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A HELLY THEOREM IN THE SETTING OF BANACH SPACES

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. There is given an extension of the classical Helly and Helly-Bray theorem to functions with values in Banach spaces. An application is given to a moment problem. From this, a representation theorem for weakly compact mappings of the space of continuous functions into a Banach space is derived.

1. Introduction

Helly's theorem had been of some importance a long time above all in the probability theory in connection with a problem of moments of distributions.

Recall that in this connection, real-valued nondecreasing functions f on the interval $[a,b]$ of the real line are considered and that the following facts are true.

(1) (First Helly's theorem) *Given a uniformly bounded sequence (f_n) of real-valued nondecreasing functions, there exists a subsequence (f_{n_k}) of (f_n) converging to a real-valued nondecreasing function f on $[a,b]$.*

(2) (Second Helly's theorem) *Given a sequence (f_n) of real-valued nondecreasing functions on $[a,b]$, converging to a real-valued nondecreasing function f, then, for every continuous function g on $[a,b]$, we have*

$$\lim_{n\to\infty}\int_a^b g(t)\,df_n(t)=\int_a^b g(t)\,df(t)\,.$$

We extend these theorems to normed space valued functions satisfying some weak compactness conditions. Then an application to a certain moment problem

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in Banach spaces is given. From this, as an application, we obtain a well-known representation theorem for weakly compact mappings of the space of continuous functions into a Banach space.

It can be shown that moment problem theorem is equivalent to a representation theorem.

2. Helly and Helly-Bray theorems

Let X be a Banach space and g a function on an interval $[a,b]$ with values in X. We denote by X' the topological dual of X.

We recall now some definitions and facts (see [5]).

Definition 1. We will say that g has *bounded semi-variation* if the set

$$V_g = \left\{ \sum_{j=1}^{k} \alpha_j \big(g(t_j) - g(t_{j-1})\big) \colon k \in \mathbb{N}^*,\ a = t_0 < t_1 < \cdots < t_k = b,\ \left|\alpha_j\right| \le 1 \right\}$$

is a bounded set in X.

Proposition 1. *A function g has bounded semi-variation if and only if elements of the set $\{x' \circ g \colon x' \in X',\ \|x'\| \le 1\}$ have uniformly bounded variations.*

P r o o f. a) Let us suppose that g has bounded semi-variation and let $\|x'\| \le 1$.

For any division $a = t_0 < t_1 < \cdots < t_k = b$ of interval $[a,b]$ and for any j, there exists α_j such that $|\alpha_j| = 1$ and $\left|x'\big(g(t_j) - g(t_{j-1})\big)\right| = \alpha_j x'\big(g(t_j) - g(t_{j-1})\big)$.

It follows that

$$\sum_{j=1}^{k} \left| x'\big(g(t_j) - g(t_{j-1})\big) \right| = x'\left(\sum_{j=1}^{k} \alpha_j \big(g(t_j) - g(t_{j-1})\big) \right) \le \|x'\| K \le K\,,$$

where K is a bound for the norm of elements of V_g.

b) We now suppose that the elements of the set $\{x' \circ g \colon x' \in X', \|x'\| \le 1\}$ have uniformly bounded variations. We have

$$\left\| \sum_{j=1}^{k} \alpha_j \big(g(t_j) - g(t_{j-1})\big) \right\| = \sup_{\|x'\| \le 1} \left| x'\left(\sum_{j=1}^{k} \alpha_j \big(g(t_j) - g(t_{j-1})\big) \right) \right|,$$

and

$$\left| x'\left(\sum_{j=1}^{k} \alpha_j \big(g(t_j) - g(t_{j-1})\big)\right)\right| = \left|\sum_{j=1}^{k} \alpha_j x'\big(g(t_j) - g(t_{j-1})\big)\right|$$
$$\leq \sum_{j=1}^{k} \left| x'\big(g(t_j) - g(t_{j-1})\big)\right|$$

and the last sum is bounded by a constant independent of x'. □

The following definitions are rather classical. We recall it for sake of completeness.

Definition 2. We will say that a real-valued function f defined on interval $[a,b]$ has a *Riemann-Stieltjes integral with respect to* g if there exists an element $A \in X$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any division $\{t_0, t_1, \dots, t_k\}$, $a = t_0 < t_1 < \dots < t_k = b$ of the interval $[a,b]$ such that $|t_j - t_{j-1}| < \delta$, and for any choice of s_j with $t_{j-1} < s_j < t_j$ we have

$$\left\| A - \sum_{j=1}^{k} f(s_j)\big(g(t_j) - g(t_{j-1})\big)\right\| \leq \varepsilon .$$

In that case, the element A will be unique and denoted by $\int f\,dg$.

We will also denote by $t(D)$ the maximum of $|t_j - t_{j-1}|$.

