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CONFIDENCE INTERVALS FOR VARIANCE COMPONENTS IN BALANCED ANOVA MODELS

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. Testing hypothesis concerning the unknown variance components is an important part of data analysis. Under the standard normality assumptions F-tests exist for testing the significance of the variance components in many problems. But there are some applications, in which F-tests do not exist. Testing variance components in mixed linear models were given in Mathew and Zhou [*Some tests for variance components using generalized p-values*, Technometrics **36** (1994), 394–402]. Our purpose is to construct the confidence intervals for variance components using generalized p-values.

1. Introduction

The purpose of this article is to construct the confidence intervals for variance components. While the conventional methods not always give exact solutions, we demonstrate some new approach to solve problems for which exact F-tests do not exist. This approach is based on the concept of a generalized p-value introduced by Tsui and Weerahandi ([4]). Burdick and Graybill ([1]) derived the approximate confidence intervals for linear combination of variance components. We will compare method using p-values with the method of Burdick and Graybill by simulations and we will observe the probability of covering of variance component by confidence interval.

In this article we suppose a balanced mixed model that has n random effects. The effects are normal independent random variables with mean values equal to zeros and with required variance components. Let σ_l^2, $l = 1, \dots, n$, denote variance components of the effects. Let SS_l, $l = 1, \dots, n$, denote ANOVA sums of squares. Under normality assumptions we have

$$SS_l \sim E(MS_l)\chi^2_{\nu_l},$$

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where $E(MS_l) = E(SS_l/\nu_l)$ are expected values of the mean sum of squares, $\chi^2_{\nu_l}$ are independent central χ^2 random variables with ν_l, $l = 1, \dots, n$, degrees of freedom. Then $\nu_l MS_l/E(MS_l)$ for $l = 1, \dots, n$ are independently distributed as chi-squared random variables with ν_l degrees of freedom.

2. Definition of generalized p-value and confidence intervals based on p-value

As in the case of hypothesis testing, exact confidence intervals for statistical problems involving nuisance parameters are available only in special cases. For example, exact confidence intervals for the second moment of a normal distribution are not available. The Behrens-Fisher problem is probably the most well known problem in which exact confidence intervals based on minimal sufficient statistics do not exist.

To describe the underlying problem in more detail, consider a population with cumulative distribution function $F(\mathbf{x}, \boldsymbol{\nu})$, where $\mathbf{X}$ is a random vector, $\boldsymbol{\nu} = (\theta, \boldsymbol{\delta}')'$ is a vector of unknown parameters, θ is a parameter of interest and $\boldsymbol{\delta}$ is a vector of nuisance parameters. Let Ξ be the sample space and $\mathbf{x}$ be the observed value of $\mathbf{X}$. We are interested in finding interval estimates of θ based on observed values of $\mathbf{X}$. The problem is to construct confidence intervals of the form $\big[A(\mathbf{x}), B(\mathbf{x})\big] \subset \Theta$, where $A(\mathbf{x})$ and $B(\mathbf{x})$ are functions of the observed data $\mathbf{x}$. In order to understand the restrictive nature of conventional confidence intervals, let us examine the conventional definitions. To fix ideas, consider a random sample $\mathbf{X} = (X_1, \dots, X_n)'$ taken from a distribution depending on a parameter θ. Let $A(\mathbf{X})$ and $B(\mathbf{X})$ be two statistics satisfying the equation

$$P\big(A(\mathbf{X}) \leq \theta \leq B(\mathbf{X})\big) = \alpha\,, \tag{1}$$

where α is a prespecified constant between 0 and 1. Let $a = A(\mathbf{x})$ and $b = B(\mathbf{x})$ be the observed values of the two statistics. Then $[a, b]$ is called confidence interval for θ with the confidence coefficient α.

Although this approach to constructing interval estimates is simple and easy to implement, in most applications involving nuisance parameters it is not easy to find $A(\mathbf{X})$ and $B(\mathbf{X})$ that satisfy (1) for all possible values of the nuisance parameters. The idea is to make this possible by making probability statement relative to the observed values. We allow the functions $A()$ and $B()$ to depend not only on the observable random vector $\mathbf{X}$ but also on the observed data $\mathbf{x}$. We'll define a generalized test variable (see [5]).

