musical structure and performance — A statistical approach*, Statist. Sci. **14** (1999), 47–79.

[3] BERAN, J.—MAZZOLA, G.: *Musical composition and performance — statistical decomposition and interpretation.* Preprint, 2001.

[4] BERESIN, R.: *MELODIA: a program for performance rules testing, teaching, and piano scores performing*, in: Proceedings of the X Italian Colloquium on Music Informatics (CIM), AIMI, Padova, 1993.

[5] BERESIN, R.—VECCHIO, C.: *Analysis and synthesis of the performing action of a real pianist by means of artificial neural networks*, in: Proceedings of the 3rd International Conference of Music Perception and Cognition (I. Deliège, ed.), Liège, 1994, pp. 353–354.

[6] GABRIELSSON, A.: *Music Performance*, in: The Psychology of Music (2nd ed.) (D. Deutsch, ed.), Academic Press, San Diego, 1999.

[7] LUSSY, M.: *Traité de l'expression musicale — accents, nuances et mouvements*, Huegel, Paris, 1874.

[8] MAZZOLA, G.—ZAHORKA, O.: *Tempo curves revisited: hierarchies of performance fields*, Comput. Music J. **18** (1994).

[9] MAZZOLA, G.: *Inverse Performance Theory*, in: Proceedings of the ICMC 1995, ICMA, Banff, 1995.

[10] MAZZOLA, G.: *Topos of Music*, Birkhäuser, Basel 2001, (to appear).

[11] MAZZOLA, G.—ZAHORKA, O.: *Tempo curves revisited: hierarchies of performance fields*, Comp. Music J. **18** (1994), 40–52.

[12] MAZZOLA, G.—ZAHORKA, O.: *Geometry and Logic of Musical Performance I, II, III. SNSF Research Reports (469pp.)*, Universität Zürich, Zürich, 1993–1995.

[13] MAZZOLA, G.—ZAHORKA, O.: *RUBATO on the Internet.* `http://www.ifi.unizh.ch/groups/mml/musicmedia`, Univ. Zürich, Zürich, 1996.

[14] MATOUŠEK, J.—NEŠETŘIL, J.: *Invitation to Discrete Mathematics*, Clarendon Press, Oxford, 1998.

[15] TODD, M.: *Toward a cognitive theory of expression: The performance and perception of rubato*, Contemp. Music Review **4** (1989), 405–416.

[16] TODD, M.: *The dynamics of dynamics: A model of musical expression*, J. Acoust. Soc. Amer. **91** (1992), 3540–3550.

[17] WIDMER, G.: *Learning Expression at Multiple Structural Levels*, in: Proceedings of the 1994 International Computer Music Conference, pp. 95–101.

[18] WIDMER, G.: *Using AI and Machine Learning to Study Expressive Music Performance*, Project Survey and First Report. Preprint, 2001.

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