Tatra Mt. Math. Publ. **24** (2002), 29–35

TATRA MOUNTAINS
*Mathematical Publications*

# QUASICONTINUITY, CLIQUISHNESS AND THE BAIRE PROPERTY OF FUNCTIONS OF TWO VARIABLES

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ABSTRACT. In this article we show that if $X$, $Y$ are topological spaces satisfying some additional conditions, then every function $f: X \times Y \to \mathbb{R}$ having upper and lower semi-quasicontinuous sections $f_x$ and upper (lower) semi-quasicontinuous [cliquish] {with the Baire property} sections $f^y$ is upper (lower) semi-quasicontinuous [cliquish] {with the Baire property} as the function of two variables.

Let $\mathbb{R}$ be the set of all reals and let $(X, T_X)$ be a topological space.

A function $h: X \mapsto \mathbb{R}$ is *quasicontinuous* [resp. *upper semi-quasicontinuous*] {*lower semi-quasicontinuous*} at a point $x \in X$ if for every positive real $\eta$ and for every open set $I$ containing $x$ there is a nonempty open set $J \subset I$ such that $h(J) \subset \big(h(x) - \eta, h(x) + \eta\big)$ [resp. $h(J) \subset \big(-\infty, h(x) + \eta\big)$ $\big\{h(J) \subset \big(h(x) - \eta, \infty\big)\big\}$] (see [7, 8]).

The function $h: X \mapsto \mathbb{R}$ is said *cliquish at a point* $x \in X$ if for every positive real $\eta$ and for every open set $I$ containing $x$ there is a nonempty open set $J \subset I$ such that $\operatorname{osc}_J h < \eta$ (see [8]).

There are upper (lower) semi-quasicontinuous functions which are not cliquish ([1]). However, if $f: X \to \mathbb{R}$ is an upper (lower) semi-quasicontinuous function and $(X, T_X)$ is a Baire space or if $f: X \to \mathbb{R}$ is locally bounded below (upper) then $f$ is cliquish ([1]).

But the following remark is true:

**Remark 1.** If a function $f: X \to \mathbb{R}$ is upper and lower semi-quasicontinuous at each point $x \in X$, then the function $f$ is cliquish.

2000 Mathematics Subject Classification: 26B05, 54C30, 54H05.
Keywords: quasicontinuity, cliquishness, Baire property, section, density topology, function of two variables.

Supported by the Bydgoszcz Pedagogical University grant 2000.

P r o o f . Fix a nonempty set $U \in T_X$ and a positive real $\eta$. Since $f$ is upper semi-quasicontinuous, there are a nonempty set $V \in T_X$ and a real $b$ such that

$$V \subset U \quad \text{and} \quad f(V) \subset (-\infty, b)\,.$$

Let $x \in V$ be a point such that

$$f(x) > \sup_V f - \frac{\eta}{3}\,. \tag{$*$}$$

Since $f$ is lower semi-quasicontinuous at $x$, there is a nonempty set $W \in T_X$ such that

$$W \subset V \quad \text{and} \quad f(W) \subset \Big(f(x) - \frac{\eta}{3}, \sup_V f\Big]\,. \tag{$**$}$$

From $(*)$ and $(**)$ it follows that

$$\operatorname{osc}_W f \le \frac{2\eta}{3} < \eta$$

and the proof is finished. □

Denote by $\mathcal{B}(X)$ the $\sigma$-field of the subsets of $X$ with the Baire property, i.e., the $\sigma$-field generated by the family composed of all open subsets of $X$ and all meager (i.e., of the first category) subsets of $X$.

The family of all functions $f\colon X \to \mathbb{R}$ which are measurable with respect to the $\sigma$-field $\mathcal{B}(X)$ (said functions with the Baire property) will be denoted by $\mathcal{B}(X, \mathbb{R})$.

**Remark 2.** ([11]) Every cliquish function $f\colon X \to \mathbb{R}$ has the Baire property.

If each point $x$ of the space $X$ has an open neighbourhood $U(x)$ satisfying the second countability axiom and if a topological space $(Y, T_Y)$ is a Baire space, then every function $f\colon X \times Y \to \mathbb{R}$ having quasicontinuous sections $f^y(t) = f(t, y)$, $t \in X$ and $y \in Y$, and quasicontinuous (cliquish) sections $f_x(u) = f(x, u)$, $x \in X$ and $u \in Y$, is quasicontinuous (cliquish) on $X \times Y$ as the function of two variables ([8] and [3]).

