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INSENSITIVITY REGIONS FOR TESTING HYPOTHESES IN MIXED MODELS WITH CONSTRAINTS

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. Insensitivity regions for the risk of the test of linear hypotheses on mean value parameters, for the value of the power function and for threshold ellipsoids are determined in a regular mixed linear model with constraints.

1. Introduction

Let us consider a linear regression model with inaccurate variance component $\boldsymbol{\vartheta}$. If only approximate value of the parameter $\boldsymbol{\vartheta}$ instead of its true value is known, it is of some interest to know whether an uncertainty in $\boldsymbol{\vartheta}$ can or cannot destroy the optimum property of chosen statistical characteristics. An insensitivity region for a chosen characteristic gives the answer to this question. It is a region defined in the space of the parameter $\boldsymbol{\vartheta}$ such that a movement of this parameter inside this region does not cause any essential change of a chosen statistical characteristic and vice versa.

In [3], [4], [7], [9], [10] problems connected with estimators, confidence ellipsoids and tests of linear hypotheses in the case of a regular mixed model without constraints are studied. In [5], [6] a problem connected with estimators in an universal model with or without constraints is solved. Insensitivity regions for estimators and confidence ellipsoids in a regular mixed linear model with constraints can be found in [11].

The aim of this paper is to continue in paper [11] and generalize the other types of insensitivity regions in a regular mixed model with constraints, i.e., an insensitivity region for the risk of the test, for the power function and for the threshold ellipsoid.

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2. Notations and auxiliary statements

Let

$$\begin{gathered}\mathbf{Y} \sim N_n\big(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\big), \qquad \boldsymbol{\beta} \in \mathbf{B} = \{\boldsymbol{\beta}\colon \boldsymbol{\beta} \in \mathbb{R}^k,\ \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}\}, \\ \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}} = \{\boldsymbol{\vartheta} \in \mathbb{R}^p\colon \vartheta_1 > 0, \dots, \vartheta_p > 0\},\end{gathered} \tag{2.1}$$

where $\mathbf{Y}$ is an n-dimensional random vector (observation vector), $\mathbf{X}_{n\times k}$ a known matrix (design matrix), $\boldsymbol{\beta}$ an unknown vector (parameter of the first order), $\mathbf{B}_{q\times k}$ a known matrix, $\mathbf{b}$ a known q-dimensional vector such that $\mathbf{b} \in \mathcal{M}(\mathbf{B})$ and $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ a covariance matrix, where $\boldsymbol{\vartheta}$ is an unknown vector (parameter of the second order) and $\mathbf{V}_1, \dots, \mathbf{V}_p$ are known symmetric, positive semidefinite matrices. This model is called a mixed linear model with constraints.

In the following, the model (2.1) will be supposed to be regular, i.e., the rank of the matrices $\mathbf{X}$ and $\mathbf{B}$ is $r(\mathbf{X}) = k < n$, $r(\mathbf{B}) = q < k$ and $\boldsymbol{\Sigma}(\boldsymbol{\vartheta})$ is positive definite for all $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$.

Let $\boldsymbol{\beta}^*$ and $\boldsymbol{\vartheta}^*$ be true value of the vector parameter $\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$, respectively.

Let $\mathbf{A}$ be an arbitrary matrix of the type $m \times n$, $\mathbf{W}$ a $m\times m$ positive definite matrix. Let $\mathcal{M}(\mathbf{A}) = \{\mathbf{A}\mathbf{u}\colon \mathbf{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$ and $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{u} \in \mathbb{R}^n\colon \mathbf{A}\mathbf{u} = \mathbf{0}\} \subset \mathbb{R}^n$. Then $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}^{-1}} = \mathbf{A}(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^{-}\mathbf{A}'\mathbf{W}^{-1}$ is the $\mathbf{W}^{-1}$-projection matrix on $\mathcal{M}(\mathbf{A})$ ($^-$ means g-inverse). The symbol $\mathbf{M}_{\mathbf{A}}^{\mathbf{W}^{-1}}$ means $\mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\mathbf{W}^{-1}}$. If $\mathbf{W} = \mathbf{I}$, then symbols $\mathbf{P}_{\mathbf{A}}$ and $\mathbf{M}_{\mathbf{A}}$ are used instead of $\mathbf{P}_{\mathbf{A}}^{\mathbf{I}}$ and $\mathbf{M}_{\mathbf{A}}^{\mathbf{I}}$, respectively.

Lemma 2.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric, positive semidefinite $n\times n$ matrices. Then $\mathcal{M}(\mathbf{A}, \mathbf{B}) = \mathcal{M}(\mathbf{A} + \mathbf{B})$.*

Proof. For the proof cf. [2], p. 445. □

Lemma 2.2. *Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be elements of a finite dimension vector space $\mathcal{H}$ with the inner product $\langle\cdot,\cdot\rangle$. Then the Gram matrix $\mathbf{G}$ generated by the n-tuple $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, i.e., $\{\mathbf{G}\}_{i,j} = \langle \mathbf{x}_i, \mathbf{x}_j\rangle$, $i, j = 1, \dots, n$, has the following properties*

(*i*) $\forall \mathbf{u} \in \mathcal{H}\colon \big(\langle \mathbf{u}, \mathbf{x}_1\rangle, \dots, \langle \mathbf{u}, \mathbf{x}_n\rangle\big)' \in \mathcal{M}(\mathbf{G})$,

(*ii*) $\mathbf{G}$ *is at least positive semidefinite matrix.*

Proof. For the proof cf. [2], p. 442. □

Lemma 2.3. *Let the regular mixed linear model with constraints (2.1) be under consideration. Then $\boldsymbol{\vartheta}^*$ –LBLUE (locally best linear unbiased estimator) of the parameter $\boldsymbol{\beta}$ is*

$$\begin{aligned}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) &= \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{Y} \\ &\quad - \mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big(\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big)^{-1}\mathbf{b}, \\ \mathrm{var}_{\vartheta^*}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big] &= \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+},\end{aligned}$$

where

$$\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+ = \mathbf{C}^{-1}(\boldsymbol{\vartheta}^*) - \mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\left(\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\right)^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*),$$
$$\mathbf{C}(\boldsymbol{\vartheta}^*) = \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}.$$