Definition 3. We will call a *repartition of points* in the interval $[a,b]$ a division $\{t_0, t_1, \dots, t_k\}$ with $a = t_0 < t_1 < \dots < t_k = b$ and a choice of points (s_j) such that $t_{j-1} < s_j < t_j$ for $0 \leq j \leq k$.

If D is a repartition of points in the interval $[a,b]$, then we define $s_D = \sum_{j=1}^{k} f(s_j)\big(g(t_j) - g(t_{j-1})\big)$.

Proposition 2. *If f is a continuous real-valued function defined on $[a,b]$ and if $g\colon [a,b] \to X$ has bounded semi-variation, then the Riemann-Stieltjes integral $\int f\,dg$ exists.*

Proof. (see [5]) By continuity of f, for fixed $\varepsilon > 0$, there exists $\delta > 0$ such that $|t - t'| < \delta \Rightarrow \big|f(t) - f(t')\big| < \varepsilon$.

Let D and D' be two repartitions of points in the interval $[a,b]$ such that D is a refinement of D' and $t(D) < \delta$. If D is defined by points t_j, s_j and D' by points t'_r, s'_r, for each j, the interval $\big[t_{j-1}, t_j\big]$ is included in some interval $\big[t'_{r-1}, t'_r\big]$. In that case we let $s''_j = s'_r$. We have

$$s_D = \sum_{j=1}^{k} f(s_j)\big(g(t_j) - g(t_{j-1})\big)$$

and

$$s_{D'} = \sum_{r=1}^{k'} f(s'_r)\big(g(t'_r) - g(t'_{r-1})\big) = \sum_{j=1}^{k} f(s''_j)\big(g(t_j) - g(t_{j-1})\big) .$$

It follows that

$$\|s_D - s_{D'}\| = \left\| \sum_{j=1}^{k} \big(f(s_j) - f(s''_j)\big)\big(g(t_j) - g(t_{j-1})\big) \right\| \le \varepsilon K$$

which shows that the net $(s_D)_{D\in\mathcal{D}}$ is Cauchy in X. Its limit obviously satisfies the condition of the Riemann-Stieltjes integral. □

DEFINITION 4. We will say that g has *weakly compact semi-variation* if g has bounded semi-variation and if the set V_g is included in a weakly compact set W of X.

In [3], functions of weakly compact semivariation are called functions of the Sirvint's class.

It is clear that, when g has weakly compact semi-variation, the weakly compact set W may be supposed to be absolutely convex.

Moreover, for any continuous function $f\colon [a,b] \to \mathbb{R}$, the net $(s_D)_{D\in\mathcal{D}}$ defined in the proof of the previous theorem is such that $s_D \in \|f\| V_g$. It follows that $\int f\,dg \in \|f\| W$ if $V_g \subseteq W$.

PROPOSITION 3. *If $V_g \subseteq W$, then $g\big([a,b]\big) \subseteq W + g(a)$.*

P r o o f . Trivial. □

DEFINITION 5. A sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly of bounded (resp. weakly compact) semi-variation if there exists a bounded (resp. weakly compact) set containing V_{g_n} for any n.

The following proposition is proved in the same way as Proposition 1.

PROPOSITION 4. *Elements of the sequence $(g_n)_{n\in\mathbb{N}}$ are uniformly of bounded semi-variation if and only if the elements of the set $\{x' \circ g_n \colon \|x'\| \le 1,\ n\in\mathbb{N}\}$ have uniformly bounded variations.*

THEOREM 5. *Let X be a Banach space and $(g_n)_{n\in\mathbb{N}}$ a sequence of functions from $[a,b]$ into X such that*

a) the sequence $\big(g_n(a)\big)_{n\in\mathbb{N}}$ is weakly relatively compact,

b) $(g_n)_{n\in\mathbb{N}}$ is uniformly of weakly compact semi-variation.

Then there exists a subsequence of the sequence $(g_n)_{n\in\mathbb{N}}$ weakly converging to a function $g\colon [a,b] \to X$ of weakly compact semi-variation with the same weakly compact subset as the sequence $(g_n)_{n\in\mathbb{N}}$.

P r o o f . As each function of the sequence $(g_n)_{n\in\mathbb{N}}$ is of weakly compact semivariation, by a result of [3], these functions have separable range. So, we may suppose, without lost of generality, that the space X is separable.

In that case, we can choose a total sequence $(x'_k)_{k\in\mathbb{N}}$ in the dual space X'.

By the classical Helly's first theorem (see [8]), and using the classical diagonal process, we can pick a subsequence $(g_{n_i})_{i\in\mathbb{N}}$ of the sequence $(g_n)_{n\in\mathbb{N}}$ such that for all $k\in\mathbb{N}$, the sequence $\big(x'_k\circ g_{n_i}(t)\big)_{i\in\mathbb{N}}$ converges to some $g_{x'_k}(t)$ for every $t\in[a,b]$.

