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CLASSIFYING ALGEBRAIC SCHEMES FOR MUSICAL MANIFOLDS

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ABSTRACT. Classification of global musical objects deals with the determination of isomorphism classes in adequate categories of global compositions, the type of objects which play the role of musical manifolds. It relies on two constructions: coefficient systems of affine functions and resolutions of global compositions. These structures are introduced and discussed. As a main result, we derive classifying schemes, and shortly comment on their musical meaning. From the musicological point of view, this is one of the most difficult chapters of mathematical music theory since the relation between classification and musicology, in particular: esthetics, is quite implicit. But classification is a deep concern since it reveals the a priori extent of a structural framework and therefore its power as an expressive tool of artistic activity will be in composition, performance, or understanding.

Introduction and notations

Classification of musical structures, i.e., the determination of isomorphism classes in adequate categories, is highly controversial in musicology by three reasons:

- It is commonly believed that classifying musical objects on whatever level are contrarious to the individual expressivity of compositions as they are cultivated since the Renaissance.
- Classification is misunderstood as a purely bureaucratic activity of list compilation.
- Due to a catastrophical lack of technical tools, traditional musicology has only rarely been able to control the variety of their objects.

The third point shows a disdain of detailed technical work which is psychologically comprehensible and must scientifically be blamed for a major scientific retard even with respect to other humanities such as linguistics.

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We have already vaporized these mystifications on several occasions, e.g., [5, 8], and we will stress one main argument here again: Classification is nothing else as the task of totally understanding an object. This is Y o n e d a's lemma in its full philosophical implication, in fact, the isomorphism class of an object X is equivalent to that of its contravariant Yoneda functor $\mathrm{Hom}(-,X)$. The latter boils down to the sum of all perspectives $Y \to X$ (morphisms) under which X may be 'observed'. Such a result is in a complete harmony with A d o r n o's, V a l é r y's, and B ä t s c h m a n n's insights in the theory of arts. They state that understanding works of art means a synthesis of all their interpretative perspectives, see [8, 10] for details. Results and methods regarding classification of musical structures have already been applied to the theory of the string quartet, composition, and performance [8, 6, 7, 9].

Throughout this paper, we work in the category ${}_R\mathrm{Mod}$ of left modules over a commutative ring R with R-affine morphisms. For two R-modules M, N, the set of R-affine morphisms is denoted by $M@_R N$. Recall that such a morphism has the shape $f = e^n.f_0$, where $e^n(x) = n+x$ is the translation by n on N, and where $f_0\colon M \to N$ is the linear part of f. The set of R-linear homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M,N)$. Observe that for certain universal constructions we admit the empty module (!) in the category ${}_R\mathrm{Mod}$.

1. Local and global compositions

In this paper, we fix an 'address' module A in ${}_R\mathrm{Mod}$. This is a kind of 'point of view' through which all other modules will be seen in the above sense of the Yoneda lemma: Understanding the isomorphism class of a module M amounts to understand its contravariant functor $@_R M$, and therefore all points of view $B@_R M$ from variable 'addresses' B. Here are the central local objects of mathematical music theory [8]:

Definition 1. For the address module A, an *A-addressed objective finite local composition* is a pair $(K, A@_R M)$ with a finite subset $K \subset A@_R M$. The subset K is called the *support* of the local composition, and M its *ambient space.* The category $\mathrm{ObLoc}^{\mathrm{finite}}_{@A}$ has the A-addressed finite objective local compositions as its objects while a morphism $f\colon (K, A@_R M) \to (L, A@_R N)$ in $\mathrm{ObLoc}^{\mathrm{finite}}_{@A}$ is a set map $f\colon K \to L$ which extends to a Yoneda-map $A@h\colon A@_R M \to A@_R N\colon u \mapsto h.u$ for an affine morphism $h\colon M \to N$.

Since we do not deal with infinite or non-objective (functorial) local compositions in this paper, and since the address never changes here (see [10] for these general points of view), we will omit these specifications and simply speak about local compositions, the address A being tacitly assumed.

If $(K, A@_R M)$ is a local composition, we consider the R-submodule $R.K$ of the R-module $A@_R M$. It is defined as the module generated by all differences $k - k'$, $k, k' \in K$; and we set $R.\emptyset = 0$. This defines a functor $R.?: \mathrm{ObLoc}^{\mathrm{finite}}_{@A} \to {}_R\mathrm{Mod}$ with values in the subcategory of ${}_R\mathrm{Mod}$ of left R-modules with *linear* homomorphisms. To a morphism $f\colon (K, A@_R M) \to (L, A@_R N)$, we associate the homomorphism $R.f\colon R.K \to R.L$ which is well-defined by the rule $R.f(k - k') = f(k) - f(k')$, $k, k' \in K$, on the generators of $R.K$. We call a local composition $(K, A@_R M)$ embedded if $K \subset R.K$.

Example 1. For the zero address $A = 0$, we have $A@_R M \xrightarrow{\sim} M$, and local compositions are those objects considered in classical mathematical music theory [10].

Example 2. For $M = A = \mathbb{Z}_{12}$, we obtain the case of self-addressed Riemann function theory (the musicologist, not the mathematician!) as it was developed in [11].

Local compositions are glued together to define 'musical manifolds':

Definition 2. An *A-addressed global objective finite composition* is defined by the following data:

(i) a set G and a finite, non-empty covering I of G,

(ii) a family $(K_t, A@_R F_t)_{t \in T}$ of A-addressed objective finite local compositions,

(iii) a surjection $I_?\colon T \to I\colon t \mapsto I_t$,

(iv) a bijection $\phi_t\colon K_t \xrightarrow{\sim} I_t$ for each $t \in T$,

(v) for each couple $s, t \in T$, the induced bijection

$$\phi_{s,t} := \phi_t^{-1} \circ \phi_s\colon \phi_s^{-1}(I_s \cap I_t) \to \phi_t^{-1}(I_s \cap I_t)$$

(restricted to the respective domains and codomains) defines an isomorphism

$$\phi_{s,t}\colon \big(\phi_s^{-1}(I_s \cap I_t),\, A@_R F_s\big) \xrightarrow{\sim} \big(\phi_t^{-1}(I_s \cap I_t),\, A@_R F_t\big)$$

of local compositions.

The data (ii) – (v) are called an *A-addressed atlas Φ for the covering I of G*. The bijections ϕ_t (or — by abuse of language — the local compositions K_t) are called the *charts of the atlas Φ*.

Two A-addressed atlases Φ, Ψ for the covering I of G are called *equivalent* iff their disjoint sum[1] $\Phi \amalg \Psi$ is an A-addressed atlas for the covering I of G.

[1] $\Phi \amalg \Psi$ has index set $T_\Phi \amalg T_\Psi$, and the atlas corresponding to the coproduct surjection $T_\Phi \amalg T_\Psi \to I$.