P r o o f . Proof can be found in [1], p. 128. □

Let the null hypothesis concerning the true value $\boldsymbol{\beta}^*$ of the parameter $\boldsymbol{\beta}$ be

$$H_0\colon \mathbf{H}\boldsymbol{\beta}^* + \mathbf{h} = \mathbf{0}\,, \tag{2.2}$$

where $\mathbf{h}$ is a known r-dimensional vector and $\mathbf{H}_{r\times k}$ is a known matrix with the rank $r(\mathbf{H}) = r \le k$ such that the matrix $\mathbf{H}\operatorname{var}_{\vartheta^*}\left[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\right]\mathbf{H}'$ is regular. This matrix is regular iff $r(\mathbf{H}', \mathbf{B}') = r + q$. (cf. [11]). Let the alternative hypothesis be

$$H_a\colon \mathbf{H}\boldsymbol{\beta}^* + \mathbf{h} = \boldsymbol{\xi} \ne \mathbf{0}\,. \tag{2.3}$$

LEMMA 2.4. *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2) *and* (2.3) *be under consideration. Let*

$$T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) = \left(\mathbf{H}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) + \mathbf{h}\right)'\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{-1}\left(\mathbf{H}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) + \mathbf{h}\right).$$

(i) If H_0 is true, then the statistic $T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*)$ has the central chi-square distribution with r degrees of freedom (i.e., $T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) \sim \chi^2_r(0)$).

(ii) If H_0 is not true, then $T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*)$ has the noncentral chi-square distribution with r degrees of freedom (i.e., $T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) \sim \chi^2_r(\delta)$) and the parameter of its noncentrality is

$$\delta = (\mathbf{H}\boldsymbol{\beta}^* + \mathbf{h})'\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{-1}(\mathbf{H}\boldsymbol{\beta}^* + \mathbf{h})\,.$$

P r o o f . Both statements follow from the second fundamental theorem of the least squares theory given in [12], p. 155. □

The statistic $T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*)$ will be used for testing the hypothesis H_0 against H_a. Let $\chi^2_r(0, 1-\alpha)$ be $(1-\alpha)$-quantile of the central chi-square distribution with r degrees of freedom. The risk of the test is the probability that the true null hypothesis H_0 will be rejected. Let the risk of the test be equal to α. It implies that if H_0 is true and $T_{\mathbf{H}}(\mathbf{y}, \boldsymbol{\vartheta}^*) \ge \chi^2_r(0, 1-\alpha)$, where $\mathbf{y}$ is a realization of an observation vector $\mathbf{Y}$, then H_0 is rejected with the risk (probability) α. The power function of this test at a point $\boldsymbol{\xi} = \mathbf{H}\boldsymbol{\beta}^* + \mathbf{h}$ is

$$\beta(\boldsymbol{\xi}) = P\left\{T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) \ge \chi^2_r(0, 1-\alpha)|\boldsymbol{\xi}\right\} = P\left\{\chi^2_r(\delta) \ge \chi^2_r(0, 1-\alpha)|\boldsymbol{\xi}\right\}.$$

3. Linear hypotheses concerning the parameter of the mean value

A small change of $\boldsymbol{\vartheta}^*$ into $\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}$, $\boldsymbol{\Delta} \in \mathbb{R}^P$, causes a change of $\boldsymbol{\vartheta}^*$-LBLUE of the parameter $\boldsymbol{\beta}$ which makes a change of the test statistic $T_{\mathbf{H}}$.

In the following we will solve two problems. The first one is connected with the risk of the test. Let ε be a given probability expressing the maximum tolerable increase of the risk of the test caused by the fact that the true value $\boldsymbol{\vartheta}^*$ is not known precisely. For a given risk α of the test and for a given probability ε we determine a region $\mathcal{R}_\varepsilon$ of all shifts $\boldsymbol{\Delta}$ of the parameter $\boldsymbol{\vartheta}^*$ such that they do not cause increase of the risk α larger than ε.

Another problem is concerning the power of the test at a given point $\boldsymbol{\xi}$. Now, let ε be a given probability expressing the maximum tolerable reduction of the power of the test at a point $\boldsymbol{\xi}$ caused by the fact that $\boldsymbol{\vartheta}^*$ is unknown. Thus for a given point $\boldsymbol{\xi}$ and a given probability ε we determine a region $\mathcal{H}_{\varepsilon,\boldsymbol{\xi}}$ of all shifts $\boldsymbol{\Delta}$ around $\boldsymbol{\vartheta}^*$ such that they do not cause decrease of the power of the test at a chosen point $\boldsymbol{\xi}$ larger than ε.

LEMMA 3.1. *Let us have the random variable*

$$\zeta_{T_H} = \boldsymbol{\Delta}' \frac{\partial T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}}\Big|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}^*},$$

which expresses the change of test statistic $T_{\mathbf{H}}$ caused by the change $\boldsymbol{\Delta}$ of the parameter $\boldsymbol{\vartheta}^*$. Then

$$\begin{aligned}
\zeta_{T_H} = &- 2\big(\mathbf{H}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) + \mathbf{h}\big)'\Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big)^{-1}\mathbf{H} \\
&\times \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{\Sigma}(\boldsymbol{\Delta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{v}(\mathbf{Y}, \boldsymbol{\vartheta}^*) \\
&- \big(\mathbf{H}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) + \mathbf{h}\big)'\Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big)^{-1}\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+ \\
&\times \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{\Sigma}(\boldsymbol{\Delta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X} \\
&\times \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big)^{-1}\big(\mathbf{H}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) + \mathbf{h}\big),
\end{aligned}$$

where $\boldsymbol{v}(\mathbf{Y}, \boldsymbol{\vartheta}^*) = \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)$. Further, its mean value is

$$E[\zeta_{T_H}] = -\boldsymbol{\Delta}'\begin{pmatrix} \mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_1) \\ \vdots \\ \mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_p) \end{pmatrix} - \boldsymbol{\Delta}'\begin{pmatrix} \boldsymbol{\xi}'\mathbf{Z}_1\boldsymbol{\xi} \\ \vdots \\ \boldsymbol{\xi}'\mathbf{Z}_p\boldsymbol{\xi} \end{pmatrix},$$

where $\mathrm{Tr}(\mathbf{U}_{\mathbf{H}})$ means the trace of the matrix $\mathbf{U}_{\mathbf{H}}$ and

$$\begin{aligned}
\mathbf{U}_{\mathbf{H}} = {}&\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big)^{-1}\mathbf{H} \\
&\times \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)
\end{aligned}$$

and for $i = 1, \dots, p$

$$\mathbf{Z}_i = \Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\Big)^{-1}\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i$$
$$\times\, \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\Big)^{-1}.$$