For each $t\in[a,b]$, the sequence $\big(g_{n_i}(t)\big)_{i\in\mathbb{N}}$ is weakly relatively compact, so it has a subsequence weakly converging to some element. Let us denote such an element by g_t.

As the sequence $\big(x'_k\circ g_{n_i}(t)\big)_{i\in\mathbb{N}}$ converges to $g_{x'_k}(t)$, we have $g_{x'_k}(t)=x'_k(g_t)$. Since the sequence $(x'_k)_{k\in\mathbb{N}}$ is total, the sequence $\big(g_{n_i}(t)\big)_{i\in\mathbb{N}}$ weakly converges to g_t, which we shall write $g(t)$. This defines the limit function g.

Let us denote by W a weakly compact set containing V_{g_n} for each n.

As $\sum\limits_{j=1}^{r}\alpha_j\big(g(t_j)-g(t_{j-1})\big)$ is the weak limit of $\Big(\sum\limits_{j=1}^{r}\alpha_j\big(g_{n_i}(t_j)-g_{n_i}(t_{j-1})\big)\Big)_{i\in\mathbb{N}}$, we have $V_g\subseteq W$ which shows that g is of weakly compact semivariation with the same weakly compact subset as the sequence $(g_n)_{n\in\mathbb{N}}$. □

THEOREM 6. (Helly-Bray Theorem). *If the sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly of weakly compact semi-variation, then there exists a subsequence (g_{n_k}) of (g_n) weakly converging on a given dense subset D of $[a,b]$ to a function g of weakly compact semi-variation, such that for any continuous function $f\colon[a,b]\to\mathbb{R}$, we have*

$$\lim_{n\to\infty}\int f\,dg_{n_k}=\int f\,dg\,.$$

P r o o f . Without loss of generality, we may suppose that $g_n(a)=0$. From Theorem 6, there exists a subsequence (g_{n_k}) of (g_n) weakly converging on D to a function g of weakly compact semi-variation.

By Proposition 1, $\int f\,dg_{n_k}$ and $\int f\,dg$ exist and are elements of $\|f\|W$ for some weakly compact set W.

It follows that the sequence $\Big(\int f\,dg_{n_k}\Big)_{k\in\mathbb{N}}$ has a weakly converging subsequence. As $\lim\limits_{k\to\infty}x'\Big(\int f\,dg_{n_k}\Big)=x'\Big(\int f\,dg\Big)$ by the classical Helly-Bray theorem, the subsequence of $\Big(\int f\,dg_{n_k}\Big)_{k\in\mathbb{N}}$ has to converge weakly to $\int f\,dg$. □

3. Application to Moment Problem

DEFINITION 6. A sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach space X is a *moment sequence* if there exists a function $f\colon [0,1] \to X$, with bounded semivariation, such that for all k, we have:

$$a_k = \int_0^1 t^k \, df(t).$$

DEFINITION 7. We say that a sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach space X is completely weakly compact if there exists a weakly compact set W in X such that

$$\sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k \in W, \qquad n = 0, 1, \dots,$$

$$\Delta^n a_k = \sum_{j=0}^{n} (-1)^j C_n^j a_{k+j},$$

where $|\alpha_k^n| \le 1$ are complex numbers.

In that case, it is obvious that there exists a constant M such that

$$\left\| \sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k \right\| \le M, \qquad \text{for } n \in \mathbb{N}.$$

THEOREM 7. (Moment theorem). *A sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach space X is the moment sequence of a function f of a weakly compact semivariation if and only if the sequence $(a_k)_{k\in\mathbb{N}}$ is completely weakly compact.*

P r o o f . Let us suppose that f is of weakly compact semivariation, and denote by W the weakly compact set associated to f. If the sequence (a_k) is the moment sequence of f, then the sequence (a_k) is completely weakly compact. In fact, if we define $x^{(n)}(t) = \sum\limits_{k=0}^{n} C_n^k \alpha_k^n t^k (1-t)^{n-k}$ for a fixed family of complex numbers (α_k^n) such that $|\alpha_k^n| \le 1$, then, by the remark after Definition 4, we have

$$|x^n(t)| = \left| \sum_{k=0}^{n} C_n^k \alpha_k^n t^k (1-t)^{n-k} \right| \le 1$$

and

$$\sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k = \int_0^1 x^n(t) \, df(t) \in W.$$

