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ON SOME CLASSES OF MEASURABLE FUNCTIONS WITH BAIRE PROPERTY

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ABSTRACT. In this article we investigate the pointwise and the discrete convergence in the class $\mathcal{A}$ of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that there is a set $A(f)$ of measure zero and of the first category for which $f|(\mathbb{R}\setminus A(f))$ is continuous. Moreover, we examine the functions $F\colon \mathbb{R}^2 \to \mathbb{R}$ continuous with respect to y and with respect to the sections $F^y \in \mathcal{A}$.

Let $\mathbb{R}$ be the set of all reals. Denote by $\mathcal{N}$ the σ-ideal of all subsets $A \subset \mathbb{R}$ of measure (of Lebesgue) zero, by $\mathcal{K}$ the σ-ideal of all subsets $A \subset \mathbb{R}$ of the first category and by $\mathcal{I}$ the intersection $\mathcal{N} \cap \mathcal{K}$.

I. Functions of one variable

Let $\mathcal{A}$ be the family of all functions $f\colon \mathbb{R} \to \mathbb{R}$ for which there are sets $A(f) \subset \mathbb{R}$ such that $A(f) \in \mathcal{I}$ and the restrictions $f/\big(\mathbb{R} \setminus A(f)\big)$ are continuous. Evidently every function $f \in \mathcal{A}$ is measurable (in the Lebesgue sense) and has the Baire property, but there are measurable functions with the Baire property which do not belong to $\mathcal{A}$.

If $f, g \in \mathcal{A}$ then $\big(A(f) \cup A(g)\big) \in \mathcal{I}$ and the restrictions $f/\big(\mathbb{R} \setminus \big(A(f) \cup A(g)\big)\big)$ and $g/\big(\mathbb{R} \setminus \big(A(f) \cup A(g)\big)\big)$ are continuous. Consequently, the functions $f+g$, $f-g$, fg, $\max(f,g)$, $\min(f,g)$, cf for $c \in \mathbb{R}$ and the composition $x \mapsto h\big(f(x)\big)$ for every continuous function $h\colon \mathbb{R} \to \mathbb{R}$, belong to $\mathcal{A}$.

Similarly we can observe that if a sequence of functions $f_n \in \mathcal{A}$, $n = 1, 2, \dots$, uniformly converges to a function f, then f belongs also to $\mathcal{A}$.

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Remark 1. In the set $\mathcal{M}_B$ of all measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ with Baire property we can introduce the metric of the uniform convergence:

$$\rho_C(f,g) = \min\bigl(1, \sup_x |f(x) - g(x)|\bigr).$$

Then the family $\mathcal{A}$ is a closed, nowhere dense subset of $\mathcal{M}_B$.

P r o o f . Evidently $\mathcal{A}$ is closed. So, it suffices to prove that for all $f \in \mathcal{A}$ and for all $\varepsilon > 0$ there exists $g \in \mathcal{M}_B \setminus \mathcal{A}$ such that $\rho_C(f,g) < \varepsilon$.

Suppose that $f \in \mathcal{A}$ and $\varepsilon > 0$. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of all intervals with rational endpoints. There are Cantor sets A_n, B_n, $n = 1,2,\dots$, of positive Lebesgue measure, such that, for all $n = 1,2,\dots$,

$$A_n \cup B_n \subset I_n \Bigm\backslash \bigcup_{k<n} (A_k \cup B_k),$$

$$A_n \cap B_n = \emptyset.$$

Let

$$h(x) = \begin{cases} \frac{1}{n} & \text{for } x \in A_n,\ n = 1,2,\dots, \\ 0 & \text{in the remaining case}. \end{cases}$$

As h is Baire class one it follows that $h \in \mathcal{M}_B$, but $h \notin \mathcal{A}$. So,

$$g = f + \frac{\varepsilon h}{2} \in \mathcal{M}_B \setminus \mathcal{A} \qquad \text{and} \qquad \rho_C(f,g) = \frac{\varepsilon}{2} < \varepsilon.$$

This completes the proof. □

We will investigate the pointwise and the discrete limits of sequences of functions from $\mathcal{A}$.

