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ON SOME PROPERTIES OF $\mathcal{I}$-APPROXIMATE DINI DERIVATIVES

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ABSTRACT. E. Łazarow and W. Wilczyński defined in [E. Łazarow: *On the Baire class of $\mathcal{I}$-approximate derivatives*, Radovi Matematicki **5** (1989), 15–27] $\mathcal{I}$-approximate Dini derivatives of a real function of a real variable, having (locally) the Baire property. In the paper, $\mathcal{I}$-approximate Dini derivatives of an arbitrary real function (without assuming its Baire property) are considered and some properties of the $\mathcal{I}$-approximate Dini derivatives are presented.

Let $\mathbb{R}$ denote the real line and $\mathbb{N}$—the set of all positive integers.

For $a \in \mathbb{R}$ and $A \subset \mathbb{R}$, let

$$a \cdot A = \{ax \colon x \in A\} \qquad \text{and} \qquad A - a = \{x - a \colon x \in A\}.$$

Let $\mathcal{S}$ be the σ-field of subsets of $\mathbb{R}$ having the Baire property, and $\mathcal{I}$-the σ-ideal of subsets of $\mathbb{R}$ of the first category.

Definition 1. ([6]) We shall say that $x \in \mathbb{R}$ is an *$\mathcal{I}$-density point* of a set $A \in \mathcal{S}$ ($\mathcal{I}$–$d(A,x) = 1$) if and only if, for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers, there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\left\{t \colon \chi_{n_{m_p}\cdot(A-x)\cap[-1,1]}(t) \not\to 1\right\} \in \mathcal{I}.$$

We shall say that x is an *$\mathcal{I}$-dispersion point* of a set $A \in \mathcal{S}$ ($\mathcal{I}$–$d(A,x) = 0$) if and only if x is an $\mathcal{I}$-density point of the set $\mathbb{R} \setminus A$.

If, in the above definition, the interval $[-1,1]$ is replaced either by $[0,1]$ or by $[-1,0]$, we obtain the definitions of right-hand $\mathcal{I}$-density and $\mathcal{I}$-dispersion points ($\mathcal{I}$–$d^+(A,x) = 1$ and $\mathcal{I}$–$d^+(A,x) = 0$) or left-hand $\mathcal{I}$-density and $\mathcal{I}$-dispersion points ($\mathcal{I}$–$d^-(A,x) = 1$ and $\mathcal{I}$–$d^-(A,x) = 0$), respectively.

For each set $A \in \mathcal{S}$, let

$$\Phi(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{I}\text{-density point of } A\}.$$

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Recall that, for any sets $A, B \in \mathcal{S}$, the following properties hold:

(1) $\Phi(A) \triangle A \in \mathcal{I}$,

(2) if $A \triangle B \in \mathcal{I}$, then $\Phi(A) = \Phi(B)$,

(3) $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,

(4) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

where $A \triangle B$ denotes the symmetric difference of the sets A and B (see [6]).

Further, the family $\mathcal{T}_I = \{A \in \mathcal{S} \colon A \subset \Phi(A)\}$ is a topology on the real line, called the *$\mathcal{I}$-density topology* (see [6]).

The topology $\mathcal{T}_I$ is stronger than the natural topology on the real line.

In the paper, $\mathrm{Int}_{\mathcal{T}_I}(A)$, $\mathrm{cl}_{\mathcal{T}_I}(A)$ denote the interior and the closure of a set A with respect to the $\mathcal{I}$-density topology, respectively.

We shall need the following

Theorem 2. ([4]) *A point 0 is an $\mathcal{I}$-dispersion point of an open set $G \subset \mathbb{R}$ if and only if, for each $n \in \mathbb{N}$, there exist $k \in \mathbb{N}$ and $\delta > 0$, $\delta \in \mathbb{R}$, such that for any $h \in (0, \delta)$ and $i \in \{1, \dots, n\}$, there exist $j_r, j_l \in \{1, \dots, k\}$ such that*

$$G \cap \left(\frac{(i-1)k + j_r - 1}{nk} h, \ \frac{(i-1)k + j_r}{nk} h \right) = \emptyset$$

and

$$G \cap \left(-\frac{(i-1)k + j_l}{nk} h, \ -\frac{(i-1)k + j_l - 1}{nk} h \right) = \emptyset .$$

A set $B \subset \mathbb{R}$ is said to be a *Baire kernel* of a set A if and only if the set B has the Baire property, B is contained in A and, for each set $C \subset A \setminus B$ having the Baire property, the set C is of the first category.