An A-addressed objective global composition is a covering I of G (often abbreviated as G^I) together with an equivalence class of A-addressed atlases for G^I.

Since we never change addresses and do not deal with infinite or non-objective (functorial) global compositions, we will omit these specifications and simply speak about global compositions in the sequel.

Given a global composition G^I, the *abstract nerve* or simplicial complex $n(G^I)$ of G^I, as well as its *geometric realization* $N(G^I)$ are defined. Recall $N(G^I)$ a union of affine k-implices $|\sigma|$ associated with the abstract k-implices $\sigma \in n(G^I)$ such that for any two different $|\sigma|, |\tau|$, the interiors are disjoint, i.e., $|\sigma|^o \cap |\tau|^o = \emptyset$.

The category $\mathrm{ObGlob}^{\mathrm{finite}}_{@A}$ of A-addressed objective finite global compositions is defined via its morphisms as follows:

Definition 3. Suppose we are given two (A-addressed objective finite global) compositions G^I and H^J. Then a morphism from G^I to H^J is a pair (f, ι) where

1. $f\colon G \to H$ is a set map,
2. $\iota\colon I \to J$ is a set map such that $f(i) \subset \iota(i)$ for all covering sets $i \in I$,
3. for any atlases $(K_t, A@_R E_t)_T$ for G and $(L_s, B@_R F_s)_S$ for H, if we take the chart isomorphisms $\phi_t\colon K_t \xrightarrow{\sim} I_t$ and $\psi_s\colon L_s \xrightarrow{\sim} J_s$ for some pair t, s of indexes which correspond under the map ι (i.e., $\iota(I_t) = J_s$), then the induced maps $f_t\colon K_t \to L_s$ define morphisms f_t of A-addressed objective local compositions.

If no misunderstanding is possible, this morphism will be denoted by $f^\iota\colon G^I \to H^J$. The composition of two morphisms $f^\iota\colon G^I \to H^J$, $g^\kappa\colon H^J \to E^L$ is defined by $g^\kappa \cdot f^\iota = g\,.\,f^{\kappa\cdot\iota}$.

Example 3. A typical example of a global composition is the interpretation of a local composition. We start with a local composition $(K, A@_R M)$ and a covering $I = \{K_1, K_2 \ldots K_n\}$ of its support and then we will define evident charts $(K_i, A@_R M)$ which glue together by the identity on the space M. The interpretation $X^{(3)}$ of a diatonic scale $(X, \mathbb{Z}_{12})$ by seven triadic degrees is a classical interpretation with $n(X^{(3)})$ being a Moebius strip [8]. A global composition is called *interpretable* iff it is isomorphic to an interpretation. Not all global compositions are interpretable, criteria for interpretability are known in terms of flasque coefficient systems of affine functions, see [4, 10].

Clearly, the abstract nerve $n(G^I)$ is functorial in G^I with values in the category of simplicial complexes. So the abstract nerve of a global composition is a first combinatorial invariant; a fortiori, the geometric nerve $N(G^I)$ is a topological invariant of a global composition.

2. Resolutions by global standard compositions

The standard compositions represent compositions with "notes in general position", i.e., their configuration is as "free" as possible by 'occasional' coincidences. In fact, the standard composition is a geometric realization of the composition's nerve and thus depends only on combinatorial information. There is a natural projection from the standard object onto the generating composition. Here is the formal construction:

Given a module A in ${}_R\mathrm{Mod}$ and a natural number $0 \leq n$, we denote $A^{\sqcup n}$ the affine $n+1$-fold coproduct $\coprod_{n+1} A$ of A. By construction, there is an isomorphism $A^{\sqcup n} \xrightarrow{\sim} R^n \oplus A^{n+1}$. We denote the canonical basis of R^n by $(e_1, \dots e_n)$, and for any element $a \in A$ and $0 \leq i \leq n$, we set $a_i = (0, \dots, a, \dots 0)$ for the $n+1$-tuple in A^{n+1} having a at its $(i+1)$-th position and zero else; the zero element is denoted by e_0. We have the inclusion morphisms

$$\sigma_i \colon A \to A^{\sqcup n} \tag{1}$$

for $0 \leq i \leq n$, with

$$\sigma_i(a) = \begin{cases} (0, a_0) & \text{(linear) if } i = 0, \\ (e_i, a_i) & \text{(affine) if } i > 0. \end{cases} \tag{2}$$

This defines a local A-addressed composition ${}_A\Delta_n \subset A@_R A^{\sqcup n}$ which is called the *A-addressed local standard composition of dimension n*. By construction, it has the following property: If M is any R-module, and if $s. = (s_0, \dots s_n)$ is any sequence of A-addressed points[2] in M, with associated local composition $S = \{s_0, \dots s_n\} \subset A@_R M$, then there is exactly one morphism of local compositions

$$(s.) \colon {}_A\Delta_n \to S \colon \sigma_i \mapsto s_i \qquad \text{for} \quad i = 0, \dots n. \tag{3}$$

This morphism is in fact defined by the universal property of the coproduct and is mediated by the following affine function $f \colon A^{\sqcup n} \to M$: Write $s_i = e^{t_i} \cdot s_{i,0}$. Then we have

$$\begin{aligned} f(e_0) &= t_0, \\ f(e_i) &= t_i - t_0 \text{ (linear) for } i > 0, \\ f(a_i) &= s_{i,0}(a) \text{ (linear) for } i \geq 0, \end{aligned}$$

and the formula $s_i = f \cdot \sigma_i$ is immediate.

To define global "free" objects among the A-addressed objective compositions with finite charts, we consider the natural weight function $\nu \colon n(G^I) \to \mathbb{N}$ with $\nu(\Sigma) = \mathrm{card}(\cap\Sigma) - 1$, where we set $\cap\Sigma = \cap_{s \in \Sigma} s$. The pair $n^\star(G^I) = \big(n(G^I), \nu\big)$

[2]In the Yoneda language, these are the morphisms in $A@_R M$.

is an object in the category of naturally weighted simplicial complexes whose morphisms are the simplicial maps which commute according to the weight functions. We shall represent the naturally weighted nerve $n^\star(G^I)$ by an isomorphic *standard representative* $n^\star$ induced by a covering of the natural interval $[0,m] = \{0,1,2,3,\dots,m = \mathrm{card}(G)-1\}$ of natural numbers.

For $n^\star$, we define the *global standard composition* ${}_A\Delta_{n^\star}$ at address A by the interpretation of the local standard composition ${}_A\Delta_m$ which is given by the present covering of $[0,m]$. We are also given a *standard atlas* of ${}_A\Delta_{n^\star}$. In fact, for any subset $q = \{t_0, \dots t_c\} \subset [0,m]$ of $c+1$ elements, we have the canonical injection $i_q\colon {}_A\Delta_c \to {}_A\Delta_m$ via $\sigma_j \mapsto \sigma_{t_j}$. This defines the standard atlas.