Moreover, under some additional assumptions on $X$ and $Y$, the quasicontinuity of all sections $f^y$, $y \in Y$, and the Baire property of all sections $f_x$, $x \in X$, imply the Baire property of $f$ on $X \times Y$ ([4]).

In this article I generalize these theorems replacing the quasicontinuity of the sections $f^y$ by a very general property $\mathcal{E}(X, \mathbb{R})$, where $\mathcal{E}(X, \mathbb{R})$ is the family of all functions $g\colon X \to \mathbb{R}$ which are upper and lower semi-quasicontinuous at each point $x \in X$.

**Remark 3.** If $X = \mathbb{R}$ and $T_X$ is the Euclidean topology in $\mathbb{R}$, then the family $Q(\mathbb{R})$ of all quasicontinuous functions $g\colon \mathbb{R} \to \mathbb{R}$ is a nowhere dense subset in the space $\mathcal{E}(\mathbb{R}, \mathbb{R})$ with the metric

$$\rho_C(g, h) = \min\Big(1, \sup_{x \in \mathbb{R}} \big|g(x) - h(x)\big|\Big)$$

of the uniform convergence.

P r o o f . Since the set $Q(\mathbb{R})$ is closed in the space $\mathcal{E}(\mathbb{R},\mathbb{R})$ ([8], p. 282), it suffices to prove that for every quasicontinuous function $f\colon \mathbb{R}\to\mathbb{R}$ and for each positive real $\eta$ there is a function $g\in\mathcal{E}(\mathbb{R},\mathbb{R})\setminus Q(\mathbb{R})$ such that $\rho_C(f,g)<\eta$.

For this let $x$ be a continuity point of $f$ and let

$$h(t)=\begin{cases}0 & \text{for } t<x\,,\\ \frac{\eta}{4} & \text{for } t=x\,,\\ \frac{\eta}{2} & \text{for } t>x\,.\end{cases}$$

Then the function $g=f+h$ belongs to $\mathcal{E}(\mathbb{R},\mathbb{R})$ and is not quasicontinuous at the point $x$. Moreover,

$$\rho_C(f,g)=\frac{\eta}{2}<\eta\,,$$

and the proof is completed. □

In Theorems 1, 2 and 3 we suppose that $(Y,T_Y)$ is a Baire topological space and $(X,T_X)$ is a topological space such that for every point $x\in X$ there is an open neighbourhood $U(x)$ having a countable basis of open sets.

**THEOREM 1.** *Let $f\colon X\times Y\to\mathbb{R}$ be a function such that the sections $f^y\in\mathcal{E}(X,\mathbb{R})$ for $y\in Y$. If the sections $f_x$, $x\in X$, are upper (lower) semi-quasicontinuous, then $f$ is upper (lower) semi-quasicontinuous as the function of two variables.*

In the proof of this theorem we apply the following lemma.

**LEMMA 1.** *Let $g\colon X\to\mathbb{R}$ be a function belonging to $\mathcal{E}(X,\mathbb{R})$. If there are a nonempty open set $V\subset X$ and a real $a$ such that the closure $\mathrm{cl}\big(g^{-1}\big((a,\infty)\big)\cap V\big)\supset V$ $\big(\mathrm{cl}\big(g^{-1}\big((-\infty,a)\big)\cap V\big)\supset V\big)$, then $g(x)\ge a$ $(g(x)\le a)$ on $V$.*

P r o o f . Suppose, by a contradiction, that there are a nonempty open set $V\subset X$, a point $u\in V$ and a real $a$ such that

$$\mathrm{cl}\big(g^{-1}\big((a,\infty)\big)\cap V\big)\supset V \qquad \text{and} \qquad g(u)<a\,. \tag{1}$$

But the function $g$ is upper semi-quasicontinuous at $u$, so there is a nonempty open set $U\subset V$ such that $g(x)<a$ on $U$, a contradiction with (1). This completes the proof. □

P r o o f o f T h e o r e m 1. Fix a point $(x,y)\in X\times Y$, an open neighbourhood $U$ of $(x,y)$, and a positive real $\eta$. By our hypothesis there are open sets $V\subset X$ and $W\subset Y$ such that

$$x\in V,\quad y\in W,\quad V\times W\subset U$$

and in $V$ there is a countable basis $(V_n)$ of open sets in $V$.