A set $B \subset \mathbb{R}$ is said to be a *Baire cover* of a set A if and only if the set B has the Baire property, A is contained in B and, for each set $C \subset B \setminus A$ having the Baire property, the set C is of the first category.

It is obvious that if sets A, B are Baire kernels of a set C, then $A \triangle B \in \mathcal{I}$ as a subset of the union $(C \setminus A) \cup (C \setminus B)$ and if sets A, B are Baire covers of a set C, then $A \triangle B \in \mathcal{I}$ as a subset of the union $(A \setminus C) \cup (B \setminus C)$.

It is easy to observe that, for each set $A \subset \mathbb{R}$, the interior of A with respect to the $\mathcal{I}$-density topology is a Baire kernel of A and the closure of A with respect to the $\mathcal{I}$-density topology is a Baire cover of A.

Definition 3. We shall say that $x \in \mathbb{R}$ is an *exterior right-hand $\mathcal{I}$-density point* of a set $A \subset \mathbb{R}$ ($\mathcal{I}$–$d_e^+(A, x) = 1$) if and only if there exists a Baire cover B of the set A, such that

$$\mathcal{I}\text{–}d^+(B, x) = 1 .$$

We shall say that $x \in \mathbb{R}$ is an *exterior right-hand $\mathcal{I}$-dispersion point* of a set $A \subset \mathbb{R}$ ($\mathcal{I}\text{–}d_e^+(A,x) = 0$) if and only if there exists a Baire cover B of the set A, such that
$$\mathcal{I}\text{–}d^+(B,x) = 0\,.$$

We shall say that $x \in \mathbb{R}$ is an *interior right-hand $\mathcal{I}$-density point* of a set $A \subset \mathbb{R}$ ($\mathcal{I}\text{–}d_i^+(A,x) = 1$) if and only if there exists a Baire kernel B of the set A, such that
$$\mathcal{I}\text{–}d^+(B,x) = 1\,.$$

We shall say that $x \in \mathbb{R}$ is an *interior right-hand $\mathcal{I}$-dispersion point* of a set $A \subset \mathbb{R}$ ($\mathcal{I}\text{–}d_i^+(A,x) = 0$) if and only if there exists a Baire kernel B of the set A, such that
$$\mathcal{I}\text{–}d^+(B,x) = 0\,.$$

In a similar way we define exterior and interior left-hand $\mathcal{I}$-density points ($\mathcal{I}\text{–}d_e^-(A,x) = 1$, $\mathcal{I}\text{–}d_i^-(A,x) = 1$) and exterior and interior left-hand $\mathcal{I}$-dispersion points ($\mathcal{I}\text{–}d_e^-(A,x) = 0$, $\mathcal{I}\text{–}d_i^-(A,x) = 0$) of any subsets of the real line.

At first, we give some properties for exterior and interior right-hand $\mathcal{I}$-density and $\mathcal{I}$-dispersion points. We can prove analogous properties for exterior and interior left-hand $\mathcal{I}$-density and $\mathcal{I}$-dispersion points.

It is easy to observe that $\mathcal{I}\text{–}d_e^+(A,x) = 1$ ($\mathcal{I}\text{–}d_e^+(A,x) = 0$) if and only if, for each Baire cover B of A,
$$\mathcal{I}\text{–}d^+(B,x) = 1 \quad (\mathcal{I}\text{–}d^+(B,x) = 0)\,,$$
and $\mathcal{I}\text{–}d_i^+(A,x) = 1$ ($\mathcal{I}\text{–}d_i^+(A,x) = 0$) if and only if, for each Baire kernel B of A,
$$\mathcal{I}\text{–}d^+(B,x) = 1 \quad (\mathcal{I}\text{–}d^+(B,x) = 0)\,.$$