We will say that a sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$, $n = 1,2,\dots$, discretely converges to the limit f ($f = d - \lim_{n\to\infty} f_n$) if

$$\forall_x \, \exists_{n(x)} \, \forall_{n>n(x)} \quad f_n(x) = f(x).$$

For any family $\mathcal{P}$ denote by $B_d(\mathcal{P})$ ($B(\mathcal{P})$) the family of all discrete limits (of pointwise limits) of sequences of functions from the family $\mathcal{P}$.

THEOREM 1. *The function $f \in B_d(\mathcal{A})$ if and only if there are a set A and a function $g\colon A \to \mathbb{R}$ such that $\mathbb{R} \setminus A \in \mathcal{I}$, $g = f/A$ and there is a sequence of continuous functions $g_n\colon A \to \mathbb{R}$, $n = 1,2,\dots$, with $g = d - \lim_{n\to\infty} g_n$.*

P r o o f . Suppose that a sequence of functions $f_n \in \mathcal{A}$, $n = 1,2,\dots$, discretely converges to a function f. Then the set

$$B = \bigcup_{n=1}^{\infty} A(f_n) \in \mathcal{I}$$

and the restricted functions $f_n/(\mathbb{R}\setminus B)$, $n=1,2,\dots$, are continuous. So, the set $A=\mathbb{R}\setminus B$ and the function $g=f/A=d-\lim_{n\to\infty} f_n/A$ satisfy all requirements.

Inversely, if a sequence of continuous functions $g_n\colon A\to\mathbb{R}$, $n=1,2,\dots$, discretely converges to a function $g=f/A$ and $\mathbb{R}\setminus A\in\mathcal{I}$, then the functions

$$f_n(x)=\begin{cases} g_n(x) & \text{for } x\in A\,,\\ f(x) & \text{otherwise on } \mathbb{R}\,,\end{cases}$$

for $n=1,2,\dots$, belong to $\mathcal{A}$ and

$$f=d-\lim_{n\to\infty} f_n\,.$$

□

EXAMPLE 1. There are pairwise disjoint nowhere dense perfect sets $C_{i,k}$, $i=1,2,3$ and $k=1,2,\dots$, of positive measure such that for every open interval I there are positive integers k_1, k_2, k_3 such that

$$C_{1,k_1}\cup C_{2,k_2}\cup C_{3,k_3}\subset I\,.$$

Then the function

$$f(x)=\begin{cases} \frac{1}{k} & \text{for } x\in C_{1,k},\ k=1,2,\dots,\\ -\frac{1}{k} & \text{for } x\in C_{2,k},\ k=1,2,\dots,\\ 0 & \text{otherwise on } \mathbb{R}\,,\end{cases}$$

is of the first Baire class, but it is not the discrete limit of any sequence of functions from $\mathcal{A}$.

Indeed, for every set $B\in\mathcal{I}$ and for each open interval I, $I\setminus B\neq\emptyset$ and the restricted function $f/(\mathbb{R}\setminus B)$ is discontinuous on a dense subset of $I\setminus B$. So, by Theorem 13 from [1] the restricted function $f/(\mathbb{R}\setminus B)$ is not the discrete limit of any sequence of continuous functions.

THEOREM 2. *A function $f\in B(\mathcal{A})$ if and only if there are a set A with $\mathbb{R}\setminus A\in\mathcal{I}$ and a function $g\colon A\to\mathbb{R}$ of the first Baire class such that $g=f/A$.*

Proof. The proof is analogous as that of Theorem 1 for the discrete convergence. □

The limit of a sequence of functions from $B(\mathcal{A})$ is measurable and has the Baire property.

THEOREM 3. *If $f\colon\mathbb{R}\to\mathbb{R}$ is measurable and has the Baire property then there is a sequence of functions $f_n\in B(\mathcal{A})$, $n=1,2,\dots$, such that $f=\lim_{n\to\infty} f_n$.*

Proof. Since f is measurable, there is an G_δ-set B of measure zero and a function $g\colon\mathbb{R}\to\mathbb{R}$ of the second Baire class such that $f/(\mathbb{R}\setminus B)=g/(\mathbb{R}\setminus B)$.