Since the section $f_x$ is upper semi-quasicontinuous at $y$, there is a nonempty open set $P \subset W$ such that

$$f(x,u) < f(x,y) + \frac{\eta}{4} \qquad \text{for} \quad u \in P.$$

The sections $f^u$, $u \in P$, are also upper semi-quasicontinuous at $x$, so for each point $u \in P$ there is a nonempty open set $V_{n(u)} \subset V$ such that

$$f(v,u) < f(x,u) + \frac{\eta}{4} \qquad \text{for} \quad v \in V_{n(u)}.$$

Since $(Y, T_Y)$ is a Baire space, the set $P$ is of the second category. So there is a positive integer $n$ such that the set

$$P_n = \{u \in P \colon n(u) = n\}$$

is of the second category. There is a nonempty open set $Q \subset \operatorname{int}(\operatorname{cl}(P_n))$, where $\operatorname{int}$ denotes the interior operation.

If $(v,u) \in V_n \times P_n$ then

$$f(v,u) < f(x,u) + \frac{\eta}{4} < f(x,y) + \frac{\eta}{4} + \frac{\eta}{4} = f(x,y) + \frac{\eta}{2}.$$

Consequently, by Lemma 1, for $(v,u) \in V_n \times Q$ we obtain

$$f(v,u) \le f(x,y) + \frac{\eta}{2} < f(x,y) + \eta$$

and the proof of the upper semi-quasicontinuity is completed. The proof of the lower semi-quasicontinuity is analogous. □

As an immediate corollary we obtain:

**THEOREM 2.** *If the sections* $f_x \in \mathcal{E}(Y, \mathbb{R})$, $x \in X$, *and* $f^y \in \mathcal{E}(X, \mathbb{R})$, $y \in Y$, *then also* $f \in \mathcal{E}(X \times Y, \mathbb{R})$.

In connection with Theorem 1 remind the following Example ([9]):

EXAMPLE 1. If $X = Y = \mathbb{R}$ and $T_X = T_Y$ is Euclidean topology in $\mathbb{R}$, then there is a set $A \subset \mathbb{R}^2$ such that for each straight line $l$ the cardinality $\operatorname{card}(l \cap A) \le 2$, and $A$ is not measurable in the Lebesgue sense and $A$ has not the Baire property. Then the characteristic function $f$ of the set $A$ has not the Baire property, but its sections $f_x$ and $f^y$, $x, y \in \mathbb{R}$, are upper semicontinuous.

**THEOREM 3.** *Let $f\colon X \times Y \to \mathbb{R}$ be a function such that the sections $f_x \in \mathcal{E}(Y, \mathbb{R})$ for $x \in X$ and the sections $f^y$ are cliquish for $y \in Y$. Then the function $f$ is cliquish as the function of two variables.*

Proof. Fix a point $(x, y) \in X \times Y$, a positive real $\eta$ and an open neighbourhood $U \subset X \times Y$ of the point $(x, y)$. There are open sets $V \in T_X$ and $W \in T_Y$ such that

$$(x, y) \in V \times W \subset U$$

and in $V$ there is a countable basis $(V_n)$ of open sets in $V$. Let $(I_n)$ be an enumeration of all open intervals with rational endpoints of the length $\leq \frac{\eta}{4}$.

Since the sections $f^y$, $y \in Y$, are cliquish, for each point $u \in W$ there are a set $V_{n(u)} \subset V$ and an open interval $I_{k(u)}$ such that

$$f^u(V_{n(u)}) \subset I_{k(u)}.$$

$(Y, T_Y)$ is a Baire space and the set of all pairs $(n, k)$ of positive integers is countable, so there is a pair $(i, j)$ of positive integers such that the set

$$P_{i,j} = \{u \in W\colon (n(u), k(u)) = (i, j)\}$$

is of the second category. If $Q \subset \operatorname{int}(\operatorname{cl}(P_{i,j}))$ is a nonempty open set then, by Lemma 1,

$$f(v, u) \in \operatorname{cl}(I_j) \qquad \text{for } (v, u) \in V_i \times Q$$

and the proof is completed. $\square$

In connection with Theorem 3 we remind the following Example:

EXAMPLE 2. Let $X = Y = \mathbb{R}$ and let $T_X = T_Y$ be the density topology $T_d$ in $\mathbb{R}$ ([1, 9]). Martin's Axiom implies that there is a function $f\colon \mathbb{R}^2 \to [0, 1]$ which is not cliquish (with respect to $T_d \times T_d$) and for which the sections $f_x$, $x \in \mathbb{R}$, are $T_d$-continuous and the sections $f^y$, $y \in \mathbb{R}$, are $T_d$-quasicontinuous ([5]).