The universal property of this global standard composition reads as follows. Take the category Covens of coverings of sets[3], and consider the covariant functor

$$ {}_A\mathrm{Cov}_{n^\star}\colon \mathrm{ObGlob}^{\mathrm{finite}}_{@A} \to \mathrm{Sets}\colon G^I \mapsto \mathrm{Hom}_{\mathrm{Covens}}\big(n^\star, (G, n_0(G^I))\big) \qquad (4)$$

where the covering $n^\star$ denotes the naturally weighted simplicial complex after forgetting its weight. Then we have this straightforward result:

Proposition 1. *The functor ${}_A\mathrm{Cov}_{n^\star}$ is representable by the standard global composition ${}_A\Delta_{n^\star}$, i.e., we have a bijection*

$$\mathrm{Hom}_{\mathrm{Covens}}\big(n^\star, (G, n_0(G^I))\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{ObGlob}^{\mathrm{finite}}_{@A}}\big({}_A\Delta_{n^\star}, G^I\big)$$

which is functorial in the A-addressed composition G^I.

In particular, if we take the standard covering $n^\star = n^\star(G^I)$ of the nerve of G^I and then the corresponding 'identity' morphism $\mathrm{Id}\colon n^\star \xrightarrow{\sim} \big(G, n_o(G^I)\big)$, we obtain a corresponding bijective morphism

$$\mathrm{res}_{G^I}\colon \Delta_{G^I} \to G^I \qquad (5)$$

with the notation $\Delta_{G^I} = {}_A\Delta_{n^\star(G^I)}$ this object and the morphism res_{G^I} has been called *the resolution of* G^I. Clearly, the associated simplicial morphism $n(\mathrm{res}_{G^I})\colon n^\star(\Delta_{G^I}) \to n^\star(G^I)$ is an isomorphism, but in general, res_{G^I} is not an isomorphism!

In particular, due to the universal property of the global standard compositions, every morphism $f^\iota\colon G^I \to H^J$ can uniquely be lifted to a corresponding morphism res_{f^ι} of resolutions to make the diagram

$$\begin{tikzcd}
\Delta_{G^I} \arrow[r, "\mathrm{res}_{f\ \iota}"] \arrow[d, "\mathrm{res}_{G^I}"'] & \Delta_{H^J} \arrow[d, "\mathrm{res}_{H^J}"] \\
G^I \arrow[r, "f^\iota"] & H^J
\end{tikzcd} \qquad (6)$$

[3] Its objects are coverings of sets X^I, its morphisms $X^I \to Y^J$ are compatible pairs of set maps $f\colon X \to Y$, $\phi\colon I \to J$, i.e., $f(x) \subset \phi(x)$.

commute. Therefore we have a resolution functor

$$\mathrm{res}_{@A}\colon\ \mathrm{ObGlob}^{\mathrm{finite}}_{@A} \to \mathrm{ObGlob}^{\mathrm{finite}}_{@A} \tag{7}$$

and a natural transformation

$$\delta_{@A}\colon\ \mathrm{res}_{@A} \to \mathrm{Id}_{\mathrm{ObGlob}^{\mathrm{finite}}_{@A}}\,. \tag{8}$$

The following deals with the reconstruction of the category $\mathrm{ObGlob}^{\mathrm{finite}}_{@A}$ from its subcategory of free objects.

3. Compositions from module complexes

The resolution of a global composition is a representation of its weighted nerve and thereby it includes invariant data of the composition. But something more needs to be done to yield a full set of invariants. The next step deals with this completion. It is related to functions on global compositions.

For a global composition G^I, a global affine function is a morphism $f\colon G^I \to A@_R R$, where the omitted exponent in f relates to the unique covering map into the local codomain composition $A@_R R$. In the charts it is easily seen that the set of global affine functions builds a R-module $\Gamma(G^I)$ under pointwise addition and scalar multiplication. It is also clear that the retraction $g^{\iota *}f = f \cdot g^{\iota}$ of an affine function $f\colon G^I \to A@_R R$ via a morphism $g^{\iota}\colon H^J \to G^I$ is an affine function, and the retracted function module is a submodule of $\Gamma(H^J)$.

In order to handle such submodules better on a technical level, we give an alternative description of such function modules in terms of coefficient systems from sheaf theory [2]. Given a global composition G^I, reconsider the category $n(G^I)$ of its abstract nerve.

Definition 4. A (R-)*module complex over* G^I is a covariant functor (a coefficient system in Godement's terminology)

$$M\colon n(G^I) \to {}_R\mathrm{Mod} \tag{9}$$

into the category ${}_R\mathrm{Mod}$ of R-modules and R-affine morphisms, with transition morphisms

$$M_{\sigma,\tau}\colon M(\sigma) \to M(\tau)$$

of modules for the simplex inclusions (morphisms) $\sigma \subset \tau$.

As usual in sheaf theory, we put $\Gamma M = \lim\limits_{n(G^I)} M(\sigma)$ and call it the set of *global sections.*

EXAMPLE 4. Since any morphism of global compositions $\overset{\iota}{f}/\alpha\colon G^I \to H^J$ yields a natural transformation $n(\overset{\iota}{f}/\alpha)\colon n(G^I) \to n(H^J)$, every module complex M over H^J induces a module complex on $\overset{\iota}{f}/\alpha \star M$ over G^I, with

$$\overset{\iota}{f}/\alpha \star M(\sigma) = M\big(n(\overset{\iota}{f}/\alpha)(\sigma)\big)\,.$$

EXAMPLE 5. If M is any R-module, the *constant module complex* of M is the complex with $M(\sigma) = M$ for all simplexes and identity transition. Observe that its global sections are in bijection with the set M^c, if $N(G^I)$ has c connected components.

Now we review global affine functions as patchworks of affine functions on charts of atlases. Suppose that we are given a global composition G^I. For a simplex σ of $n(G^I)$, we have the canonical local composition $\cap\sigma$, and a morphism $f\colon \cap\,\sigma \to A@_RR$ is an affine function on $\cap\sigma$. The set $n\Gamma(\sigma)$ of these functions is provided with the structure of a R-module by the usual addition and scalar multiplication of function values: $(f+g)(x) = f(x)+g(x)$, $(rf)(x) = rf(x)$.

Next suppose that we have an inclusion of simplexes $\sigma \subset \tau$ of $n(G^I)$. Then the ambient spaces of the charts of these simplexes are the same, i.e., the inclusion of local compositions $\cap\tau \subset \cap\sigma$ are in bijection with an inclusion of local compositions $K_\tau \subset K_\sigma \subset A@_RN$ for a specific module N. Since an affine function on $\cap\tau$ is the restriction of an affine morphism $A@h\colon A@_RN \to A@_RR$, f evidently extends to the restriction $A@h|_{K_\sigma}$, so the transition morphisms by restriction of affine functions are surjective. The corresponding complex of *affine functions* is denoted by $n\Gamma(G^I)$. The subcomplex $C = C_{G^I}$ of constant functions is defined by $C(\sigma) = \big\{f \in n\Gamma(G^I)(\sigma)\colon f = \text{ constant on} \cap\sigma\big\}$. The set of global sections of the function complex is denoted by $\Gamma(G^I)$ and clearly identifies with the previous definition of $\Gamma(G^I)$.

