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SCORZA DRAGONI TYPE THEOREMS

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ABSTRACT. In 1994 Z. Grande proved that Scorza Dragoni theorem does not hold for the density topology and proved some special forms of this theorem [Z. Grande: *On Scorza Dragoni's property for the density topology*, Real Anal. Exchange **20** (1994–1995), 286–290]. We show that his method can be applied in more general cases.

1. Main result

We assume that X is a nonempty set and $\mathscr{A}$ is a nonempty family of subsets of X. Moreover let I be a nonempty set, $\mathscr{T}$ and $\mathscr{I}$ be families of subsets of I, and let (Y, ϱ) be a metric space.

Definition 1. We say that a function $f\colon X \to Y$ has property $(\mathcal{A})$ at a point $x \in X$, if

($\mathcal{A}$) for each $\varepsilon > 0$, there is a set $A \in \mathscr{A}$ containing x such that $f(A) \subset K\bigl(f(x), \varepsilon\bigr)$.

Theorem 1. *Suppose that*

(1) *for every family* $\{U_s\}_{s\in S} \subset \mathscr{A}$ *with* $X = \bigcup_{s\in S} U_s$, *there is a sequence* $(s_n) \subset S$ *such that* $U \cap \bigcup_{n\in\mathbb{N}} U_{s_n} \neq \emptyset$ *for every* $U \in \mathscr{A}$.

Assume that $f\colon I \times X \to Y$ *is a function such that all sections* f_t, $t \in I$, *have the equal property* $(\mathcal{A})$ *at every point* $x \in X$, *i.e.,*

(2) *for all* $x \in X$ *and* $\varepsilon > 0$, *there is a set* $A \in \mathscr{A}$ *containing* x *such that* $f_t(A) \subset K\bigl(f_t(x), \varepsilon\bigr)$ *for all* $t \in I$,

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and that

($\mathcal{B}$) *for every* $(u_m) \subset X$, *there exists a set* $B \in \mathscr{I}$ *with the property that for all* $m \in \mathbb{N}$, $t \in I \setminus B$, *and* $\varepsilon > 0$, *there is a set* $J \in \mathscr{T}$ *containing* t *such that* $f^{u_m}(J \setminus B) \subset K\big(f^{u_m}(t), \varepsilon\big)$.

Then there exists a set $B \in \mathscr{I}$ *with the property that for every point* $(t,u) \in (I \setminus B) \times X$ *and every* $\varepsilon > 0$, *there are sets* $J \in \mathscr{T}$ *and* $A \in \mathscr{A}$ *such that* $(t,u) \in J \times A$ *and* $f\big((J \setminus B) \times A\big) \subset K\big(f(t,u), \varepsilon\big)$.

P r o o f. Fix $n \in \mathbb{N}$. For every $u \in X$, by (2), there exists a set $U_{n,u} \in \mathscr{A}$ containing u such that

$$\varrho\big(f(t,u), f(t,x)\big) < \frac{1}{4n} \qquad \text{for all } t \in I \text{ and } x \in U_{n,u}\,.$$

Clearly $X = \bigcup_{u \in X} U_{n,u}$. By (1), there is a sequence $(u_{n,k}) \subset X$ such that $U \cap \bigcup_{k=1}^{\infty} U_{n,u_{n,k}} \neq \emptyset$ for every $U \in \mathscr{A}$. Put $U_{n,k} = U_{n,u_{n,k}}$ $(k \in \mathbb{N})$. Observe that

(3) $\varrho\big(f(t,u_{n,k}), f(t,x)\big) < \frac{1}{4n}$ for all $t \in I$ and $x \in U_{n,k}$.

By ($\mathcal{B}$), we can find a set $B \in \mathscr{I}$ with the property that

(4) for all $n,k \in \mathbb{N}$, $t \in I \setminus B$, and each $\varepsilon > 0$, there is a set $J \in \mathscr{T}$ containing t such that $\varrho\big(f(t,u_{n,k}), f(v,u_{n,k})\big) < \varepsilon$ for each $v \in J \setminus B$.

Fix a point $(t,u) \in (I \setminus B) \times X$ and $\varepsilon > 0$. Choose an $n \in \mathbb{N}$ with $n > 2/\varepsilon$. We consider two cases.

CASE 1. First assume that $u \in U_{n,k}$ for some $k \in \mathbb{N}$. By (4), there is a set $J \in \mathscr{T}$ such that

(5) $J \ni t$ and $\varrho\big(f(t,u_{n,k}), f(v,u_{n,k})\big) < \frac{\varepsilon}{2}$ for each $v \in J \setminus B$.