Conversely, for $n = 1, 2, \dots$, let us define a function $f_n\colon [0,1] \to X$ by the formula:

$$f_n(x) = \sum_{j=0}^{m-1} \binom{n}{j} \Delta^{n-j} a_j \qquad \text{for} \quad x \in \left(\frac{m-1}{n}, \frac{m}{n} \right),$$

$$f_n\left(\frac{m}{n}\right) = \sum_{j=0}^{m} \binom{n}{j} \Delta^{n-j} a_j\,, \qquad f_n(0) = 0\,, \quad f_n(1) = a_0\,.$$

The functions f_n are uniformly of weakly compact semivariation. This is an easy consequence of the fact that the sequence $(a_k)_{k\in\mathbb{N}}$ is completely weakly compact. If we define the operator Λ on the space of polynomials by

$$\Lambda\left(\sum_{j=0}^{n} c_j x^j \right) = \sum_{j=0}^{n} c_j a_j\,,$$

it is clear that the Bernstein polynomials

$$B_{k,n}(x) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{j}{n}\right)^k x^j (1-x)^{n-j}$$

verify

$$\Lambda(B_{n,k}) = \int_0^1 t^k \, df_n(t)\,.$$

Using Theorem 6, we can choose a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of (f_n) and a function f of weakly compact semivariation such that $\big(f_{n_i}(x)\big)_{i\in\mathbb{N}}$ weakly converges to $f(x)$ at each continuity point $x \in (0,1)$ of f and for $x = 0$ and $x = 1$. By Theorem 6, for every k, we have:

$$\lim_{i\to\infty} \int_0^1 t^k \, df_{n_i}(t) = \int_0^1 t^k \, df(t)\,, \qquad \text{for } \sigma(X, X')\,.$$

We now show that $\lim\limits_{n\to\infty} \Lambda(B_{k,n}) = a_k$ for the norm topology of X, and the conclusion will follow. By classical algebraic computations (see [9]), it is easy to show that $a_0 = \Lambda(B_{0,n})$ and that

$$a_k = \sum_{j=k}^{n} \frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} \binom{n}{j} \Delta^{n-j} a_j\,.$$

Consequently, we obtain:

$$a_k - \Lambda(B_{k,n}) = \sum_{j=k}^{n} \left(\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \right) \binom{n}{j} \Delta^{n-j} a_j$$
$$- \sum_{j=0}^{k-1} \binom{n}{j} \left(\frac{j}{n}\right)^k \Delta^{n-j} a_j \,.$$

Let $y = \frac{j}{n}$, and observe that

$$\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k = \prod_{i=0}^{k-1} \frac{ny-i}{n-i} - y^k .$$

It follows that, given $\varepsilon > 0$, there exists n_0 such that

$$\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \le \varepsilon \qquad \text{for } n \ge n_0$$

and

$$\left\| \sum_{j=k}^{n} \left(\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \right) \binom{n}{j} \Delta^{n-j} a_j \right\| \le \varepsilon \qquad \text{for } n \ge n_0 \,.$$

It is also clear that

$$\left\| \sum_{j=0}^{k-1} \binom{n}{j} \left(\frac{j}{n}\right)^k \Delta^{n-j} a_j \right\| \le \left(\frac{k}{n}\right)^k \| a_0 \| \,.$$

Now, it is easy to conclude that if n is large enough we have

$$\left\| a_k - \Lambda(B_{k,n}) \right\| \le 2\varepsilon \| a_0 \| \,,$$

which proves the theorem. □

We obtained our moment theorem only with the help of a "Helly's theorem", not a representation theorem of a weakly compact operator. We conclude this paper by deducing a representation theorem for weakly compact operator on $C[0,1]$ into X from our moment problem theorem.

Theorem 8. (Representation theorem). *Every weakly compact linear operator L on $C\big([\alpha,\beta],\mathbb{R}\big)$, with values in a Banach space X, is representable in the form*

$$L(g) = \int_{\alpha}^{\beta} g(t)\, df(t) \,,$$

where f is a function of weakly compact semivariation from $[\alpha,\beta]$ into X.

P r o o f. It is clear that it suffices to prove the result for the interval $[0,1]$. The sequence $(a_k)_{k\in\mathbb{N}}$ defined by the formula $a_k = L(t^k)$ is completely weakly compact. In fact, we have

$$\begin{aligned}\sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k &= \sum_{j=0}^{n} \binom{n}{j} \alpha_k^n L\big(t^k(1-t)^{n-k}\big)\\ &= L\left(\sum_{j=0}^{n} \binom{n}{j} \alpha_k^n\, t^k(1-t)^{n-k}\right) \in W,\end{aligned}$$

W being a weakly compact set. By Theorem 7, there exists an f of weakly compact semivariation such that

$$L(t^k) = \int_0^1 t^k \, df(t)$$

and, by Weierstrass theorem, this equality extends to every continuous function. We recall that a weakly compact mapping from a Banach space into a normed vector space is automatically continuous. □

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