For the following construction, we select one ambient space $A@_RF_\sigma$ for every simplex σ of $n(G^I)$. Take a chart $\cap(\sigma)$ of $\cap\sigma$ in $A@_RF_\sigma$. Then we have the submodule $R.\cap(\sigma)$ of $A@_RF_\sigma$. For a morphism $\sigma \subset \tau$, we obtain a R-linear homomorphism $R.\cap(\tau) \to R.\cap(\sigma)$ which induces this homomorphism

$$\mathrm{Hom}_{\sigma,\tau}(M)\colon\ \mathrm{Hom}_R(R.\cap(\sigma), M) \to \mathrm{Hom}_R(R.\cap(\tau), M)$$

on the homomorphism modules for any given module M. This module complex is denoted by $\mathrm{Hom}_R(G^I, M)$, although it is only determined up to an isomorphism of module complexes[4] on G^I; it evaluates $\mathrm{Hom}_R(G^I, M)(\sigma) = \mathrm{Hom}_R(R.\cap(\sigma), M)$. In the particular case $M = A@_RR$, we abbreviate $\mathrm{Hom}_R(G^I, A@_RR) = G^{I*}$ and call this *the module complex of A-addressed forms on G^I*. For an affine

[4] By definition of module complexes, this is an isomorphism of functors.

function $f\colon \cap\sigma \to A@_R R$, we have a R-linear homomorphism $R\,.\,f\colon R\,.\cap(\sigma) \to A@_R R$, and the map $f \mapsto R\,.\,f$ is R-linear and functorial in σ, i.e., we have a morphism of module complexes on G^I:

$$R_{G^I}\colon n\Gamma \to G^{I*}. \tag{10}$$

Lemma 1. *The kernel of R_{G^I} is the module C_{G^I} of constant functions. Therefore, if N is any module complex with $C \subset N \subset n\Gamma(G^I)$, we have an embedding of module complexes*

$$N/C \rightarrowtail G^{I*}. \tag{11}$$

P r o o f . The function f is in the kernel iff its values $R\,.\,f(s_i - s_j) = f(s_i) - f(s_j)$ vanish, and this in turn means that f is constant on $\cap\sigma$. □

Definition 5. Let P be a property of modules. If M is a module complex over G^I, it is said to share the property P iff all its values $M(\sigma)$ do so. In particular, M is called injective (projective) iff the module $M(\sigma)$ is so for every simplex σ.

Lemma 2. *If A is the zero address $A = 0_R$, and if $n\Gamma$ is injective (in particular, if R is a field), the morphism R_{G^I} is surjective, and therefore, we have an isomorphism $n\Gamma/C \xrightarrow{\sim} G^{I*}$.*

P r o o f . Since we have the zero address, any $\cap\sigma$ is isomorphic to an embedded local composition in the same ambient module, and any linear map $l\colon R\,.\cap\sigma \to R$ extends to a map on the ambient module. □

Let $f^\iota\colon G^I \to H^J$ be a morphism of global compositions. Take a simplex σ in $n(G^I)$, and its image σ' under the associated simplicial map. Then, each restricted morphism

$$f|_{\cap\sigma}\colon \cap\sigma \to \cap\sigma'$$

gives rise to a map

$$f^\iota \star n\Gamma(H^J)(\sigma) = n\Gamma(H^J)(\sigma') \to n\Gamma(G^I)(\sigma)$$

by right composition with this restricted morphism. Moreover, the map is R-linear. Therefore, if M is any subcomplex of $n\Gamma(H^J)$, its induced complex $\overset{\iota}{f}/\operatorname{Id}_A \star M$ is mapped R-linearly into what is called the *retracted* module complex

$$M|\overset{\iota}{f}/\operatorname{Id}_A \subset n\Gamma(G^I). \tag{12}$$

In particular, if $M = C_{H^J}$, we have $C_{H^J}|f^\iota \subset C_{G^I}$.

By these techniques in mind, the resolution functor $\mathrm{res}_{@A}$ and its associated natural transformation $\delta_{@A}$ give rise to a module complex of affine functions $\Delta n\Gamma(G^I) = n\Gamma(G^I)|\,\mathrm{res}_{G^I}$ in $n\Gamma(\Delta_{G^I})$, for each global composition G^I. Call this complex *the resolution complex of composition* G^I. Moreover, this assignment commutes with the morphism of the resolution functor, i.e., for a morphism $f^\iota\colon G^I \to H^J$, we have a canonical inclusion

$$n\Gamma(\Delta_{H^J})|\operatorname{res}_{f^\iota} \subset n\Gamma(\Delta_{G^I}) \tag{13}$$

of the the retracted resolution complex of H^J in the resolution complex of G^I. The next step deals with the reconstruction of G^I from $n\Gamma(\Delta_{G^I})$ and the related question of classification of global compositions by use of the resolution complex which is suggested by the functorial relation (13).

The generic situation from the preceding constructions is that we are given a module complex $M \subset n\Gamma(G^I)$, containing the constants C, and that we would like to construct a kind of "quotient" composition whose affine functions are those of M. First we look at the local situation.

Definition 6. Let $S \subset A@_RU$ be a local composition in the R-module U. For a submodule $L \subset \Gamma(S)$ of affine functions on S, the evaluation map $\dot{}\colon S \to A@_RL^\star$ into the A-valued points of the dual module $L^\star = \mathrm{Hom}_R(L, R)$ of L is defined by $\dot{s}(a)(l) = l(s)(a)$.

The problem is that the evaluation is not a morphism of local compositions in general. We have to investigate sufficient conditions for the existence of a morphism. In the special case which is of interest we have this guarantee: In fact, let $S = {}_A\Delta_n \subset A@_RA^{\sqcup n}$. Then, the dotted points $\dot{\sigma_i}\colon A \to L^\star$ define the universal map $H_L\colon A^{\sqcup n} \to L^\star$, and we have interpreted $\dot{}\colon\ {}_A\Delta_n \to A@_RL^\star$ as a morphism of local compositions.

Next, suppose we are given two local compositions $S \subset A@_RU$, $T \subset A@_RV$ and a morphism

$$f\colon S \to T\,,$$

together with a module $L_T \subset \Gamma(T)$ whose retract $L_T|f$ is included in a module $L_S \subset \Gamma(S)$. We then have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\cdot"] \arrow[d, "f"'] & A@_RL_S^\star \arrow[d, "A@|f^\star"] \\ T \arrow[r, "\cdot"] & A@_RL_T^\star \end{tikzcd} \tag{14}$$

where $|f^\star$ is R-dual of the canonical linear map $|f\colon L_T \to L_S$.