Then by (3), for each $(v,x) \in (J \setminus B) \times U_{n,k}$ we have

$$\begin{aligned}
&\varrho\big(f(t,u), f(v,x)\big)\\
&\quad \le \varrho\big(f(t,u), f(t,u_{n,k})\big) + \varrho\big(f(t,u_{n,k}), f(v,u_{n,k})\big) + \varrho\big(f(v,u_{n,k}), f(v,x)\big)\\
&\qquad\qquad < \frac{1}{4n} + \frac{\varepsilon}{2} + \frac{1}{4n} < \frac{\varepsilon}{8} + \frac{\varepsilon}{2} + \frac{\varepsilon}{8} = \frac{3\varepsilon}{4} < \varepsilon\,.
\end{aligned}$$

CASE 2. Now let $u \notin \bigcup_{k \in \mathbb{N}} U_{n,k}$. By (2), there is a set $A \in \mathscr{A}$ containing u such that

$$\varrho\big(f(t,u), f(t,x)\big) < \frac{\varepsilon}{12} \qquad \text{for all } t \in I \text{ and } x \in A\,.$$

Let $k \in \mathbb{N}$ be such that $A \cap U_{n,k} \neq \emptyset$, and choose a $y \in A \cap U_{n,k}$. Choose a set $J \in \mathscr{T}$ so that condition (5) holds. Proceeding as in the previous case we can show that

$$\varrho\big(f(t,y), f(v,y)\big) < \frac{3\varepsilon}{4} \qquad \text{for each} \quad v \in J \setminus B\,.$$

Consequently, for each $(v,x) \in (J \setminus B) \times A$ we have

$$\begin{aligned} &\varrho\big(f(t,u), f(v,x)\big) \\ &\le \varrho\big(f(t,u), f(t,y)\big) + \varrho\big(f(t,y), f(v,y)\big) + \varrho\big(f(v,y), f(v,u)\big) + \varrho\big(f(v,u), f(v,x)\big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad < \frac{\varepsilon}{12} + \frac{3\varepsilon}{4} + \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \varepsilon\,. \end{aligned}$$

□

2. Corollaries

The following theorems are corollaries of Theorem 1. The first two were proved by Z. Grande [2], while Theorems 4 and 5 are new.

Theorem 2. *Let (X, T_X) be a topological space such that for every T_X-open cover of X, $\{U_s\}_{s\in S}$, there is a sequence $(s_n) \subset S$ such that $\bigcup_{n\in\mathbb{N}} U_{s_n}$ is dense in X.*

Assume $I \subset \mathbb{R}$ is an interval, (Y, ϱ) is a metric space and $f\colon I \times X \to Y$ is a function such that all sections f^x, $x \in X$, are Lebesgue measurable and all sections f_t, $t \in I$, are T_X-equicontinuous at every point $x \in X$. Then for every $\varepsilon > 0$ there is a closed set $K \subset I$ such that $\mu(I \setminus K) < \varepsilon$ and the restriction $f \restriction (K \times X)$ is $(T_e \restriction K) \times T_X$-continuous.

Theorem 3. *Let (X, T_X) and I be as in Theorem 2, and let (Y, ϱ) be a separable metric space. Assume that $f\colon I \times X \to Y$ is a function such that all sections f^x, $x \in X$, have the Baire property and all sections f_t, $t \in I$, are T_X-equicontinuous at every point $x \in X$. Then there exists a residual set $K \subset I$ such that the restriction $f \restriction (K \times X)$ is $(T_e \restriction K) \times T_X$-continuous.*

Theorem 4. *Let (X, T_X) be a topological space such that for every T_X-semi-open cover of X, $\{U_s\}_{s\in S}$, there is a sequence $(s_n) \subset S$ such that $\bigcup_{n\in\mathbb{N}} U_{s_n}$ is dense in X, and let I and (Y, ϱ) be as in Theorem 2. Assume that $f\colon I \times X \to Y$ is a function such that all sections f^x, $x \in X$, are Lebesgue measurable and all sections f_t, $t \in I$, are T_X-equi-quasi-continuous at every point $x \in X$. Then for every $\varepsilon > 0$ there exists a closed set $K \subset I$ such that $\mu(I \setminus K) < \varepsilon$ and the restriction $f \restriction (K \times X)$ is $(T_e \restriction K) \times T_X$-quasi-continuous.*

THEOREM 5. *Let (X, T_X) and I be as in Theorem 4, and let (Y, ϱ) be a separable metric space. Assume that $f\colon I \times X \to Y$ is a function such that all sections f^x, $x \in X$, have the Baire property and all sections f_t, $t \in I$, are T_X-equi-quasi-continuous at every point $x \in X$. Then there is a residual set $K \subset I$ such that the restriction $f \restriction (K \times X)$ is $(T_e \restriction K) \times T_X$-quasi-continuous.*

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