Definition 1. A *generalized test variable* $T = T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$ is any function with the following three properties:

1. The probability distribution of $T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$ is free of nuisance vector parameters $\boldsymbol{\delta}$.
2. Observed value $t = T(\mathbf{x}, \mathbf{x}, \boldsymbol{\nu})$ does not depend on unknown parameters.
3. For fixed $\mathbf{x}$ and $\boldsymbol{\delta}$, and for any given t, $P\big(T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu}) \geq t | \theta\big)$ is monotonic function of θ.

Definition 2. $T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$ is *stochastically increasing in* θ, when $P\big(T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu} > t)\big)$ is nondecreasing in θ for fixed $\mathbf{x}$ and $\boldsymbol{\delta}$.

Consider the problem of testing left-sided null hypothesis of the form $H_0\colon \theta \leq \theta_0$ against $H_1\colon \theta > \theta_0$, where θ is the parameter of interest and θ_0 is a specified value of the parameter. Assume that $T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$ is stochastically increasing in θ.

Definition 3. Let $T = T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$ be stochastically increasing generalized test variable. Then

$$p = P\big(T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu}) \geq t_{\mathrm{obs}} | \theta = \theta_0\big),$$

where $t_{\mathrm{obs}} = T(\mathbf{x}, \mathbf{x}, \boldsymbol{\nu}_0)$, $\boldsymbol{\nu}_0 = (\theta_0, \boldsymbol{\delta}')'$, *is generalized* p*-value for testing* H_0 and function

$$\pi(\theta) = P\big(T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu}) \geq t_{\mathrm{obs}} | \theta\big)$$

is called *a data-based power function of generalized test variable* $T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$.

If $T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$ is stochastically decreasing in θ, then $p = P\big(T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu}) \leq t_{\mathrm{obs}} | \theta = \theta_0\big)$ is generalized p-value for testing H_0. Moreover, if the null hypothesis H_0 is right-sided, then generalized p-value for testing H_0 can be computed using formula $p = P\big(T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu}) \leq t_{\mathrm{obs}} | \theta = \theta_0\big)$, or the formula $p = P\big(T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu}) \geq t_{\mathrm{obs}} | \theta = \theta_0\big)$ depending on whether $T(\mathbf{X}, \mathbf{x}, \boldsymbol{\nu})$ is stochastically increasing or decreasing in θ. In each of these cases, the generalized p-value serves to measure how well data supports or contradicts the null hypothesis. The smaller values favor the alternative hypothesis and larger values favor the null hypothesis.

To find a generalized test variable is a difficult task. This problem could be reduced by known and used methods. Likewise to other problems of a statistical inference, first we reduce problem by a principle of sufficiency. If a set of minimal sufficient statistics consists of more than one statistic, number of statistics can be reduced by a principle of invariance and similarity. The generalized test variable will be based on this set.

When the form of a p-value of a one-sided test is available, generalized confidence intervals for θ can be deduced directly from its power function. Assume that the p-value is based on a generalized test variable whose distribution function is stochastically increasing in θ. Let p_0 be a p-value constructed for testing one-sided null hypothesis. When the form of the p-value is known in terms of θ_0, the power function $\pi(\theta)$ of the test is obtained by replacing θ_0 by θ. Then a set

$$\{\theta\colon \alpha_1 \le \pi(\theta) \le \alpha_2\}, \quad \alpha_1 + \alpha_2 = 1,$$

leads to an exact $100(\alpha_2-\alpha_1)$ % confidence region for θ.

3. Confidence interval for variance components based on p-value in ANOVA model

Zhou and Matthew ([2]) considered a problem that deals with hypothesis testing for variance components in balanced mixed linear models where exact F-tests do not exist. They suggested the solution of this problem using generalized p-values.

Suppose a model defined in Section 1. The generalized testing problem is

$$H_0\colon \sigma_1^2 \le \sigma_{10}^2 \quad \text{versus} \quad H_1\colon \sigma_1^2 > \sigma_{10}^2,$$

where σ_{10}^2 is a given constant. Let SS_k, $k = 1,\dots,m$, denote the required ANOVA sums of squares such that appear in the unbiased estimator of the variance component of interest, i.e., σ_1^2. We have

$$SS_k \sim E(MS_k)\chi^2_{\nu_k},$$

where $E(MS_1) = \left(a_1\sigma_1^2 + \sum\limits_{l=2}^{n} a_l\sigma_l^2\right)$ and $E(MS_k) = \left(\sum\limits_{l=2}^{n} b_{kl}\sigma_l^2\right)$ and where a_l and b_{kl} are known nonnegative scalars. Let the unbiased analysis of variance estimator of σ_1^2 based on SS_k be of the form

$$\begin{aligned}\widehat{\sigma_1^2} &= \frac{1}{a_1}\left(\frac{SS_1}{\nu_1} + \sum_{k=2}^{r}\frac{SS_k}{\nu_k} - \sum_{k=r+1}^{m}\frac{SS_k}{\nu_k}\right)\\ &= \frac{1}{a_1}\left(\sum_{k=1}^{r} MS_k - \sum_{k=r+1}^{m} MS_k\right),\end{aligned}$$

first r sums of squares appear with the positive sign and the other $(m-r)$ sums of squares appear with the negative sign at the unbiased estimator of σ_1^2.