This construction yields a morphism $\dot{f}\colon \dot{S} \to \dot{T}$ of local compositions in the ambient spaces $L_S^\star$, $L_T^\star$, respectively. By this technique we may associate a global composition with a module complex $N \subset n\Gamma({}_A\Delta_{n^\star})$ of affine functions in the standard composition ${}_A\Delta_{n^\star}$ of a standard covering $n^\star$.

At the end, suppose we are given such a module complex N of affine functions, and suppose that the restriction transition morphisms $N(\tau) \to N(\sigma)$, $\tau \subset \sigma$, are all surjective. We know that this is the case for retracted function modules from resolution morphisms.

In the following discussion of classification, we will tacitly assume that all module complexes of affine functions have surjective transition morphisms.

With this, if we apply the construction from diagram (14) to the situation where $S = {}_A\Delta_{n(\sigma)} \subset A@A^{\sqcup n(\sigma)}$, and $T = {}_A\Delta_{n(\tau)} \subset A@A^{\sqcup n(\tau)}$ for simplexes $\tau \subset \sigma$ of ${}_A\Delta_{n^\star}$, and with $L_S = N(\sigma)$, $L_T = N(\tau)$, then we have injective vertical arrows in the corresponding commutative diagram

$$\begin{tikzcd} {}_A\Delta_{n(\sigma)} \arrow[r, "\cdot"] \arrow[d, "inclusion"'] & A@_R N(\sigma)^\star \arrow[d, "A@\,\mathrm{res}^\star"] \\ {}_A\Delta_{n(\tau)} \arrow[r, "\cdot"] & A@_R N(\tau)^\star \end{tikzcd} \tag{15}$$

where res is the restriction map. We write $\cap\sigma = {}_A\dot{\Delta}_{n(\sigma)}$, and therefore we get a surjective morphism $({}_A\Delta_{n(\sigma)} \to \cap\sigma)_{n^\star}$ of diagrams of local compositions over the nerve $n^\star$. Setting ${}_A\Delta_{n^\star}/N = \mathrm{colim}_{n^\star} \cap\sigma$, we have a commutative diagram of sets

$$\begin{tikzcd} {}_A\Delta_{n(\sigma)} \arrow[r, "\cdot"] \arrow[d] & \bigcap\sigma \arrow[d] \\ {}_A\Delta_{n^\star} \arrow[r, "/N=\mathrm{colim}^\cdot"] & {}_A\Delta_{n^\star}/N \end{tikzcd} \tag{16}$$

induced by the dot morphisms of local compositions.

Definition 7. Call a module complex $N \subset n\Gamma(G^I)$ *separating*, iff for every zero-simplex (chart) $\sigma \in n_0(G^I)$, the dot map $\sigma \to A@_R N(\sigma)^\star$ is injective.

Intuitively, this means that for any pair of points in any chart, there is a function of N in this chart which separates these points. So, if in the above situation $N \subset n\Gamma({}_A\Delta_{n^\star})$ is separating, the dot maps are all bijective onto their images since the vertical maps in diagram (14) are injective. Therefore, the colimit diagram (16) has bijective horizontal arrows, and the images $\cap\sigma$ are injected into the limit ${}_A\Delta_{n^\star}/N$. So these images cover the limit and the images of the zero-dimensional simplexes build a canonical atlas of a global A-addressed composition, i.e., the diagram (16) becomes a bijective morphism of A-addressed global compositions.

So, if N is separating, we have constructed a canonical global composition and a bijective morphism from the free object to a global composition which is defined by the functions of N.

Definition 8. We call this composition ${}_A\Delta_{n^\star}/N$ the *N-quotient* of ${}_A\Delta_{n^\star}$. The morphism ${}_A\Delta_{n^\star} \to {}_A\Delta_{n^\star}/N$ from diagram (16) is denoted by $/N$.

In particular, if $\mathrm{res}_{G^I} : \Delta_{G^I} \to G^I$ is the resolution of the composition G^I, we have the resolution complex $\Delta n\Gamma(G^I)$. Clearly, the resolution complex is separating iff the complex of affine functions $n\Gamma(G^I)$ exist. So there is a $\Delta n\Gamma(G^I)$-quotient if the complex of affine functions $n\Gamma(G^I)$ is separating; we also say that G^I *is separating.* Of course, this exist a property which is invariant under bijective morphisms among global compositions.

Proposition 2. *If the global composition G^I admits a projective atlas, i.e., an atlas whose charts have projective R-modules, then G^I is separating.*

Proof. In this case, the canonical bidual map $U \to U^{\star\star}$ is injective for each ambient space U of a projective atlas. Now, if $s \neq t$ are two points in the chart $\sigma \subset A@_R U$, there is $a \in A$ with $s(a) \neq t(a)$, and we may take a linear form $h \in U^\star$ such that $h\big(s(a)\big) \neq h\big(t(a)\big)$ which means that the induced affine form $l = A@h|\sigma$ separates s from t. □

Proposition 3. *Suppose that G^I is separating. Then we are given a commutative triangle of covering set (!) isomorphisms*

$$\begin{tikzcd} & \Delta_{G^I} \arrow[dl, "/\Delta n\Gamma(G^I)"'] \arrow[dr, "\mathrm{res}_{G^I}"] & \\ \Delta_{G^I}/\Delta n\Gamma(G^I) \arrow[rr, "f"'] & & G^I \end{tikzcd} \tag{17}$$

which stem from morphisms of global compositions except — possibly — for f.

To see for conditions when f is a morphism of global compositions, we may concentrate on the local situation. This means that we are given a separating local composition $S = \{s_0, \dots s_n\} \subset A@_R U$ of cardinality $n+1$. For the module $\Gamma(S)$ of affine functions $g\colon S \to A@_R R$ on S, we have the corresponding commutative triangle of set bijections:

$$\begin{tikzcd} & {}_A\Delta_n \arrow[dl, "/\Delta\Gamma(S)"'] \arrow[dr, "."] & \\ {}_A\Delta_n/\Delta\Gamma(S) \arrow[rr, "f"'] & & S \end{tikzcd} \tag{18}$$

which becomes a triangle of morphisms of local compositions if we can derive f from an affine morphism $F\colon \Delta\Gamma(S)^\star \to U$ of ambient modules. On a level of ambient modules, we have the following commutative square of affine morphisms:

$$\begin{tikzcd} A^{\sqcup n} \arrow[r, "\sqcup\dot{S}"] \arrow[d, "\sqcup S"'] & \Gamma(S)^\star \arrow[d, "q^\star"] \\ U \arrow[r, "b"] & (U@_R R)^\star \end{tikzcd} \tag{19}$$

where the $b(u)(t) = t(u)$ is the extension of the bidual, where $q\colon U@_R R \to \Gamma(S)$ is the canonical surjection, and where $\sqcup S$, $\sqcup\dot{S}$ are the universal morphisms associated with $S, \dot{S}$, respectively.