The variance of ζ_{T_H} *is*

$$\mathrm{var}_{\vartheta^*}[\zeta_{T_H}] = \boldsymbol{\Delta}'(\mathbf{A}_0 + \mathbf{D}_{\boldsymbol{\xi}})\boldsymbol{\Delta}\,,$$

where

$$\begin{aligned} \mathbf{A}_0 &= 2\mathbf{S}_{\mathbf{U_H}} + 4\mathbf{C}_{\mathbf{U_H},[\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}]^{+}+\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}}\,, \\ \{\mathbf{S}_{\mathbf{U_H}}\}_{i,j} &= \mathrm{Tr}(\mathbf{U_H}\mathbf{V}_i\mathbf{U_H}\mathbf{V}_j)\,, \qquad i,j = 1,\dots,p\,, \end{aligned} \tag{3.1}$$

and

$$\Big\{\mathbf{C}_{\mathbf{U_H},[\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}]^{+}+\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}}\Big\}_{i,j}$$
$$= \mathrm{Tr}\Big[\mathbf{U_H}\mathbf{V}_i\big(\big[\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}\big]^{+} + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}\big)\mathbf{V}_j\Big]\,,$$
$$\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'} = \mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big(\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big)^{-1}\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*).$$

Further

$$\{\mathbf{D}_{\boldsymbol{\xi}}\}_{i,j} = 4\boldsymbol{\xi}'\mathbf{K}_{ij}\boldsymbol{\xi}\,, \qquad i,j = 1,\dots,p\,,$$

$$\mathbf{K}_{ij} = \big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\big)^{-1}\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)$$
$$\times\, \mathbf{V}_i\Big(\big[\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}\big]^{+} + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}\Big)\mathbf{V}_j$$
$$\times\, \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\big)^{-1}.$$

P r o o f . For the proof see [11], where a similar problem for a confidence ellipsoid is solved. □

The idea of solving the mentioned problems of the risk of the test and its power function is based on applying the Chebyshev inequality under the special conditions given in Lemma 3.1.

It is necessary to realize that "the dangerous movement" of the test statistic $T_{\mathbf{H}}$, which is caused by the change $\boldsymbol{\Delta}$ of the parameter $\boldsymbol{\vartheta}^*$, depends on the hypothesis. Under H_0, the movement of $T_{\mathbf{H}}$ to the right, i.e., $\zeta_{T_H}(\mathbf{y}) \to +\infty$, might cause an increase of the risk of the test. The movement to the left is not interesting, since the risk of the test decreases. On the other hand, under

H_a, "the dangerous movement" of $T_{\mathbf{H}}$ is to the left, i.e., $\zeta_{T_H}(\mathbf{y}) \to -\infty$, which causes a decrease of the power.

Let the test statistic $T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta})$ be linearly approximated by

$$T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) \approx T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) + \zeta_{T_H}.$$

With respect to the Chebyshev inequality

$$P\left\{ \left| \zeta_{T_H} - E[\zeta_{T_H}] \right| \ge t\sqrt{\operatorname{var}_{\vartheta^*}[\zeta_{T_H}]} \right\} \le \frac{1}{t^2}, \qquad t > 0. \tag{3.2}$$

If t is sufficiently large, the (3.2) is sufficiently near to 0. It implies that inequalities

$$\zeta_{T_H} \le E[\zeta_{T_H}] + t\sqrt{\operatorname{var}_{\vartheta^*}[\zeta_{T_H}]}, \tag{3.3}$$

$$\zeta_{T_H} \ge E[\zeta_{T_H}] - t\sqrt{\operatorname{var}_{\vartheta^*}[\zeta_{T_H}]} \tag{3.4}$$

are valid with a sufficiently high probability.

In the following we must distinguish whether H_0 is true (the problem concerning the risk of the test) or H_0 is not true (the problem concerning the power function). Firstly, let H_0 be true. Then for ζ_{T_H} limited by the inequality (3.3) we get

$$\begin{aligned} &P\{\chi_r^2(0) + \zeta_{T_H} \ge \chi_r^2(0, 1-\alpha)\} \\ &\le P\left\{\chi_r^2(0) + E[\zeta_{T_H}] + t\sqrt{\operatorname{var}_{\vartheta^*}[\zeta_{T_H}]} \ge \chi_r^2(0, 1-\alpha)\right\}. \end{aligned} \tag{3.5}$$

If we admit that the risk α increases with a given probability ε, we obtain the condition

$$P\{T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) + \zeta_{T_H} \ge \chi_r^2(0, 1-\alpha)\} \le \alpha + \varepsilon. \tag{3.6}$$

From the last two inequalities (3.5) and (3.6) it follows that

$$E[\zeta_{T_H}] + t\sqrt{\operatorname{var}_{\vartheta^*}[\zeta_{T_H}]} \le \delta_\varepsilon,$$

where

$$\delta_\varepsilon = \chi_r^2(0, 1-\alpha) - \chi_r^2(0, 1-\alpha-\varepsilon)$$

is a sufficient condition that ζ_{T_H} caused by the change $\boldsymbol{\Delta}$ of the parameter $\boldsymbol{\vartheta}^*$ is acceptable. Hence this inequality determines those changes $\boldsymbol{\Delta}$ of $\boldsymbol{\vartheta}^*$ which make only a tolerable change of the risk of the test.

Analogously we can proceed in the case of the power of the test. Let the alternative hypothesis H_a be true (i.e., $\mathbf{H}\boldsymbol{\beta}^* + \mathbf{h} = \boldsymbol{\xi} \ne \mathbf{0}$). From (3.4) it follows

$$\begin{aligned} &P\{\chi_r^2(\delta) + \zeta_{T_H} \ge \chi_r^2(0, 1-\alpha) \,|\, \boldsymbol{\xi}\} \\ &\ge P\left\{\chi_r^2(\delta) + E[\zeta_{T_H}] - t\sqrt{\operatorname{var}_{\vartheta^*}[\zeta_{T_H}]} \ge \chi_r^2(0, 1-\alpha) \,|\, \boldsymbol{\xi}\right\}, \end{aligned} \tag{3.7}$$

where $\delta = \boldsymbol{\xi}'\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\big)^{-1}\boldsymbol{\xi}$. If the power of the test $\beta(\boldsymbol{\xi})$ can decrease with a given probability ε, we get

$$P\big\{T_{\mathbf{H}}(\mathbf{Y},\boldsymbol{\vartheta}^*) + \zeta_{T_H} \geq \chi_r^2(0, 1-\alpha)\,|\boldsymbol{\xi}\big\} \geq \beta(\boldsymbol{\xi}) - \varepsilon\,. \tag{3.8}$$

Finally, from (3.7) and (3.8) it follows that

$$E[\zeta_{T_H}] - t\sqrt{\operatorname{var}_{\vartheta^*}[\zeta_{T_H}]} \geq -\delta_{\varepsilon,\boldsymbol{\xi}}\,,$$

where

$$\delta_{\varepsilon,\boldsymbol{\xi}} = \chi_r^2(\delta, 1-\beta(\boldsymbol{\xi})+\varepsilon) - \chi_r^2(0, 1-\alpha)$$

is a sufficient condition that ζ_{T_H} acceptable, i.e., changes $\boldsymbol{\Delta}$ of $\boldsymbol{\vartheta}^*$ make only a tolerable change of the power of the test at a point $\boldsymbol{\xi}$.