From the above and the facts that the interior of a set $A \subset \mathbb{R}$ with respect to the $\mathcal{I}$-density topology is a Baire kernel of A, and the closure of A with respect to the $\mathcal{I}$-density topology is a Baire cover of A it immediately follows that $\mathcal{I}\text{–}d_e^+(A,x) = 1$ ($\mathcal{I}\text{–}d_e^+(A,x) = 0$) if and only if x is a right-hand $\mathcal{I}$-density point (a right-hand $\mathcal{I}$-dispersion point) of the closure of A with respect to the $\mathcal{I}$-density topology, and, similarly, $\mathcal{I}\text{–}d_i^+(A,x) = 1$ ($\mathcal{I}\text{–}d_i^+(A,x) = 0$) if and only if x is a right-hand $\mathcal{I}$-density point (a right-hand $\mathcal{I}$-dispersion point) of the interior of A with respect to the $\mathcal{I}$-density topology.

It is clear that each interior right-hand $\mathcal{I}$-density point of the set $A \subset \mathbb{R}$ is an exterior right-hand $\mathcal{I}$-density point of this set and each exterior right-hand $\mathcal{I}$-dispersion point of the set A is an interior right-hand $\mathcal{I}$-dispersion point of this set.

It is obvious that a point $x \in \mathbb{R}$ is an exterior right-hand $\mathcal{I}$-density point (an exterior right-hand $\mathcal{I}$-dispersion point) of a set A if and only if x is an interior right-hand $\mathcal{I}$-dispersion point (an interior right-hand $\mathcal{I}$-density point) of the set $\mathbb{R} \setminus A$.

If $A, B \subset \mathbb{R}$ and $A \subset B$, then it is easily seen that each interior (exterior) right-hand $\mathcal{I}$-density point of the set A is an interior (exterior) right-hand $\mathcal{I}$-density point of the set B and each interior (exterior) right-hand $\mathcal{I}$-dispersion point of the set B is an interior (exterior) right-hand $\mathcal{I}$-dispersion point of the set A.

The notion of an exterior one-hand $\mathcal{I}$-dispersion point is used to define $\mathcal{I}$-approximate Dini derivatives for an arbitrary real function.

Definition 4. Let f be a real function defined in some neighbourhood of a point $x_0 \in \mathbb{R}$.

The *upper right-hand $\mathcal{I}$-approximate Dini derivative* of a function f at a point x_0 ($\mathcal{I}$–$D^+f(x_0)$) is defined as the greatest lower bound of the set

$$\left\{\alpha\colon \mathcal{I}\text{–}d_e^+\left(\left\{t > x_0 : \frac{f(t) - f(x_0)}{t - x_0} > \alpha\right\}, x_0\right) = 0\right\}.$$

The *lower right-hand $\mathcal{I}$-approximate Dini derivative* of f at x_0 ($\mathcal{I}$–$D_+f(x_0)$) is defined as the least upper bound of

$$\left\{\alpha\colon \mathcal{I}\text{–}d_e^+\left(\left\{t > x_0\colon \frac{f(t) - f(x_0)}{t - x_0} < \alpha\right\}, x_0\right) = 0\right\}.$$

The left-hand $\mathcal{I}$-approximate Dini derivatives are defined similarly and denoted by $\mathcal{I}$–$D^-f(x_0)$ and $\mathcal{I}$–$D_-f(x_0)$.

Further, we shall consider only the upper right-hand $\mathcal{I}$-approximate Dini derivative, because we can obtain analogous properties for other $\mathcal{I}$-approximate Dini derivatives if we replace the function f by

1^0 $g(x) = -f(x)$ then $\mathcal{I}$–$D_+f(x) = -(\mathcal{I} - D^+g(x))$,

2^0 $g(x) = f(-x)$ then $\mathcal{I}$–$D_-f(x) = -(\mathcal{I}\text{–}D^+g(-x))$,

3^0 $g(x) = -f(-x)$ then $\mathcal{I}$–$D^-f(x) = \mathcal{I}$–$D^+g(-x)$, respectively.

By taking into consideration the characteristic function of the Bernstein set, it is easy to see that the $\mathcal{I}$-approximate Dini derivatives of the function which does not have the Baire property may not have this property, either. We shall prove that the Baire property of the function f implies the Baire measurability of its $\mathcal{I}$-approximate Dini derivatives.