If U is finitely generated and projective, the bidual $U \to U^{\star\star}$ is an isomorphism, and the dual $r\colon (U@_R R)^\star \to U^{\star\star}$ of the inclusion $U^\star \to U@_R R$ defines a left-inverse i of b. Therefore, we are given the commutative triangle

$$\begin{tikzcd} & A^{\sqcup n} \arrow[dl, "\sqcup\dot{S}"'] \arrow[dr, "\sqcup S"] & \\ \Gamma(S)^\star \arrow[rr, "i\cdot q^\star"'] & & U \end{tikzcd} \tag{20}$$

which yields the required morphism for f in diagram (18). This yields

Theorem 1. *If the composition G^I has a finitely generated, projective atlas*[5]*, the factorization diagram of Proposition 3 is a diagram of global compositions.*

P r o o f. In fact, by Proposition 2, G^I is separating, and by the local diagram (20), applied to the charts of this situation, the map f in the diagram of Proposition 3 is a morphism of global compositions. □

The factorization morphism f has a chance to become an isomorphism if we succeed in constructing an ambient space morphism in the other direction. A sufficient condition for such inverse morphism is that the surjection $q\colon U@_R R \to \Gamma(S)$ has a section $p\colon \Gamma(S) \to U@_R R$. This implies that on the duals, one has $\mathrm{Id}_{\Gamma(S)^\star} = p^\star \cdot q^\star$, and we may take $p^\star \cdot b$ to go back from U to $\Gamma(S)^\star$. This condition is certainly satisfied if $\Gamma(S)$ is projective. We say that the global composition *has projective functions* iff its complex of affine functions is projective on the zero simplexes. So we have:

Theorem 2. *If G^I has a finitely generated projective atlas and projective functions, then the horizontal arrow f in the factorization diagram of Proposition 3 is an isomorphism.*

This condition is evidently satisfied if R is semi-simple, i.e., a finite product of commutative fields. This means that in this case we are able to reconstruct G^I

[5] All charts have finitely generated projective modules.

from its retracted affine functions on the resolution. Moreover, in this case, the retracted module complex can also be recovered from the quotient composition, i.e.,

$$\Delta n\Gamma(G^I) = n\Gamma\big({}_A\Delta_{G^I}/\Delta n\Gamma(G^I)\big)\,|/\Delta n\Gamma(G^I) \tag{21}$$

so that the question of characterizing those module complexes of affine functions in ${}_A\Delta_{n^\star}$ which could give rise to compositions having this free object as their resolution, remains.

Clearly, the first condition on such a module complex is that it is separating. Secondly, we may suppose that it is finitely generated projective, i.e., its modules on the zero-dimensional simplexes are so. The first requirement is, in particular, the case for the global compositions in Theorem 2 above since the charts have finitely generated projective ambient modules. The second requirement is that one we had in Theorem 2. Also, under these conditions, the quotient composition has charts $\Gamma(S)^\star$ which are again finitely generated projective and separating. The third and last requirement is obvious: N should contain the constant functions. So we proceed to the analysis of module complexes $N \subset n\Gamma({}_A\Delta_n)$ which are separating, finitely generated projective, and contain the constant functions; call these complexes *representative.*

4. Orbits of module complexes

By means of the representation of a composition via its module complex of functions on the purely combinatorial standard composition, we obtain a classification frame for global compositions.

Let us first recall the overall situation of resolutions. For a global composition G^I, we have the functorially associated standard resolution $\Delta_{G^I} = {}_A\Delta_{n^\star}$ with the standard covering $n^\star$ of G^I, and the resolution bijection $\mathrm{res}_{G^I}: \Delta_{G^I} \to G^I$. If a global composition H^J is isomorphic to G^I via a morphism $f^\iota: G^I \to H^J$, we have a commutative diagram with horizontal isomorphisms:

$$\begin{tikzcd} {}_A\Delta_{n^\star} \arrow[r, "\mathrm{res}_{f^\iota}"] \arrow[d, "\mathrm{res}_{G^I}"'] & {}_A\Delta_{n^\star} \arrow[d, "\mathrm{res}_{H^J}"] \\ G^I \arrow[r, "f^\iota"] & H^J \end{tikzcd} \tag{22}$$

By the universal property of the standard compositions, the automorphism group $s_{A,n^\star}$ of the standard composition ${}_A\Delta_{n^\star}$ identifies with a subgroup of the symmetric group $\mathfrak{S}_{m+1}$ of permutations of ${}_A\Delta_{n^\star}$ if the standard covering is defined on the integer interval $[0,m]$ as discussed above. By retraction (see formula (12)), this group acts from the right on the set $\mathrm{Rep}_{A,n}$ of representative module complexes on ${}_A\Delta_{n^\star}$

$$\mathrm{ret}\colon\ \mathrm{Rep}_{A,n^\star} \times S_{A,n^\star} \to \mathrm{Rep}_{A,n^\star}\colon (N,g) \mapsto N|g\,. \tag{23}$$

So, if our isomorphic global compositions G^I, H^J have a finitely generated projective atlas and projective functions, their retracted function complexes N_{G^I}, N_{H^J} on the standard composition ${}_A\Delta_{n^\star}$ are representative, and we have $N_{G^I} = N_{H^J}|\mathrm{res}_f$.

Conversely, if we are given a representative module complex N on ${}_A\Delta_{n^\star}$, we have a bijective morphism

$$r\colon {}_A\Delta_{n^\star} \to {}_A\Delta_{n^\star}/N \tag{24}$$

(see Definition 8). Since the charts of the quotient have ambient spaces $N(\sigma)^\star$, the quotient has a finitely generated projective atlas. What are the functions of the quotient? Since the simplicial complexes are isomorphic, it is sufficient to consider the local situation on a chart. So we are given a representative module complex of affine functions $N \subset \Gamma({}_A\Delta_n)$ in a local standard composition. An affine function f on the image $\Delta \subset A@_R N^\star$ is induced by an affine morphism $F\colon N^\star \to R$, and since N is finitely generated projective, we may write $F = e^t \cdot n^\star$, with the bidual $n^\star$ of an element $n \in N$. So the retraction of the induced function f on σ_i, and on an argument $a \in A$ evaluates

$$f(\sigma_i)(a) = t + n(\sigma_i)(a) \tag{25}$$

and, since the constants are in N, we may absorb the constant t in n, which means $n\Gamma({}_A\Delta_{n^\star}/N)|r = N$. Since this retraction is isomorphic to the function module of the quotient, the latter is also representative.