Now, we can define both types of insensitivity regions precisely. In the sequel the notation

$$\mathbf{a}_0 = \big[\operatorname{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_1), \cdots, \operatorname{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_p)\big]', \tag{3.9}$$

$$\mathbf{a}_{\boldsymbol{\xi}} = \mathbf{a}_0 + \big[\boldsymbol{\xi}'\mathbf{Z}_1\boldsymbol{\xi}, \cdots, \boldsymbol{\xi}'\mathbf{Z}_p\boldsymbol{\xi}\big]' \tag{3.10}$$

will be used. Let functions $\Phi(\boldsymbol{\Delta})$ and $\Psi_{\boldsymbol{\xi}}(\boldsymbol{\Delta})$ be defined as follows

$$\Phi(\delta\boldsymbol{\vartheta}) = -\boldsymbol{\Delta}'\mathbf{a}_0 + t\sqrt{\boldsymbol{\Delta}'\mathbf{A}_0\boldsymbol{\Delta}}\,,$$

$$\Psi_{\boldsymbol{\xi}}(\delta\boldsymbol{\vartheta}) = -\boldsymbol{\Delta}'\mathbf{a}_{\boldsymbol{\xi}} - t\sqrt{\boldsymbol{\Delta}'(\mathbf{A}_0 + \mathbf{D}_{\boldsymbol{\xi}})\boldsymbol{\Delta}}\,.$$

Definition 3.2. Let

$$\mathcal{R}_\varepsilon = \big\{\boldsymbol{\Delta} \in \mathbb{R}^p\colon\ \Phi(\boldsymbol{\Delta}) \leq \delta_\varepsilon\big\}\,,$$

where δ_ε is given by the relationship

$$P\big\{\chi_r^2(0) \geq \chi_r^2(0, 1-\alpha) - \delta_\varepsilon\big\} = \alpha + \varepsilon\,.$$

The set $\mathcal{R}_\varepsilon$ is called the *insensitivity region of the risk of the test.*

Definition 3.3. Let $\boldsymbol{\xi} = \mathbf{H}\boldsymbol{\beta}^* + \mathbf{h}$ and let

$$\mathcal{H}_{\varepsilon,\boldsymbol{\xi}} = \big\{\boldsymbol{\Delta} \in \mathbb{R}^p\colon \Psi_{\boldsymbol{\xi}}(\boldsymbol{\Delta}) \geq -\delta_{\varepsilon,\boldsymbol{\xi}}\big\}\,,$$

where $\delta_{\varepsilon,\boldsymbol{\xi}}$ is given by the relationship

$$P\big\{\chi_r^2(\delta) \geq \chi_r^2(0, 1-\alpha) + \delta_{\varepsilon,\boldsymbol{\xi}}\big\} = \beta(\boldsymbol{\xi}) - \varepsilon,$$

$$\delta = \boldsymbol{\xi}'\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{H}'\big)^{-1}\boldsymbol{\xi}\,.$$

The set $\mathcal{H}_{\varepsilon,\boldsymbol{\xi}}$ is called the *insensitivity region of the power of the test at a point* $\boldsymbol{\xi}$.

Lemma 3.4. *Matrices* $\mathbf{S}_{\mathbf{U_H}}$, $\mathbf{C}_{\mathbf{U_H},[\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}]^+ + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{XC}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}}$ *and* $\mathbf{D}_{\boldsymbol{\xi}}$ *are at least p.s.d.*

P r o o f . Obviously $\mathbf{U_H}$ is a positive semidefinite matrix. Thus there exists such a matrix $\mathbf{J}$ that $\mathbf{U_H} = \mathbf{JJ}'$. Hence $\mathbf{S}_{\mathbf{U_H}}$ is the Gram matrix of p-tuple $\{\mathbf{J}'\mathbf{V}_1\mathbf{J}, \dots, \mathbf{J}'\mathbf{V}_p\mathbf{J}\}$ in the vector space $\mathcal{S}$ of all symmetric matrices with the inner product $\langle \mathbf{A}, \mathbf{B}\rangle = \operatorname{Tr}(\mathbf{AB})$, $\forall \mathbf{A}, \mathbf{B} \in \mathcal{S}$, and according to Lemma 2.2 $\mathbf{S}_{\mathbf{U_H}}$ is at least p.s.d.

The matrix $\mathbf{D}_{\boldsymbol{\xi}}$ can be rewritten in the form

$$\begin{aligned}\{\mathbf{D}_{\boldsymbol{\xi}}\}_{i,j} = 4\operatorname{Tr}\Big\{ &\Big(\big[\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}\big]^+ + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{XC}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}\Big)\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X} \\ &\times \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big)^{-1}\boldsymbol{\xi}\boldsymbol{\xi}' \\ &\times \Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big)^{-1}\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\Big\}.\end{aligned}$$

In a similar way it can be proved that $\mathbf{C}_{\mathbf{U_H},[\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}]^+ + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{XC}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}}$ and $\mathbf{D}_{\boldsymbol{\xi}}$ are also the Gram matrices in proper vector space. □

Corollary 3.5. *Let* $\mathbf{a}_0$ *and* $\mathbf{a}_{\boldsymbol{\xi}}$ *be given by* (3.9) *and* (3.10) *and let* $\mathbf{A}_0$ *and* $\mathbf{D}_{\boldsymbol{\xi}}$ *be defined as in Lemma 3.1. Then* $\mathbf{a}_0, \mathbf{a}_{\boldsymbol{\xi}} \in \mathcal{M}(\mathbf{S}_{\mathbf{U_H}})$ *and* $\mathcal{M}(\mathbf{S}_{\mathbf{U_H}}) \subset \mathcal{M}(\mathbf{A}_0) \subset \mathcal{M}(\mathbf{A}_0 + \mathbf{D}_{\boldsymbol{\xi}})$. *Further, equations* $(t^2\mathbf{A}_0 - \mathbf{a}_0\mathbf{a}_0')\mathbf{x}_0 = \mathbf{a}_0\delta_\varepsilon$ *and* $(t^2(\mathbf{A}_0 + \mathbf{D}_{\boldsymbol{\xi}}) - \mathbf{a}_{\boldsymbol{\xi}}\mathbf{a}_{\boldsymbol{\xi}}')\mathbf{x}_0 = \mathbf{a}_{\boldsymbol{\xi}}\delta_{\varepsilon,\boldsymbol{\xi}}$ *are consistent with respect to* $\mathbf{x}_0$.