Note that by the convenient ordering MS_k we are able to construct the unbiased ANOVA estimator of other variance components σ_l^2, $l = 2,\dots,n$. In the balanced case, the unbiased estimator of any variance component, based on the sums of squares, is unique because the sums of squares are complete and sufficient.

Let ss_k be the observed values of SS_k. Then the generalized test variable for testing H_0 against H_1 is of the form

$$T(SS_1,\ldots,SS_m,ss_1,\ldots,ss_m,\sigma_1^2,\ldots,\sigma_n^2)=\frac{a_1\sigma_1^2+\sum\limits_{k=r+1}^{m}E(MS_k)\frac{ss_k}{SS_k}}{\sum\limits_{k=1}^{r}E(MS_k)\frac{ss_k}{SS_k}}.$$

The following holds true:

(1) The distribution of $T\sim\frac{a_1\sigma_1^2+\sum_{k=r+1}^{m}\frac{ss_k}{\chi^2_{\nu_k}}}{\sum_{k=1}^{r}\frac{ss_k}{\chi^2_{\nu_k}}}$ is free of the nuisance parameters $\sigma_2^2,\ldots,\sigma_n^2$.

(2) The observed value $t=T(ss_1,\ldots,ss_m,ss_1,\ldots,ss_m,\sigma_1^2,\ldots,\sigma_n^2)=1$ does not depend on the unknown parameters $\sigma_1^2,\ldots,\sigma_n^2$.

(3) $P(T\geq t|\ \sigma_1^2)$ is stochastically increasing in σ_1^2.

Then

$$p=P(T>1|\sigma_1^2=\sigma_{10}^2)=P\left(\sum_{k=r+1}^{m}\frac{ss_k}{\chi^2_{\nu_k}}-\sum_{k=1}^{r}\frac{ss_k}{\chi^2_{\nu_k}}>-a_1\sigma_{10}^2\right)$$

is the generalized p-value for testing H_0 against H_1. In general, the p-value cannot be obtained analytically and usually it have to be obtained through simulation. We will compute it by method for evaluation of the distribution function of a linear combination of independent inverted gamma random variables ([8]). $100(\alpha_2-\alpha_1)$ % generalized confidence interval for σ_1^2 is given by

$$\{\sigma_1^2\colon \alpha_1\leq P(T>1|\ \sigma_1^2)\leq\alpha_2\},\qquad \text{where } \alpha_1,\alpha_2\in(0,1).$$

4. Confidence intervals on linear combinations of variance components by method of Graybill and Burdick

In some situations there exist neither exact confidence intervals for a given variance component nor their linear combination. For example, for balanced one-way random model with variance components σ_a^2, σ_e^2 we are able to construct a confidence interval neither for σ_a^2 nor for $\sigma_a^2+\sigma_e^2$. Burdick and Graybill (1992) derived approximate confidence intervals for various linear combinations and ratios of variance components for balanced or unbalanced ANOVA models too. Suppose a mixed model defined in Section 1. A problem that occurs in statistical applications is that of finding confidence intervals for linear combinations of variance components. We are interested in finding the confidence intervals for the parameter

$$d = \sum_{i=1}^{p} c_i \vartheta_i - \sum_{j=p+1}^{q} c_j \vartheta_j ,$$

where $c_i, c_j \geq 0$ and ϑ_i are linear combinations of variance components.