Lemma 5. *Let $h\colon \mathbb{R} \to \mathbb{R}$. If $D \subset \mathbb{R}$ is a residual set such that the function $h|_D$ is continuous, then, for each open interval $(a, b) \subset [0, \infty)$, the sets*

$$A = \left\{x \in \mathbb{R}\colon \big(x + (a, b)\big) \cap \{t > x\colon h(t) > h(x)\} \notin \mathcal{I}\right\} \cap D,$$
$$B = \left\{x \in \mathbb{R}\colon \big(x - (a, b)\big) \cap \{t < x\colon h(t) < h(x)\} \notin \mathcal{I}\right\} \cap D.$$

are open relative to D.

P r o o f. We only show that the set A is open relative to D. It is sufficient to prove that, for each point $x_0 \in A$, there exists a number $\alpha > 0$ such that

$$x_0 \in D \cap (x_0 - \alpha, x_0 + \alpha) \subset A.$$

Let $x_0 \in A$. Then $\big(x_0 + (a,b)\big) \cap \big\{t > x_0 \colon h(t) > h(x_0)\big\} \notin \mathcal{I}$ and $x_0 \in D$. Since D is a residual set, therefore

$$\big(x_0 + (a,b)\big) \cap \big\{t > x_0 \colon h(t) > h(x_0)\big\} \cap D \neq \emptyset.$$

Hence there exists a point $t_0 \in D$ such that $x_0 + a < t_0 < x_0 + b$ and $h(t_0) > h(x_0)$.

Let $c \in \big(h(x_0), h(t_0)\big)$. By the continuity of the function $h|_D$ and by $t_0, x_0 \in D$, there exists $\alpha > 0$ such that

1^0 $x_0 + \alpha < t_0 - \alpha$,

2^0 $x_0 + a + \alpha < t_0 < x_0 + b - \alpha$,

3^0 $D \cap (x_0 - \alpha,\, x_0 + \alpha) \subset \big\{x \in D \colon h(x) < c\big\}$,

4^0 $D \cap (t_0 - \alpha, t_0 + \alpha) \subset \big\{x \in D \colon h(x) > c\big\}$.

We shall prove that

$$D \cap (x_0 - \alpha,\, x_0 + \alpha) \subset A.$$

Let $x \in D \cap (x_0 - \alpha,\, x_0 + \alpha)$. We need to show that

$$\big(x + (a,b)\big) \cap \big\{t > x \colon h(t) > h(x)\big\} \notin \mathcal{I}.$$

We observe that

$$D \cap \big(x + (a,b)\big) \cap (t_0 - \alpha, t_0 + \alpha) \subset \big(x + (a,b)\big) \cap \big\{t > x \colon h(t) > h(x)\big\}.$$

Indeed, if $y \in D \cap (t_0 - \alpha, t_0 + \alpha)$, then, from 4^0 and 3^0 we have $h(y) > c$ and $h(x) < c$, so $h(y) > h(x)$, and from 1^0 it follows that $y > x$. By 2^0,

$$t_0 \in \big(x + (a,b)\big) \cap (t_0 - \alpha, t_0 + \alpha)$$

and, in consequence, the set

$$\big(x + (a,b)\big) \cap \big\{t > x \colon h(t) > h(x)\big\}$$

is of the second category. □

THEOREM 6. *If a function $f \colon \mathbb{R} \to \mathbb{R}$ has the Baire property, then its $\mathcal{I}$-approximate Dini derivatives have the Baire property, too. Moreover, if f is continuous, then they are of the Baire class 3.*

P r o o f. We prove the theorem for a function $\mathcal{I}$–D^+f only. Let a be a real number. Let, for $m \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$A_m(x) = \big\{t \geq x \colon f(t) - f(x) \leq \big(a + \tfrac{1}{m}\big)(t - x)\big\}.$$

It is easy to see that

$$\mathcal{I}\text{–}D^{+}f(x) \le a$$

if and only if, for each $m \in \mathbb{N}$, x is a right-hand $\mathcal{I}$-density point of the set $A_m(x)$.