Then $g = \lim_{n\to\infty} g_n$, where $g_n\colon \mathbb{R} \to \mathbb{R}$ $(n = 1,2,\ldots)$ is of the first Baire class. The function f has the Baire property, so by Kuratowski theorem ([5], Th. 8.1) there are an F_σ-set D of the first category and a continuous function $h\colon (\mathbb{R}\setminus D) \to \mathbb{R}$ such that $f/(\mathbb{R}\setminus D) = h$. Since h is continuous relative to an G_δ-set $\mathbb{R}\setminus D$, there is a function $\varphi\colon \mathbb{R}\to\mathbb{R}$ of Baire 1 such that $\varphi/(\mathbb{R}\setminus D) = h$.

Let $D = \bigcup_{n=1}^{\infty} D_n$, where D_n are closed and $D_n \subset D_{n+1}$ for all $n = 1,2,\ldots$, and let $f_n\colon \mathbb{R}\to\mathbb{R}$ $(n = 1,2,\ldots)$ be the function defined by the formula:

$$f_n(x) = \begin{cases} \varphi(x) & \text{for } x \in \mathbb{R}\setminus D_n\,, \\ g_n(x) & \text{for } x \in D_n\setminus B\,, \\ f(x) & \text{for } x \in D_n \cap B\,. \end{cases}$$

Observe that for $n = 1,2,\ldots$, the set $A_n = D_n \cap B \in \mathcal{I}$. Moreover, the restricted function $f/(\mathbb{R}\setminus A_n)$ is in the first Baire class since the restricted functions $f_n/(D_n\setminus B)$ and $f_n/(\mathbb{R}\setminus D_n)$ are of Baire 1.

So, by Theorem 2, the functions $f_n \in B(\mathcal{A})$ for $n = 1,2,\ldots$ and the proof is completed. $\square$

II. Functions of two variables

Now we will investigate functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that the sections $f_x(t) = f(x,t)$, $x,t\in\mathbb{R}$, are continuous.

We will say that a function $f\colon \mathbb{R}^2\to\mathbb{R}$ has the property $(\mathcal{E})$ ($f\in\mathcal{E}$) if there is a set $A\in\mathcal{I}$ such that the restricted function $f/\big((\mathbb{R}\setminus A)\times\mathbb{R}\big)$ is continuous.

Theorem 4. *If $f\colon \mathbb{R}^2\to\mathbb{R}$ is a function such that the sections f_x, $x\in\mathbb{R}$, are continuous and the sections $f^y(t) = f(t,y)$, $t,y\in\mathbb{R}$, belong to $\mathcal{A}$ then $f\in B(\mathcal{E})$.*

P r o o f. For $n = 1,2,\ldots$, $x\in\mathbb{R}$ and $y\in\left[\frac{k-1}{n},\frac{k}{n}\right]$, where $k = 0,-1,1,-2,2,\ldots$, let

$$f_n(x,y) = f\left(x,\frac{k-1}{n}\right) + \frac{f\big(x,\frac{k}{n}\big) - f\big(x,\frac{k-1}{n}\big)}{\frac{1}{n}}\cdot\left(y - \frac{k-1}{n}\right).$$

Fix a positive integer n. Since the sections f^y, $y\in\mathbb{R}$, belong to $\mathcal{A}$, for each integer k there is a set $A_k\in\mathcal{I}$ such that the function

$$\mathbb{R}\setminus A_k \ni x \longrightarrow f\left(x,\frac{k}{n}\right)$$

is continuous. The set

$$A = \bigcup_{k=-\infty}^{\infty} A_k \in \mathcal{I}$$

and the restricted function

$$f_n / \big((\mathbb{R} \setminus A) \times \mathbb{R} \big)$$

is evidently continuous. So $f_n \in \mathcal{E}$. From the continuity of the sections f_x, $x \in \mathbb{R}$, we obtain the equality

$$f = \lim_{n \to \infty} f_n ,$$

which completes the proof. □

The class $B(\mathcal{E})$ is contained in the class $\mathcal{H}$ of all functions $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that the sections f_x, $x \in \mathbb{R}$, are continuous and the sections f^y, $y \in \mathbb{R}$, are measurable and have the Baire property. In the class $\mathcal{H}$ we can introduce the metric of the uniform convergence ρ_C (see page 1).