P r o o f . The part concerning the vector $\mathbf{a}_0$ follows from Lemmas 2.1, 2.2 and 3.4. As concern the vector $\mathbf{a}_{\boldsymbol{\xi}}$, let us denote $\mathbf{a}_1 = (\boldsymbol{\xi}'\mathbf{Z}_1\boldsymbol{\xi}, \dots, \boldsymbol{\xi}'\mathbf{Z}_p\boldsymbol{\xi})'$. Then $\mathbf{a}_{\boldsymbol{\xi}} = \mathbf{a}_0 + \mathbf{a}_1$. From Lemma 3.4 it follows that $\mathbf{S}_{\mathbf{U_H}}$ is the Gram matrix generated by the p-tuple $\{\mathbf{J}'\mathbf{V}_1\mathbf{J}, \dots, \mathbf{J}'\mathbf{V}_p\mathbf{J}\}$ in the vector space of symmetric matrices with the inner product defined by $\langle \mathbf{A}, \mathbf{B}\rangle = \operatorname{Tr}(\mathbf{AB})$, where

$$\mathbf{J} = \boldsymbol{\Sigma}^{-1}\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big)^{-\frac{1}{2}}.$$

Further, $\boldsymbol{\xi}'\mathbf{Z}_i\boldsymbol{\xi} = \operatorname{Tr}(\mathbf{V}_i\mathbf{L})$, $i = 1, \dots, p$, where $\mathbf{L}$ is the p.s.d. matrix (its explicit form can be easily obtained from Lemma 3.1). Then there exists such a matrix

$$\mathbf{K} = \boldsymbol{\Sigma}^{-1}\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big)^{-1}\boldsymbol{\xi}$$

that $\mathbf{L} = \mathbf{KK}'$. Hence

$$\boldsymbol{\xi}'\mathbf{Z}_i\boldsymbol{\xi} = \operatorname{Tr}(\mathbf{V}_i\mathbf{KK}') = \operatorname{Tr}(\mathbf{J}'\mathbf{V}_i\mathbf{JD}), \qquad i = 1, \dots, p,$$

where $\mathbf{D} = \big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big)^{-\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}'\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big)^{-\frac{1}{2}}$ and from Lemma 2.2 it follows that $\mathbf{a}_1 \in \mathcal{M}(\mathbf{S}_{\mathbf{U_H}})$. □

Theorem 3.6. *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2), (2.3) *be under consideration. Let* $\mathbf{a}_0$ *and* $\mathbf{a}_{\boldsymbol{\xi}}$ *be given by* (3.9) *and* (3.10) *and let* $\mathbf{A}_0$ *and* $\mathbf{D}_{\boldsymbol{\xi}}$ *be given as in Lemma 3.1.*

(*i*) *The boundary of the set* $\mathcal{R}_\varepsilon$ *is*

$$\overline{\mathcal{R}}_\varepsilon = \left\{ \boldsymbol{\Delta} \colon (\boldsymbol{\Delta} - \mathbf{x}_0)'(t^2\mathbf{A}_0 - \mathbf{a}_0\mathbf{a}_0')(\boldsymbol{\Delta} - \mathbf{x}_0) = \frac{\delta_\varepsilon^2 t^2}{t^2 - \mathbf{a}_0'\mathbf{A}_0^-\mathbf{a}_0} \right\},$$

where $\mathbf{x}_0 = \frac{\delta_\varepsilon}{t^2 - \mathbf{a}_0'\mathbf{A}_0^-\mathbf{a}_0}\mathbf{A}_0^-\mathbf{a}_0$ *and* $\delta_\varepsilon = \chi_r^2(0, 1-\alpha) - \chi_r^2(0, 1-\alpha-\varepsilon)$.

(*ii*) *Let* $\mathbf{A}_{\boldsymbol{\xi}} = \mathbf{A}_0 + \mathbf{D}_{\boldsymbol{\xi}}$. *Then the boundary of the set* $\mathcal{H}_{\varepsilon,\boldsymbol{\xi}}$ *is*

$$\overline{\mathcal{H}}_{\varepsilon,\boldsymbol{\xi}} = \left\{ \boldsymbol{\Delta} \colon (\boldsymbol{\Delta} + \mathbf{x}_1)'(t^2\mathbf{A}_{\boldsymbol{\xi}} - \mathbf{a}_{\boldsymbol{\xi}}\mathbf{a}_{\boldsymbol{\xi}}')(\boldsymbol{\Delta} + \mathbf{x}_1) = \frac{\delta_{\varepsilon,\boldsymbol{\xi}}^2 t^2}{t^2 - \mathbf{a}_{\boldsymbol{\xi}}'\mathbf{A}_{\boldsymbol{\xi}}^-\mathbf{a}_{\boldsymbol{\xi}}} \right\},$$

where $\mathbf{x}_1 = \frac{\delta_{\varepsilon,\boldsymbol{\xi}}}{t^2 - \mathbf{a}_{\boldsymbol{\xi}}'\mathbf{A}_{\boldsymbol{\xi}}^-\mathbf{a}_{\boldsymbol{\xi}}}\mathbf{A}_{\boldsymbol{\xi}}^-\mathbf{a}_{\boldsymbol{\xi}}$, $\delta = \boldsymbol{\xi}'\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big)^{-1}\boldsymbol{\xi}$ *and* $\delta_{\varepsilon,\boldsymbol{\xi}} = \chi_r^2(\delta, 1-\beta(\boldsymbol{\xi})+\varepsilon) - \chi_r^2(0, 1-\alpha)$.