First, we assume that the function f is continuous. Then, by Theorem 2,

$$\begin{aligned}\{x \in \mathbb{R} : \mathcal{I}\text{–}D^{+}f(x) \le a\} &= \bigcap_{m\in\mathbb{N}} \{x \in \mathbb{R} \colon \mathcal{I}\text{–}d^{+}(A_m(x), x) = 1\} \\ &= \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} \bigcap_{h\in(0,\frac{1}{p})} \bigcap_{i\in\{1,\dots,n\}} \bigcup_{j\in\{1,\dots,k\}} Q_m^{nkphij},\end{aligned}$$

where

$$Q_m^{nkphij} = \left\{x \in \mathbb{R} \colon \left[x + \frac{(i-1)k+j-1}{nk}h,\ x + \frac{(i-1)k+j}{nk}h\right] \subset A_m(x)\right\}.$$

The continuity of f implies that every set $A_m(x)$ is closed, therefore all the sets Q_m^{nkphij} are closed. Hence the set

$$\{x \in \mathbb{R} \colon \mathcal{I}\text{–}D^{+}f(x) \le a\}$$

is of type $\mathcal{F}_{\sigma\delta}$ and

$$\{x \in \mathbb{R} \colon \mathcal{I}\text{–}D^{+}f(x) > a\}$$

is of type $\mathcal{G}_{\delta\sigma}$ and, consequently,

$$\{x \in \mathbb{R} \colon \mathcal{I}\text{–}D^{+}f(x) \ge a\} = \bigcap_{m\in\mathbb{N}} \left\{x \in \mathbb{R} \colon \mathcal{I}\text{–}D^{+}f(x) > a + \frac{1}{m}\right\}$$

is of type $\mathcal{G}_{\delta\sigma\delta}$ and the function $\mathcal{I} - D^{+}f$ is of the Baire class 3.

Now, we assume that f has the Baire property. Then there exists a residual set B such that the function $f_{/B}$ is continuous. It is sufficient to show that, for each $a \in \mathbb{R}$, the sets

$$\{x \in B \colon \mathcal{I}\text{–}D^{+}f(x) \le a\}, \quad \{x \in B \colon \mathcal{I}\text{–}D^{+}f(x) \ge a\}$$

have the Baire property.

Let $a \in \mathbb{R}$. We may assume that the set $\{x \in B \colon \mathcal{I}\text{–}D^{+}f(x) \le a\}$ is of the second category. By Theorem 2,

$$\begin{aligned}B \cap \{x \in \mathbb{R} \colon \mathcal{I}\text{–}D^{+}f(x) \le a\} &= \bigcap_{m\in\mathbb{N}} \left(B \cap \{x \in \mathbb{R} \colon \mathcal{I}\text{–}d^{+}(A_m(x), x) = 1\}\right) \\ &= \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} \bigcap_{h\in(0,\frac{1}{p})} \bigcap_{i\in\{1,\dots,n\}} \bigcup_{j\in\{1,\dots,k\}} \left(B \cap P_m^{nkphij}\right),\end{aligned}$$

where

$$P_m^{nkphij} = \left\{x \in \mathbb{R} \colon \left(x + \frac{(i-1)k + j-1}{nk} h\,,\, x + \frac{(i-1)k+j}{nk} h\right) \Big\backslash A_m(x) \in \mathcal{I}\right\}.$$

From Lemma 5 it follows that the sets $B \setminus P_m^{nkhij}$ are open relative to B, therefore the sets $B \cap P_m^{nkhij}$ are closed relative to B. Thus the sets

$$\left\{x \in B \colon \mathcal{I}\text{–}D^+ f(x) \le a\right\}, \quad \left\{x \in B \colon \mathcal{I}\text{–}D^+ f(x) \ge a\right\}$$

have the Baire property. □

The results presented in papers [3] and [4] show how much we can learn about the function f by studying the existence and the behaviour of its $\mathcal{I}$-approximate derivative. We shall demonstrate that a certain knowledge of the function can be obtained by studying of the behaviour of its $\mathcal{I}$-approximate Dini derivatives.