Method of Graybill and Burdick is derived from the fact, that the unbiased estimator of linear combination $d = \sum_{i=1}^{p} c_i \vartheta_i - \sum_{j=p+1}^{q} c_j \vartheta_j$ is $\widehat{d} = \sum_{i=1}^{p} c_i MS_i - \sum_{j=p+1}^{q} c_j MS_j$, because MS_i is the unbiased ANOVA estimator of ϑ_i. Then the approximate $(1-2\alpha)100$ % confidence interval for d is of the form $[L, U]$. L is the lower bound, where

$$L = \widehat{d} - \sqrt{l} ,$$

$$\widehat{d} = \sum_{i=1}^{p} c_i MS_i - \sum_{j=p+1}^{q} c_j MS_j ,$$

$$l = \sum_{i=1}^{p} G_i^2 c_i^2 MS_i^2 - \sum_{j=p+1}^{q} H_j^2 c_j^2 MS_j^2$$

$$+ \sum_{i=1}^{p} \sum_{j=p+1}^{q} G_{ij} c_i c_j MS_i MS_j + \sum_{i=1}^{p-1} \sum_{t>1}^{p} G_{it}^* c_i c_t MS_i MS_t .$$

$G_i, H_j, G_{ij}, G_{it}^*$ are defined:

$$G_i = 1 - \frac{1}{F_{\alpha;\, \nu_i\, ,\, \infty}}, \ (i = 1, \dots, p), \quad H_j = \frac{1}{F_{1-\alpha;\, \nu_j\, ,\, \infty}} - 1 , \quad (j = p+1, \dots, q) ,$$

$$G_{ij} = \frac{(F_{\alpha;\, \nu_i,\, \nu_j} - 1)^2 - G_i^2 F_{\alpha;\, \nu_i,\, \nu_j}^2 - H_j^2}{F_{\alpha;\, \nu_i,\, \nu_j}} ,$$

$$G_{it}^* = \left[\left(1 - \frac{1}{F_{\alpha;\, \nu_i + \nu_t\, , \infty}} \right)^2 \frac{(\nu_i + \nu_t)^2}{\nu_i \nu_t} - \frac{G_i^2 \nu_i}{\nu_t} - \frac{G_t^2 \nu_t}{\nu_i} \right] \Big/ (p-1) ,$$

$t = i+1, \dots, p$, where $F_{\alpha;\nu_i,\nu_j}$ is the critical value of F distribution with ν_i, ν_j degrees of freedom and α is a significance level. The upper bound is

$$U = \widehat{d} + \sqrt{u} ,$$

where

$$u = \sum_{i=1}^{p} H_i^2 c_i^2 MS_i^2 + \sum_{j=p+1}^{q} G_j^2 c_j^2 MS_j^2$$

$$+ \sum_{i=1}^{p} \sum_{j=p+1}^{q} H_{ij} c_i c_j MS_i MS_j + \sum_{i=1}^{q-1} \sum_{u>j}^{q} H_{uj}^* c_j c_u MS_j MS_u .$$

$H_i, G_j, H_{ij}, H^*_{ju}$ are defined:

$$G_j = 1 - \frac{1}{F_{\alpha;\,\nu_j,\,\infty}}\,, \quad (j = p{+}1, \ldots, q)\,, \quad H_i = \frac{1}{F_{1-\alpha;\,\nu_i,\,\infty}} - 1\,, \quad (i = 1, \ldots, p)\,,$$

$$H_{ij} = \frac{(1 - F_{1-\alpha;\,\nu_i,\,\nu_j})^2 - H_i^2 F^2_{1-\alpha;\,\nu_i,\,\nu_j} - G_j^2}{F_{1-\alpha;\,\nu_i,\,\nu_j}}\,,$$

$$H^*_{ju} = \left[\left(1 - \frac{1}{F_{\alpha;\,\nu_j+\nu_u,\infty}}\right)^2 \frac{(\nu_j + \nu_u)^2}{\nu_j \nu_u} - \frac{G_j^2 \nu_j}{\nu_u} - \frac{G_u^2 \nu_u}{\nu_j}\right] \Big/ (q - p - 1)\,,$$

$$u = j{+}1, \ldots, q\,.$$

5. Simulation study

In this section we will compute the confidence intervals using both methods. We will compare results by probabilities of coverage of the confidence intervals. The model was generated by simulations.

Example 1. Consider a balanced two-way random effects model with interactions. This model is of the form:

$$Y_{ijk} = \mu + A_i + B_j + (AB)_{ij} + E_{ijk}\,,$$

$$i = 1, \ldots, I\,; \quad j = 1, \ldots, J\,; \quad k = 1, \ldots, K > 1\,,$$

where μ is an unknown constant and A_i, B_j, $(AB)_{ij}$ and E_{ijk} are normal independent random variables with mean values equal to zeros and with variance components σ^2_A, σ^2_B, σ^2_{AB} and σ^2_E.