P r o o f . From Definitions 3.2 and 3.3, the statements can be proved by solving the equation $\Phi(\boldsymbol{\Delta}) = \delta_\varepsilon$ and $\Psi_{\boldsymbol{\xi}}(\boldsymbol{\Delta}) = -\delta_{\varepsilon,\boldsymbol{\xi}}$ by applying results from Lemma 3.1, Corollary 3.5. For detail cf. [4], where a similar problem is solved. □

Theorem 3.7. *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2), (2.3) *be under consideration.*

(*i*) *If* H_0 *is true, then*

$$\boldsymbol{\Delta} \in \mathcal{R}_\varepsilon \;\Rightarrow\; P\big\{T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) \geq \chi_r^2(0, 1-\alpha)\big\} \leq \alpha + \varepsilon\,.$$

(*ii*) *If* H_0 *is not true (i.e.,* $\mathbf{H}\boldsymbol{\beta}^* + \mathbf{h} = \boldsymbol{\xi}$*), then*

$$\boldsymbol{\Delta} \in \mathcal{H}_{\varepsilon,\boldsymbol{\xi}} \;\Rightarrow\; P\big\{T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) \geq \chi_r^2(0, 1-\alpha)\big|\boldsymbol{\xi}\big\} \geq \beta(\boldsymbol{\xi}) - \varepsilon\,.$$

P r o o f . The proof follows from Definitions 3.2, 3.3 and Lemma 3.1. □

Remark 3.8. With respect to the Chebyshev inequality it seems that the proper value of the parameter t in the explicit expression of insensitivity regions lies in the interval $[3, 5]$. The procedure for determining the optimum value of the parameter t maximizing the size of the insensitivity region can be found in cf. [9], where this problem is solved for the risk of the test in the regular mixed linear model without constraints.

4. Threshold ellipsoids

In the connection with testing hypotheses we can study also the problem to find a region defined in the space of an unknown parameter $\boldsymbol{\beta}$, which enable to determine all values of an alternative hypothesis distinguishable from a null hypothesis with a sufficiently high chosen power κ (cf. [8]).

Definition 4.1. Let the regular mixed linear model with constraints (2.1) be under consideration. Let $\boldsymbol{\beta}_0$ be the value of an unknown parameter $\boldsymbol{\beta}$ assumed by the null hypothesis $H_0\colon \boldsymbol{\beta} = \boldsymbol{\beta}_0$ tested against the alternative $H_a\colon \boldsymbol{\beta} \neq \boldsymbol{\beta}_0$ under the risk α. (κ,α)-*threshold ellipsoid* for $\boldsymbol{\beta}$ is the set

$$\mathcal{T}_{\kappa,\alpha}(\boldsymbol{\beta}) = \big\{\boldsymbol{\beta} \in \mathcal{B}\colon (\boldsymbol{\beta} - \boldsymbol{\beta}_0)'\mathbf{T}(\boldsymbol{\beta} - \boldsymbol{\beta}_0) \le c^2\big\}, \qquad c \in \mathbb{R}^1,$$

where c is a real number and $\mathbf{T}_{k\times k}$ is a symmetric matrix such that the value of the power function of the used test must be exactly κ for the true value $\boldsymbol{\beta}^*$ at the boundary of $\mathcal{T}_{\kappa,\alpha}$.

Lemma 4.2. *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2) *and* (2.3) *be under consideration. Then* (κ,α)*-threshold ellipsoid is*

$$\mathcal{T}_{\kappa,\alpha}(\boldsymbol{\beta}) = \Big\{\boldsymbol{\beta} \in \mathcal{B}\colon (\mathbf{H}\boldsymbol{\beta} + \mathbf{h})'\Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big)^{-1}(\mathbf{H}\boldsymbol{\beta} + \mathbf{h}) \le \delta_{\mathrm{krit}}\Big\},$$

where δ_{krit} *is the value of the noncentrality parameter of the noncentral chi-square distribution with* r *degrees of freedom defined by the relationship*

$$P\big\{\chi_r^2(\delta_{\mathrm{krit}}) \ge \chi_r^2(0, 1-\alpha)\big\} = \kappa.$$

P r o o f. The statement follows from the Section 4b.2 in [12]. □

As in the last section, a change of the vector parameter $\boldsymbol{\vartheta}^*$ into $\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}$ can make a decrease of the power of the test at the boundary of the threshold ellipsoid. Hence the problem about the insensitivity region of the threshold ellipsoid is closely connected with the insensitivity region of the power of the test. In contrast to the problem about the power of the test from the last section, now we determine the insensitivity region not only for one chosen point $\boldsymbol{\xi}$, but for all points $\boldsymbol{\xi}$ having the same chosen power κ, i.e., for all $\boldsymbol{\xi}$ located on the boundary of the threshold ellipsoid $\mathcal{T}_{\kappa,\alpha}$.

The problem is to determine a joint region $\mathcal{H}_{\varepsilon,\boldsymbol{\xi}}$ for all $\boldsymbol{\xi}$ located on the boundary of a threshold ellipsoid $\mathcal{T}_{\kappa,\alpha}$. One possible approach is to eliminate the dependence of the mean value and the variance of ζ_{T_H} on $\boldsymbol{\xi}$. For detail see [10], where this problem is solved in the regular mixed model without constraints.

In the following we will consider points $\boldsymbol{\xi}$ located on the boundary of $\mathcal{T}_{\kappa,\alpha}$ only, i.e., all points $\boldsymbol{\xi}$, $\boldsymbol{\xi} = \mathbf{H}\boldsymbol{\beta}^* + \mathbf{h}$, which fulfil the condition

$$\boldsymbol{\xi}'\big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\big)^{-1}\boldsymbol{\xi} = \delta_{\mathrm{krit}}. \tag{4.1}$$

Definition 4.3. Let

$$\mathcal{H}_{\varepsilon,\kappa} = \left\{\boldsymbol{\Delta} \in \mathbb{R}^p \colon \ \Psi_{\boldsymbol{\xi}}(\boldsymbol{\Delta}) \geq -\delta_{\varepsilon,\kappa}\right\}$$

for all $\boldsymbol{\xi}$ fulfilling the condition (4.1), where $\delta_{\varepsilon,\kappa}$ is given by the relationship

$$P\left\{\chi_r^2(\delta_{\mathrm{krit}}) \geq \chi_r^2(0, 1-\alpha) + \delta_{\varepsilon,\kappa}\right\} = \kappa - \varepsilon\,.$$

The set $\mathcal{H}_{\varepsilon,\kappa}$ is called the *insensitivity region for the* (κ,α)*-threshold ellipsoid.*