Lemma 7. *Let $f \colon \mathbb{R} \to \mathbb{R}$ and $A \subset \left\{x \in \mathbb{R} \colon \mathcal{I}\text{–}D^+ f(x) < +\infty\right\}$. Then, for every real number a, there exists a set S_a having the Baire property, such that*

$$\left\{t \in \mathbb{R} \colon f(t) < a\right\} \cap A = S_a \cap A\,.$$

P r o o f . Let $a \in \mathbb{R}$ and $E_a = \left\{t \in \mathbb{R} \colon f(t) < a\right\}$. First, we prove that, for each open interval $(\alpha, \beta) \subset \mathbb{R}$, there exist an open interval $(c, d) \subset (\alpha, \beta)$ and a set $W \subset (c, d)$ having the Baire property, such that

$$E_a \cap A \cap (c, d) = W \cap A\,.$$

Let $(\alpha, \beta) \subset \mathbb{R}$ be an arbitrary open interval. If $(\alpha, \beta) \cap E_a \cap A = \emptyset$, then it is sufficient to take $W = \emptyset$ and $(c, d) = (\alpha, \beta)$.

Now, we assume that $(\alpha, \beta) \cap E_a \cap A \ne \emptyset$. Let $x_0 \in (\alpha, \beta) \cap E_a \cap A$. Since $x_0 \in E_a$, there exists $n \in \mathbb{N}$ such that $f(x_0) < a - \frac{1}{n}$. We show that x_0 is not an exterior right-hand $\mathcal{I}$-density point of the set

$$B = \left\{t > x_0 \colon f(t) \ge a\right\}.$$

Suppose to the contrary that $\mathcal{I}\text{–}d_e^+(B, x_0) = 1$. We observe that, for each $t \in B$,

$$\frac{f(t) - f(x_0)}{t - x_0} > \frac{a - a + \frac{1}{n}}{t - x_0} = \frac{1}{n(t - x_0)}\,. \tag{1}$$

We prove that $\mathcal{I}\text{–}D^+ f(x_0) = +\infty$, i.e., that

$$\left\{\alpha \colon \mathcal{I}\text{–}d_e^+\left(\left\{t > x_0 \colon \frac{f(t) - f(x_0)}{t - x_0} > \alpha\right\}, x_0\right) = 0\right\} = \emptyset\,.$$

Suppose to the contrary that there exists a number α_0 such that

$$\mathcal{I}\text{–}d_e^+\left(\left\{t > x_0 \colon \frac{f(t) - f(x_0)}{t - x_0} > \alpha_0\right\},\, x_0\right) = 0\,.$$

Let $\delta > 0$ be a number such that, for each $t \in (x_0, x_0 + \delta)$,

$$\frac{1}{n(t - x_0)} > \alpha_0 \,. \tag{2}$$

Since $\mathcal{I}\text{–}d_e^+(B, x_0) = 1$, therefore $\mathcal{I}\text{–}d_e^+\big(B \cap (x_0, x_0 + \delta), x_0\big) = 1$. Then, from (1) and (2) it follows that

$$\begin{aligned} B \cap (x_0, x_0 + \delta) &\subset \left\{ t > x_0 \colon \frac{f(t) - f(x_0)}{t - x_0} > \frac{1}{n(t - x_0)} \right\} \cap (x_0, x_0 + \delta) \\ &\subset \left\{ t > x_0 \colon \frac{f(t) - f(x_0)}{t - x_0} > \alpha_0 \right\}, \end{aligned}$$

and, in consequence,

$$\mathcal{I}\text{–}d_e^+\left(\left\{ t > x_0 \colon \frac{f(t) - f(x_0)}{t - x_0} > \alpha_0 \right\},\, x_0 \right) = 1\,,$$

which is impossible.

Thus $\mathcal{I}\text{–}D^+ f(x_0) = +\infty$, a contradiction to $x_0 \in A$. So, x_0 is not an exterior right-hand $\mathcal{I}$-density point of B.

Let C be a Baire kernel of a set $\{t > x_0 \colon f(t) < a\}$. Since x_0 is not an exterior right-hand $\mathcal{I}$-density point of B, therefore x_0 is not a right-hand $\mathcal{I}$-dispersion point of C. Hence the set C is not of the first category in any right-hand neighbourhood of the point x_0. Moreover, C has the Baire property, thus there exists an interval $(c, d) \subset (x_0, \beta)$ such that C is residual on (c, d). Let $W = E_a \cap (c, d)$. Obviously,

$$E_a \cap A \cap (c, d) = W \cap A\,.$$

Since $C \subset E_a$ and C is residual on (c, d), therefore the set $(c, d) \setminus E_a$ is of the first category. So, the set $W = E_a \cap (c, d)$ has the Baire property.