Analysis of variance table of this model is:

Source	degrees of freedom	MS_k	EMS_k
A	$\nu_1 = I - 1$	SS_1/ν_1	$\vartheta_1 = \sigma^2_E + K\sigma^2_{AB} + JK\sigma^2_A$
B	$\nu_2 = J - 1$	SS_2/ν_2	$\vartheta_2 = \sigma^2_E + K\sigma^2_{AB} + IK\sigma^2_B$
AB	$\nu_3 = (I-1)(J-1)$	SS_3/ν_3	$\vartheta_3 = \sigma^2_E + K\sigma^2_{AB}$
Error	$\nu_4 = IJ(K-1)$	SS_4/ν_4	$\vartheta_4 = \sigma^2_E$
Total	$IJK - 1$		

where SS_k are required sums of squares, MS_k are the mean sums of squares and EMS_k expected values of mean sums of squares.

Here, we consider the analysis of variance model with constants given in the following table:

Source	df	EMS_k
A	4	$\vartheta_1 = \sigma_E^2 + 5\sigma_{AB}^2 + 30\sigma_A^2$
B	5	$\vartheta_2 = \sigma_E^2 + 5\sigma_{AB}^2 + 25\sigma_B^2$
AB	20	$\vartheta_3 = \sigma_E^2 + 5\sigma_{AB}^2$
error	120	$\vartheta_4 = \sigma_E^2$

For each combination of variance components (see table below), we generated 2000 times the required sums of squares and constructed the 95 % confidence interval for σ_A^2. Using formulas from Graybill and Burdick theory for construction of 95 % confidence interval for σ_A^2 we obtain for given constants these probabilities of coverage:

$\sigma_{AB}^2 \diagdown \sigma_A^2$	0.1	1	10	100
0.1	0.943	0.945	0.951	0.949
1	0.952	0.951	0.946	0.952
10	0.948	0.950	0.951	0.950
100	0.952	0.950	0.952	0.951

$\sigma_E^2 = 1$. Then we derived confidence intervals using p-value. Unbiased ANOVA estimator of σ_A^2 is $\widehat{\sigma_A^2} = \frac{1}{30}(MS_1 - MS_3)$, $E(\widehat{\sigma_A^2}) = \frac{1}{30}\big(E(MS_1) - E(MS_3)\big) = \frac{1}{30}(\vartheta_1 - \vartheta_3)$ and therefore the generalized test variable for this model is of the form:

$$T(SS_1, SS_3, ss_1, ss_3, \sigma_A^2, \sigma_{AB}^2, \sigma_E^2) = \frac{30\sigma_A^2 + \vartheta_3 \frac{ss_3}{SS_3}}{\vartheta_1 \frac{ss_1}{SS_1}}.$$

The distribution of T is:

$$T \sim \frac{30\sigma_A^2 + \frac{ss_3}{\chi_{20}^2}}{\frac{ss_1}{\chi_4^2}}.$$

Using data-based power function of the generalized test variable T we compute confidence intervals for σ_A^2. We obtained the 95 % confidence interval for σ_A^2 from the inequality:

$$0.025 \le \pi(\sigma_A^2) = P\big(T > 1 |\ \sigma_A^2\big) = P\left(\frac{ss_3}{\chi_{20}^2} - \frac{ss_1}{\chi_4^2} > -30\sigma_A^2\right) \le 0.975\,.$$

Based on 2000 simulations we obtained the coverage probabilities:

$\sigma_{AB}^2 \diagdown \sigma_A^2$	0.1	1	10	100
0.1	0.952	0.952	0.949	0.949
1	0.953	0.948	0.946	0.958
10	0.954	0.955	0.945	0.957
100	0.963	0.952	0.954	0.958

About rightness of results we can make sure by test of hypothesis $H_0: p = 0.95$. From confidence interval for p we see that with 99 % probability values between 0.937 and 0.963 belong to this interval. Both methods of construction confidence intervals for variance components are good and probabilities of coverage come to 95 % probability by required 95 % confidence intervals.

6. Conclusion

In this paper is described approach in statistical problems when exact confidence intervals of hypothesis of variance components do not exist. This approach is maybe not the only one but it gives good results and exact confidence intervals for variance components. The other alternative is the method of Graybill and Burdick. This paper consists of derivation of generalized test variables and p- values in balanced ANOVA models. The unbalanced case is unsolved. It's solution suggested by Zhou and Mathew (see [2]).

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