Lemma 4.4. *Let* $\mathbf{E}_{ij}$ *be the* $p \times p$ *matrix with the* (i,j)*th entry equal to* 1 *and with other entries equal to* 0. *Let* $\mathbf{K}_{ij}$, $i,j = 1,\dots,p$, *be defined as in Lemma 3.1 and let* $\mathbf{A}_0$ *be given by* (3.1). *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2) *and* (2.3) *be under consideration. Then for all* $\boldsymbol{\xi}$ *fulfilling* (4.1) *it holds that*

$$\operatorname{var}_{\boldsymbol{\vartheta}^*}[\zeta_{T_H}] \leq \ \boldsymbol{\Delta}'(\mathbf{A}_0 + \mathbf{D}_\kappa)\boldsymbol{\Delta}\,,$$

where

$$\begin{aligned} \mathbf{D}_\kappa &= \sum_{l=1}^{s} \delta_{\mathrm{krit}}\gamma_l \mathbf{G}_l\Big(\mathbf{H}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{H}'\Big)^{-1}\mathbf{G}_l'\,,\\ \mathbf{G}_l &= \begin{pmatrix} \mathbf{g}_{l,1}' \\ \vdots \\ \mathbf{g}_{l,p}' \end{pmatrix}, \qquad l = 1,\dots,s\,,\\ \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{E}_{ij}\otimes\mathbf{K}_{ij}) &= \sum_{l=1}^{s}\gamma_l\mathbf{g}_l\mathbf{g}_l' \quad (\textit{the spectral decomposition})\,, \end{aligned} \tag{4.2}$$

where $r\left(\sum\limits_{i=1}^{p}\sum\limits_{j=1}^{p}(\mathbf{E}_{ij}\otimes\mathbf{K}_{ij})\right) = s \leq rp$, $\mathbf{g}_l \in \mathbb{R}^{rp}$, $\mathbf{g}_l'\mathbf{g}_k = \begin{cases} 1 & \textit{if } l = k\,,\\ 0 & \textit{otherwise}, \end{cases}$ $\mathbf{g}_{l,i} \in \mathbb{R}^r$, $\mathbf{g}_l = \left(\mathbf{g}_{l,1}',\dots,\mathbf{g}_{l,p}'\right)'$, $i = 1,\dots,p$, $l = 1,\dots,s$, *and* $\otimes$ *means Kronecker product.*

Proof. In the following an analogous procedure is used as in [10]. From Lemma 3.1 it is true

$$\operatorname{var}_{\boldsymbol{\vartheta}^*}[\zeta_{T_H}] = \boldsymbol{\Delta}'(\mathbf{A}_0 + \mathbf{D}_{\boldsymbol{\xi}})\boldsymbol{\Delta}$$

and the matrix $\mathbf{A}_0$ is independent of $\boldsymbol{\xi}$. Further, the term $\boldsymbol{\Delta}'\mathbf{D}_{\boldsymbol{\xi}}\boldsymbol{\Delta}$ can be rewritten in the form

$$\boldsymbol{\Delta}'\mathbf{D}_{\boldsymbol{\xi}}\boldsymbol{\Delta} = (\boldsymbol{\Delta}'\otimes\boldsymbol{\xi}')\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{E}_{ij}\otimes\mathbf{K}_{ij})(\boldsymbol{\Delta}\otimes\boldsymbol{\xi})\,.$$

Using the spectral decomposition of the matrix $\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{E}_{ij}\otimes\mathbf{K}_{ij})$ and the Schwarz inequality with the norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{-1}\mathbf{x}}$ we get

$$\begin{aligned}\boldsymbol{\Delta}'\mathbf{D}_{\boldsymbol{\xi}}\boldsymbol{\Delta} &= (\boldsymbol{\Delta}'\otimes\boldsymbol{\xi}')\sum_{r=1}^{s}\gamma_r\mathbf{g}_r\mathbf{g}_r'(\boldsymbol{\Delta}\otimes\boldsymbol{\xi}) = \sum_{r=1}^{s}\gamma_r(\boldsymbol{\Delta}'\mathbf{G}_r\boldsymbol{\xi})^2\\ &\le \boldsymbol{\Delta}'\left(\sum_{r=1}^{s}\delta_{\text{krit}}\gamma_r\mathbf{G}_r\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{-1}\mathbf{G}_r'\right)\boldsymbol{\Delta} \equiv \boldsymbol{\Delta}'\mathbf{D}_\kappa\boldsymbol{\Delta}\,.\end{aligned}$$

□

Lemma 4.5. *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2) *and* (2.3) *be under consideration. Then for all* $\boldsymbol{\xi}$ *fulfilling* (4.1) *it holds that*

$$E[\zeta_{T_H}] \ge -\boldsymbol{\Delta}'\begin{bmatrix}\mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_1) & + & \delta_{\text{krit}}\sqrt{\mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_1\mathbf{U}_{\mathbf{H}}\mathbf{V}_1)}\\ & \vdots & \\ \mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_p) & + & \delta_{\text{krit}}\sqrt{\mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_p\mathbf{U}_{\mathbf{H}}\mathbf{V}_p)}\end{bmatrix}.$$

P r o o f . From the previous, the problem is to minimize $E[\zeta_{T_H}]$, i.e., maximize the term $[\boldsymbol{\xi}'\mathbf{Z}_1\boldsymbol{\xi},\dots,\boldsymbol{\xi}'\mathbf{Z}_p\boldsymbol{\xi}]'$ subject to the condition (4.1). Using the Schwarz inequality we get for $i=1,\dots,p$

$$\begin{aligned}\mathrm{Tr}(\boldsymbol{\xi}'\mathbf{Z}_i\boldsymbol{\xi}) =\ & \mathrm{Tr}\Big[\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{-\frac12}\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{\frac12}\\ &\times\mathbf{Z}_i\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{\frac12}\left(\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)^{-\frac12}\boldsymbol{\xi}\boldsymbol{\xi}'\Big]\\ &\le \delta_{\text{krit}}\sqrt{\mathrm{Tr}\left(\mathbf{Z}_i\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\mathbf{Z}_i\mathbf{H}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{H}'\right)}\\ &= \delta_{\text{krit}}\sqrt{\mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_i\mathbf{U}_{\mathbf{H}}\mathbf{V}_i)}\,.\end{aligned}$$