**LEMMA 2.** *Let $(X, T_X)$ be a topological space with the countable $\pi$-basis and let $f\colon X \to \mathbb{R}$ be a function. If*

(i) *for every non-meager set $A \in \mathcal{B}(X)$ and for each real $\eta > 0$ there is a non-meager set $B \in \mathcal{B}(X)$ with $B \subset A$ and $\operatorname{osc}_B f \leq \eta$,*

*then $f$ has the Baire property.*

Proof. By the hypothesis for each real $\eta > 0$ we can find a countable family of sets $A_n \in \mathcal{B}(X)$ of the second category such that

$$\operatorname{osc}_{A_n} f \leq \eta, \quad A_n \cap A_m = \emptyset \qquad \text{for } n \neq m,\ n, m = 1, 2, \ldots$$

and

$$X \setminus \bigcup_{n=1}^{\infty} A_n$$

is of the first category. In each set $A_n$ fix a point $x_n \in A_n$. Putting

$$g(x) = f(x_n) \qquad \text{for } \ (x) \in A_n\,, \ n \ge 1\,,$$

and

$$g(x) = f(x) \qquad \text{for } \ (x) \in X \Big\backslash \bigcup_n A_n\,,$$

we obtain a function $g \in \mathcal{B}(X,\mathbb{R})$ with $|f-g| \le \eta$. So $f$ has the Baire property. $\square$

In the next theorem we suppose that the topological spaces $(X,T_X)$ and $(Y,T_Y)$ are Baire spaces with countable $\pi$-bases.

**Theorem 4.** *Let $f\colon X \times Y \to \mathbb{R}$ be a function such that the sections $f_x$, $x \in X$, have the Baire property and for each $y \in Y$ the section $f^y \in \mathcal{E}(X,\mathbb{R})$. Then the function $f$ has the Baire property as the function of two variables.*

P r o o f. We will prove that the function $f$ satisfies the condition (i) of Lemma 2. Take a set $A \in \mathcal{B}(X \times Y)$ of the second category and a real $\eta > 0$. Let $D$ be the set of all points

$$y \in \Pr_Y(A) = \big\{u \in Y\colon \exists_{x\in X}\ (x,u) \in A\big\}$$

such that the section

$$A^y = \big\{x \in X\colon (x,y) \in A\big\}$$

is of the second category and belongs to $\mathcal{B}(X)$. Since the sections $f^y$, $y \in Y$, are cliquish, for each point $y \in D$ there is an open set $V_{n(y)}$ and a closed interval $I(y)$ with rational endpoints such that $V_{n(y)} \setminus A^y$ is of the first category, $f^y(V_{n(y)}) \subset I(y)$ and the length $d\big(I(y)\big)$ of the interval $I(y)$ is $\le \frac{\eta}{2}$. The set $D$ is of the second category and the family of all pairs $(V_n, I)$, where $I$ is an interval with rational endpoints and $n = 1,2,\dots$, is countable, so there are a positive integer $j$ and a closed interval $I$ such that the set

$$P = \big\{y \in D\colon n(y) = j \ \text{ and } \ I(y) = I\big\}$$

is of the second category. There is a set $Q \in \mathcal{B}(Y)$ of the second category such that every set $M \subset Q \setminus P$ belonging to $\mathcal{B}(Y)$ is of the first category. Let

$$F = A \cap (V_j \times Q)\,.$$

Then $F \subset A$, $F \in \mathcal{B}(X \times Y)$ and $F$ is of the second category. If a point $(x,y) \in F$ and $f(x,y) \in \mathbb{R} \setminus I$, then there is a set $V_{n(x,y)} \subset V_j$ such that $f^y(V_{n(x,y)}) \subset \mathbb{R} \setminus I$. So, if the set $F \setminus f^{-1}(I)$ is of the second category, then there is an open set $V_k \subset V_j$ such that the set

$$G = \big\{y \in Q\colon f^y(V_k) \subset \mathbb{R} \setminus I\big\}$$

is of the second category and consequently the sections $f_x$ have not the Baire property for $x \in V_k$. This contradiction shows that the set $F \setminus f^{-1}(I)$ is of the first category and consequently the set $B = F \cap f^{-1}(I) \subset A$ is of the second category. Since $f^{-1}(B) \subset I$, we obtain $\operatorname{osc}_B f \le \eta$. So, by Lemma 2 the function $f$ has the Baire property. $\square$

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Received September 26, 2000

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