Conversely, if we are given two representative module complexes $M, N \in \mathrm{Rep}_{A,n^\star}$, and an automorphism $g \in S_{A,n^\star}$ with $N = M|g$, then we have a commutative diagram with isomorphism on the horizontal arrows

$$\begin{tikzcd}
{}_A\Delta_{n^\star} \arrow[r, "g"] \arrow[d, "r_N"'] & {}_A\Delta_{n^\star} \arrow[d, "r_M"] \\
{}_A\Delta_{n^\star}/N \arrow[r] & {}_A\Delta_{n^\star}/M
\end{tikzcd} \tag{26}$$

by the factorization of the resolution map $g \cdot r_M$ through the quotient ${}_A\Delta_{n^\star}/N$. So conversely, isomorphic representative module complexes give rise to isomorphic quotients, and we have proven:

Theorem 3. *The orbit space $\mathrm{Rep}_{A,n^\star}/S_{A,n^\star}$ is in bijection with the set of isomorphism classes of A-addressed global compositions with projective functions and finitely generated projective atlases which have a covering complex* *isomorphic to $n^\star$. This bijection is induced by the retraction of the function module complex to the resolution ${}_A\Delta_{n^\star}$, in one direction, and by the quotient composition on a given representative module complex on ${}_A\Delta_{n^\star}$, in the other.*

In particular, this classification result is valid for the global compositions having as their address a module A over a semi-simple commutative ring R.

Now we discuss the action of the automorphism group of the standard composition on module complexes since it induces isomorphism classes of global compositions.

According to the above classification in Theorem 3, we are interested in a more explicit description of the action of the automorphism group $S_{A,n^\star}$ on the set $\mathrm{Rep}_{A,n^\star}$ of representative module complexes on the standard composition ${}_A\Delta_{n^\star}$. To begin with, the module complex of affine functions on ${}_A\Delta_{n^\star}$ reads as follows: From Section 2 we recall that the standard covering $n^\star$ may be given in form of a covering of the integer interval $[0, m]$ by subsets (the zero simplexes) $\sigma = \{t_0, \dots t_{c(\sigma)}\}$ of $c(\sigma) + 1$ elements, and therefore by the standard atlas injections $i_\sigma\colon {}_A\Delta_\sigma \to {}_A\Delta_m\colon \sigma_j \mapsto \sigma_{t_j}$. On each simplex σ of dimension $c(\sigma)$ of this covering, we have a corresponding local standard composition ${}_A\Delta_{c(\sigma)}$, and its function module is

$$\Gamma({}_A\Delta_{c(\sigma)}) \xrightarrow{\sim} (A@_R R)^{c(\sigma)+1}$$

whereas the constants correspond to the diagonal submodule

$$\Delta(A@_R R) \subset (A@_R R)^{c(\sigma)+1},$$

and our module complex N must be a finitely generated module

$$\Delta(A@_R R) \subset N(\sigma) \subset (A@_R R)^{c(\sigma)+1}$$

for each simplex σ, finitely generated projective as well as separating in dimension zero. What does it mean that it is separating? It means that for any zero-dimensional simplex σ, and for any pair $i \neq j$ of indices between 0 and $c(\sigma)$, there is such an element $n \in N(\sigma)$ such that its coordinates n_i and n_j differ. Moreover, for any two simplexes $\tau \subset \sigma$, we have a commutative diagram

$$\begin{tikzcd}
\Delta(A@_R R) \arrow[r, hook] \arrow[d, "\mathrm{Id}"'] & N(\tau) \arrow[r, hook] \arrow[d, two heads, "\mathrm{res}_{\tau,\sigma}"'] & (A@_R R)^{c(\tau)+1} \arrow[d, two heads, "\mathrm{pr}"'] \\
\Delta(A@_R R) \arrow[r, hook] & N(\sigma) \arrow[r, hook] & (A@_R R)^{c(\sigma)+1}
\end{tikzcd}$$

with surjective vertical arrows induced by the projection to the right side. Further, an automorphism of the underlying global standard composition boils down to a permutation of the interval $[0, m]$ which is compatible with the covering.

In particular, this is a finite group action which on every module $N(\sigma)$ acts by induction from the permutation of components on the supporting direct sum module $(A@_R R)^{c(\sigma)+1}$.

Consider the diagonal embedding and its factorization through

$$d(x_0, \dots, x_c) = (x_1 - x_0, \dots, x_c - x_0):$$

$$0 \longmapsto A@_R R \overset{\Delta}{\longmapsto} (A@_R R)^{c+1} \xrightarrow{d} (A@_R R)^c \longrightarrow 0$$

which is an equivariant for the given permutation group action. So we have a direct equivariant decomposition $(A@_R R)^{c+1} = \Delta(A@_R R) \oplus (A@_R R)^c$ which carries over the submodules:

$$N(\sigma) = A@_R R \oplus N(\sigma)_{\text{red}}, \text{ with } N(\sigma)_{\text{red}} \subset (A@_R R)^c,$$

So the above projection diagram reduces to

$$\begin{tikzcd} N(\tau)_{\text{red}} \arrow[r, hook] \arrow[d, "\mathrm{res}_{\tau,\sigma}"'] & (A@_R R)^{c(\tau)} \arrow[d, "\mathrm{pr}"] \\ N(\sigma)_{\text{red}} \arrow[r, hook] & (A@_R R)^{c(\sigma)} \end{tikzcd}$$

It is well-known that $N(\sigma)$ is projective of finite type iff $A^\star$ and $N(\sigma)_{\text{red}}$ are so. So, if we make the general hypothesis that $A^\star$ is projective of finite type, we may concentrate on the reduced part of the module complex, i.e., on a finitely generated projective module complex N/C and on the reduced group action.

5. Classifying schemes

This section is devoted to the classification theorem: There is an algebraic scheme whose rational points represent certain isomorphism classes of global compositions. The scheme is a fine moduli space in the sense of D. Mumford.

To begin with, in the local situation of one simplex of dimension c, we consider a (covariant) Grassmann functor $X_r^{A,c}$: ComRings$|_R \to$ Sets on all commutative R-algebras. By writing $W_{A,c} = (A@_R R)^c$, if $s\colon R \to S$ is an R-algebra, we have

$$\begin{aligned} X_r^{A,c}(s) &= \mathrm{Grass}_{W_{A,c},r}(S) \qquad (27)\\ &= \{V \subset S \otimes_R W_{A,c} \colon S \otimes_R W_{A,c}/V = \text{ locally free of rank } r\}. \end{aligned}$$

Since[6] $S \otimes_R W_{A,c} \xrightarrow{\sim} ((S \otimes_R A)@_S S)^c$, this functor parametrizes S-modules of affine functions on ${}_{S\otimes_R A}\Delta_c$ which are locally free of co-rank r in the module of affine functions on the $S \otimes_R A$-addressed standard composition of dimension c.

[6]In fact, $S \otimes_R (A@_R R) \xrightarrow{\sim} S \otimes_R (A^\star \oplus R) \xrightarrow{\sim} (S \otimes_R A)^\star \oplus S \xrightarrow{\sim} (S \otimes_R A)@_S S$.