□

Theorem 4.6. *Let matrices* $\mathbf{A}_0$ *and* $\mathbf{D}_\kappa$ *be given by* (3.1) *and* (4.2). *Let* $\left[\sqrt{\mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_1\mathbf{U}_{\mathbf{H}}\mathbf{V}_1)},\cdots,\sqrt{\mathrm{Tr}(\mathbf{U}_{\mathbf{H}}\mathbf{V}_p\mathbf{U}_{\mathbf{H}}\mathbf{V}_p)}\right]' \in \mathcal{M}(\mathbf{A}_0+\mathbf{D}_\kappa)$. *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2) *and* (2.3) *be under consideration. Then the boundary of the insensitivity region for the threshold ellipsoid* $\mathcal{T}_{\kappa,\alpha}$ *is the set*

$$\overline{\mathcal{H}}_{\varepsilon,\kappa} = \left\{\boldsymbol{\Delta}\colon(\boldsymbol{\Delta}+\mathbf{x}_2)'(t^2\mathbf{A}_\kappa-\mathbf{a}_2\mathbf{a}_2')(\boldsymbol{\Delta}+\mathbf{x}_2) = \frac{\delta_{\varepsilon,\kappa}^2t^2}{t^2-\mathbf{a}_2'\mathbf{A}_\kappa^-\mathbf{a}_2}\right\},$$

where $\delta_{\varepsilon,\kappa}$ *is given by the relationship*

$$P\{\chi_r^2(\delta_{\mathrm{krit}}) \geq \chi_r^2(0, 1-\alpha) + \delta_{\varepsilon,\kappa}\} = \kappa - \varepsilon$$

and

$$\begin{aligned}
\mathbf{A}_\kappa &= \mathbf{A}_0 + \mathbf{D}_\kappa\,,\\
\mathbf{a}_2 &= \big[\mathrm{Tr}(\mathbf{U_H V}_1), \cdots, \mathrm{Tr}(\mathbf{U_H V}_p)\big]'\\
&\quad + \Big[\sqrt{\mathrm{Tr}(\mathbf{U_H V}_1\mathbf{U_H V}_1)}, \cdots, \sqrt{\mathrm{Tr}(\mathbf{U_H V}_p\mathbf{U_H V}_p)}\Big]'\delta_{\mathrm{krit}}\,,\\
\mathbf{x}_2 &= \frac{\delta_{\varepsilon,\kappa}}{t^2 - \mathbf{a}_2'\mathbf{A}_\kappa^-\mathbf{a}_2}\mathbf{A}_\kappa^-\mathbf{a}_2\,.
\end{aligned}$$

P r o o f . The proof follows from Theorem 3.6, Lemma 4.4 and Lemma 4.5. □

Remark 4.7. In Theorem 4.6, the assumption

$$\Big[\sqrt{\mathrm{Tr}(\mathbf{U_H V}_1\mathbf{U_H V}_1)}, \cdots, \sqrt{\mathrm{Tr}(\mathbf{U_H V}_p\mathbf{U_H V}_p)}\Big]' \in \mathcal{M}(\mathbf{A}_0 + \mathbf{D}_\kappa)$$

cannot be omitted. If this condition is not fulfilled, then the boundary of the insensitivity region $\mathcal{H}_{\varepsilon,\kappa}$ is given by a general quadratic form. Let

$$\mathbf{a}_2 = \mathbf{c}_0 + \mathbf{c}_1\,, \qquad \mathbf{c}_0 \perp \mathbf{c}_1\,, \qquad c_0 \in \mathcal{M}(\mathbf{A}_0 + \mathbf{D}_\kappa)\,, \tag{4.3}$$

$$\mathbf{\Delta} = \mathbf{\Delta}_0 + \mathbf{\Delta}_1\,, \qquad \mathbf{\Delta}_0 \perp \mathbf{\Delta}_1\,, \qquad \mathbf{\Delta}_0 \in \mathcal{M}(\mathbf{A}_0 + \mathbf{D}_\kappa)\,. \tag{4.4}$$

Then the expression of $\overline{\mathcal{H}}_{\varepsilon,\kappa}$ is

$$\begin{aligned}
\overline{\mathcal{H}}_{\varepsilon,\kappa} = \Big\{\mathbf{\Delta}\colon (\mathbf{\Delta}_0 + \mathbf{x}_3)'(t^2\mathbf{A}_\kappa - \mathbf{c}_0\mathbf{c}_0')(\mathbf{\Delta}_0 + \mathbf{x}_3) - \mathbf{\Delta}_1'\mathbf{c}_1\mathbf{c}_1'\mathbf{\Delta}_1&\\
+ 2\mathbf{\Delta}_1'\mathbf{c}_1\delta_{\varepsilon,\kappa} = \frac{\delta_{\varepsilon,\kappa}^2 t^2}{t^2 - \mathbf{c}_0'\mathbf{A}_\kappa^-\mathbf{c}_0}\Big\}&\,,
\end{aligned}$$

where $\mathbf{x}_3 = \frac{\delta_{\varepsilon,\kappa}}{t^2 - \mathbf{c}_0'\mathbf{A}_\kappa^-\mathbf{c}_0}\mathbf{A}_\kappa^-\mathbf{c}_0$.

P r o o f . From Definition 4.3, the boundary of $\mathcal{H}_{\varepsilon,\kappa}$ is given by the equation

$$\Psi_{\boldsymbol{\xi}}(\mathbf{\Delta}) = -\delta_{\varepsilon,\kappa}\,, \qquad \forall\boldsymbol{\xi}\,, \qquad \boldsymbol{\xi}'\Big(\mathbf{H}\big[\mathbf{M_{B'}C}(\boldsymbol{\vartheta}^*)\mathbf{M_{B'}}\big]^+\mathbf{H}'\Big)^{-1}\boldsymbol{\xi} = \delta_{\mathrm{krit}}\,.$$

The statement can be obtained by solving this equation subject to assentions of Lemmas 4.4, 4.5, and conditions (4.3) and (4.4). □

Theorem 4.8. *Let the regular mixed linear model with constraints* (2.1) *and hypotheses* (2.2), (2.3) *be under consideration. Then for all* $\boldsymbol{\xi}$ *fulfilling the condition* (4.1) *it holds that*

$$\mathbf{\Delta} \in \mathcal{H}_{\varepsilon,\kappa} \;\Rightarrow\; P\big\{T_{\mathbf{H}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \mathbf{\Delta}) \geq \chi_r^2(0, 1-\alpha)\big|\boldsymbol{\xi}\big\} \geq \kappa - \varepsilon\,.$$

P r o o f . It is an obvious consequence of Theorem 3.7. □

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