Remark 2. The class $\mathcal{E}$ is a nowhere dense subset of the metric space $(\mathcal{H}, \rho_C)$.

P r o o f. Evidently, $\mathcal{E}$ is a closed subset of $(\mathcal{H}, \rho_C)$. Fix a function $g \in \mathcal{E}$ and a positive real r. Let $f \colon \mathbb{R} \to [0,1]$ be a measurable function with the Baire property from Example 1. Then f does not belong to $\mathcal{A}$ and the function

$$h = g + \frac{rf}{2} \in \mathcal{H} \setminus \mathcal{E} \qquad \text{and} \qquad \rho_C(g,h) = \frac{r}{2} < r .$$

So the set $\mathcal{E}$ is nowhere dense in $(\mathcal{H}, \rho_C)$. □

Example 2. As it is well known [2, 3] there are functions $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that the sections f_x and f^y, $x, y \in \mathbb{R}$, are continuous and the set

$$O(f) = \big\{ (x,y) \colon \operatorname{osc} f(x,y) \geq 1 \big\}$$

is of positive measure. Then for each set $A \in \mathcal{I}$ the restricted function $f\big((\mathbb{R} \setminus A) \times \mathbb{R}\big)$ is discontinuous at all points

$$(x,y) \in \big((\mathbb{R} \setminus A) \times \mathbb{R} \big) \cap O(f)$$

and consequently $f \in B(\mathcal{E}) \setminus \mathcal{E}$.

Theorem 5. *Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a function such that the sections $f^y \in \mathcal{A}$ for each $y \in \mathbb{R}$. Suppose that there is a set $B \in \mathcal{I}$ such that the sections f_x, $x \in \mathbb{R} \setminus B$, are equicontinuous at each point $y \in \mathbb{R}$, i.e., for each y and for each* *real* $r > 0$ *there is a positive real* $s = s(y, r)$ *such that* $|f_x(t) - f_x(y)| < r$ *for each* $t \in (y - s,\ y + s)$ *and each* $x \in \mathbb{R} \setminus B$. *Then* $f \in \mathcal{E}$.

Proof. Denote by $\mathbb{Q}$ the set of all rationals. Since the sections f^y, $y \in \mathbb{R}$, belong to $\mathcal{A}$, for each $y \in \mathbb{Q}$ there is a set $A(y) \in \mathcal{I}$ such that the restricted function

$$\mathbb{R} \setminus A(y) \ni x \to f^y(x)$$

is continuous. Observe that the set

$$A = B \cup \bigcup_{y \in \mathbb{Q}} A(y) \in \mathcal{I}.$$

We will prove that the restricted function $f/\big((\mathbb{R} \setminus A) \times \mathbb{R}\big)$ is continuous. For this fix points $x \in \mathbb{R} \setminus A$, $y \in \mathbb{R}$, and a positive real r. Since the sections f_w, $w \in \mathbb{R} \setminus B$, are equicontinuous at y, there is a positive real s such that

$$\big|f(w,t) - f(w,y)\big| < \frac{r}{6} \qquad \text{for} \ \ t \in (y - s,\ y + s) \qquad \text{and} \qquad w \in \mathbb{R} \setminus B.$$

Let u be a rational belonging to $(y - s,\ y + s)$. Since the restricted function $f^u/\big(\mathbb{R} \setminus A(u)\big)$ is continuous at the point x, there is a positive real z such that

$$\big|f(v,u) - f(x,u)\big| < \frac{r}{2} \qquad \text{for} \ \ v \in (x - z,\ x + z).$$

Let

$$U = (x - z,\ x + z) \times (y - s,\ y + s).$$

Then U is a neighbourhood of the point (x, y) and for $(t, v) \in U \cap (\mathbb{R} \setminus A) \times \mathbb{R}$ we have

$$\begin{aligned} \big|f(t,v) - f(x,y)\big| &\le \big|f(t,v) - f(t,y)\big| + \big|f(t,y) - f(t,u)\big| + \big|f(t,u) - f(x,u)\big| \\ &\quad + \big|f(x,u) - f(x,y)\big| < \frac{r}{6} + \frac{r}{6} + \frac{r}{2} + \frac{r}{6} = r. \end{aligned}$$

This completes the proof. □

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