Next, we have to deal with the separation property. The S-valued ambient module is the scalar extension $S \otimes_R W_{A,c}$. For any pair $1 \leq i \leq j \leq c$, consider the difference projections

$$p_{i,j} \colon S \otimes_R W_{A,c} \to (S \otimes_R A)@_S S = \begin{cases} \mathrm{pr}_i & \text{if } i = j\,, \\ \mathrm{pr}_j - \mathrm{pr}_i & \text{if } i \neq j\,. \end{cases} \tag{28}$$

where pr_i is the projection onto the ith factor.

The separation property means that no difference projection $p_{i,j}$ vanishes on the submodule $V \subset X_r^{A,c}(s)$. For two indices $1 \leq i \neq j \leq c$, let $\Delta_{A,i,j,c}$ be the submodule of $W_{A,c}$ whose i,j entries are equal, while the other entries vanish. Define these submodules of $W_{A,c}$:

$$V_{i,j} = \begin{cases} (A@_R R)^{i-1} \oplus 0 \oplus (A@_R R)^{c-i-1}\,, & i = j\,, \\ \Delta_{A,i,j,c} \oplus (A@_R R)^{i-1} \oplus 0 \oplus (A@_R R)^{j-i} \oplus 0 \oplus A@_R R)^{c-j-1} & \text{else}\,. \end{cases}$$

We have these quotients $S \otimes_R W_{A,c}/S \otimes_R V_{i,j} \xrightarrow{\sim} (S \otimes_R A)@_S S$. Let us assume henceforth that A is locally free of rank m. Then we have $S \otimes_R V_{i,j} \in \mathrm{Grass}_{W_{A,c},m+1}(S)$. The separation condition for V then means that we have $V \subset S \otimes_R V_{i,j}$ for no $1 \leq i \neq j \leq c$. The latter condition defines a subfunctor $X_{r,i,j}^{A,c}$ of $X_r^{A,c}$ which is represented by a closed subscheme $\mathrm{Grass}_{r,i,j}^{A,c}$ of the scheme $\mathrm{Grass}_r^{A,c}$ over $\mathrm{spec}(R)$. In fact, it is the closed subscheme of the flag scheme $\mathrm{Drap}_{r,m}^{A,c}$ defined by the m-codimensional flag component being fixed to $S \otimes_R V_{i,j}$. It is a closed subscheme since a section of structural morphism of the Grassmannian is a closed immersion.

In other words, we have to take V in the (open) complement scheme $O_r^{A,c}$ of all closed subschemes $\mathrm{Grass}_{r,i,j}^{A,c}$ of $\mathrm{Grass}_r^{A,c}$. So locally, the separation property defines an open subscheme of the Grassmannian which is also invariant under all permutations of the indexes, an action which is induced by the automorphisms of the underlying standard composition.

If the zero-dimensional simplexes of the global standard composition are ${}_A\Delta_{c_i}, i = 1, 2, \ldots, k$, our module complex is given by an open subscheme $O_{r.}^{A,c.} = \prod_{i=1,2,\ldots,k} O_{r_i}^{A,c_i}$ of the fiber product of Grassmannians over $\mathrm{Spec}(S)$. Since all combinations of co-ranks are possible, we should also take the coproduct $O^{A,c.} = \coprod_{r.} O_{r.}^{A,c.}$ of all these open subschemes on which the permutation (automorphism) group of the covering acts. To obtain the effective candidates, we have to consider coincidence on restrictions. So if ρ is any simplex whose vertices are σ_1, σ_2, we need to know that $N(\sigma_1)|_\rho = N(\sigma_2)|_\rho$. This evidently is a closed, equivariant condition and we have shown that the representative module complexes over the R-algebra s which are of any rank configuration on the respective simplexes define a locally closed subscheme $C^{A,n^\star}$ of a projective scheme over $\mathrm{Spec}(R)$. Since any finite number of points is contained in an open affine subscheme in the situation of projective schemes [3, II.4.5.4], we know from [1] that the quotient scheme or orbits of the finite group action of $S_{A,n^\star}$ in the sense of the difference cokernel of the group action μ and the first projection pr_1

$$C^{A,n^\star} \times_{\mathrm{Spec}(R)} S_{A,n^*} \overset{\mathrm{pr}_1}{\underset{\mu}{\rightrightarrows}} C^{A,n^*}$$

exists. This yields:

Theorem 4. *For an addresse A which is locally free of rank m over the commutative ring R, there is a subscheme $J^{n^\star}$ of a projective* $\mathrm{spec}(R)$*-scheme of finite type such that its S-valued points $J^{n^\star}(\mathrm{Spec}(S))$ for an R-algebra S are in bijection with the classifying orbits of module complexes N in ${}_{S\otimes_R A}\Delta_{n^\star}$ which are locally free of defined co-ranks on the zero-simplexes of $n^\star$.*

In particular, if the ground ring R is semi-simple, this theorem gives the classification of any global composition which are addressed in a finitely generated R-module A.

6. Some comments on the meaning of classification and its techniques

Now, after all, what could the program of classification be? Its core value is that it deals with understanding musical works. And we should stress that our concept of a musical work is not the narrow one which restricts to those individual opera which — at least in Europe — started to emanate in the Renaissance. It includes as well general musical corpora such as scales, systems, everything that can be represented by means of global compositions.

From the precise parametric description of a work and of its ambiguities, this work appears as a point configuration in a more or less complex space. However, this configuration is already a determinate perspective which shows a multitude of relations among its ingredients. It is the composer's perspective (now including an abstract 'composer' or creator of a general musical structure like a scale). For example, the choice of tonality, instrumentation, tempo, etc., are points of view which may or may not pertain to the composition, this is a question of the epochs of creation. But their character can undoubtedly be a subject to a variation. Among others, here we do address the question of historical instrumentation for early music.

In order to understand the relations among different parts of a composition, and even to simply recognize them, a change of the given perspective is mandatory. If a never seen object must be inspected, what should we do? You walk around it. This is the most common version of Yoneda's lemma. The analogy to cartography is straightforward: The natural perspective of the landscape in which we live does not coincide with the perspective which meets our need for orientation best. To reach this goal, we preferably build maps which show the landscape from an infinitely high point.

The same happens to music. You play a piece in slow motion 'from very near', in a zoomed optics, a complex chord is arpeggiated, i.e., viewed from a skew angle, and so forth. This idea of variation of the perspective has in fact been integrated in the compositional thinking of the 20th century, perhaps most prominently by Edgar Varèse, especially in his comments on the composition "Intégrales" [12, p. 67]. There he invokes a geometric analogon of a machine which is able to project a mobile spatial object from variable space angles onto a luminous surface. This latter idea is astonishingly akin to the resolution projection from points in general position to points in special position.

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