From what has already been proved it follows that if

$$\begin{aligned} \mathcal{P} = \big\{(c, d) \colon \text{ there exists a set } W \in \mathcal{S} \text{ such that } & W \subset (c, d) \\ \text{and } & E_a \cap A \cap (c, d) = W \cap A \big\}\,, \end{aligned}$$

and $G = \bigcup_{P \in \mathcal{P}} P$, then the set G is open and dense in $\mathbb{R}$. Therefore $\mathbb{R} \setminus G$ is a closed and boundary set in $\mathbb{R}$ and, in consequence, is of the first category. By the Lindelöf Theorem, there exists a sequence of intervals $\{(c_n, d_n)\}_{n \in \mathbb{N}} \subset \mathcal{P}$ such that

$$G = \bigcup_{n \in \mathbb{N}} (c_n, d_n)\,.$$

For each $n \in \mathbb{N}$, let $W_n \subset (c_n, d_n)$ be a set having the Baire property, such that

$$E_a \cap A \cap (c_n, d_n) = W_n \cap A\,.$$

Then the set $V = \bigcup_{n\in\mathbb{N}} W_n$ has the Baire property and

$$E_a \cap A \cap G = V \cap A .$$

Hence

$$\begin{aligned} E_a \cap A &= (E_a \cap A \cap G) \cup \big(E_a \cap A \cap (\mathbb{R} \setminus G)\big) \\ &= (V \cap A) \cup \big(E_a \cap A \cap (\mathbb{R} \setminus G)\big) \\ &= A \cap \big(V \cup \big(E_a \cap (\mathbb{R} \setminus G)\big)\big\} \\ &= A \cap S_a , \end{aligned}$$

where

$$S_a = V \cup \big(E_a \cap (\mathbb{R} \setminus G)\big) .$$

The set $E_a \cap (\mathbb{R} \setminus G)$ is of the first category as a subset of $\mathbb{R} \setminus G$, therefore $E_a \cap (\mathbb{R} \setminus G)$ has the Baire property and, consequently, S_a has the Baire property, too.

Thus, for each $a \in \mathbb{R}$, there exists a set S_a having the Baire property, such that $E_a \cap A = S_a \cap A$, which finishes the proof. □

Theorem 8. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and let $A \subset \big\{x \in \mathbb{R} \colon \mathcal{I}\text{–}D^+ f(x) < +\infty\big\}$ be a set of the second category. Then there exists a set $W \subset A$ such that the set $A \setminus W$ is of the first category and the function $f|_W$ is continuous.*

Proof. From Lemma 7 it follows that, for each $a \in \mathbb{R}$, there exists a set S_a having the Baire property, such that

$$\big\{x \in \mathbb{R} \colon f(x) < a\big\} \cap A = S_a \cap A .$$

Hence, for every open set E, there exists a set S having the Baire property, such that

$$f^{-1}(E) \cap A = S \cap A .$$

Let a family $\{U_n,\ n \in \mathbb{N}\}$ be a countable basis with respect to the natural topology on the real line.

For each $n \in \mathbb{N}$, let G_n be an open set and P_n-a set of the first category, such that

$$(f|_A)^{-1}(U_n) = (G_n \triangle P_n) \cap A .$$

Put

$$P = \bigcup_{n=1}^{\infty} P_n \qquad \text{and} \qquad W = A \setminus P .$$

Obviously, the set P is of the first category. Moreover, for each $n \in \mathbb{N}$,

$$\begin{aligned} (f|_W)^{-1}(U_n) &= (f_{/A})^{-1}(U_n) \cap W = \big((G_n \triangle P_n) \cap A\big) \cap W \\ &= (G_n \triangle P_n) \cap W = (G_n \setminus P_n) \cap W \\ &= G_n \cap W . \end{aligned}$$

Therefore the function $f|_W$ is continuous, which concludes the proof. □

From the above theorem we immediately get

THEOREM 9. *If $f\colon \mathbb{R} \to \mathbb{R}$ and $\mathbb{R} \setminus \{x \in \mathbb{R} : \mathcal{I}\text{–}D^+ f(x) < +\infty\} \in \mathcal{I}$, then the function f has the Baire property.*

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