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A THEOREM OF CANTOR-BERNSTEIN TYPE FOR ORTHOGONALLY σ-COMPLETE PSEUDO MV-ALGEBRAS

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. We prove a theorem mentioned in the title of the paper; this generalizes a result of De Simone, Mundici and Navara.

1. Introduction

The notion of pseudo MV-algebra has been introduced independently by Georgescu and Iorgulescu [4], [5], and by Rachůnek [13] (in [13], the term 'generalized MV-algebra' has been applied). For the definition and the terminology, cf. Section 2 and 3 below.

In [12], a theorem of Cantor-Bernstein type for orthogonally σ-complete lattice ordered groups was presented.

A Cantor-Bernstein theorem for σ-complete MV-algebras was proved by De Simone, Mundici and Navara [2].

In this paper we generalize the result from [2]. The generalization is performed in two directions. Firstly, we show that instead of σ-completeness it suffices to apply the weaker assumption of orthogonal σ-completeness. (We remark that a σ-complete MV-algebra must be archimedean, while the orthogonal σ-completeness does not imply the archimedean property.) Secondly, instead of an MV-algebra, the more general notion of pseudo MV-algebra can be taken into account.

We apply the method of [12] dealing with direct factors of the structures under consideration.

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For related results concerning Boolean algebras, lattice ordered groups and MV-algebras cf. also [14], [15], [16], [6]–[11].

2. Preliminaries

For the terms of the theory of MV-algebras cf. the monograph Cignoli, D'Ottaviano and Mundici [1].

An element of an MV-algebra A is called boolean (cf. [2]) if $a \oplus a = a$. We denote by $\mathscr{B}(A)$ the set of all boolean elements of A. Let $a \in \mathscr{B}(A)$. The interval algebra $[0, a]$ of A is described in Section 3, where we consider also the analogous notion concerning pseudo MV-algebras.

An MV-algebra A is said to be σ-complete if each sequence of elements of A has the supremum and the infimum in A.

The following theorem is the main result of [2].

Theorem (A). ([2], Theorem 4.1.) *Let A B be σ-complete MV-algebras. Let $a \in \mathscr{B}(A)$, $b \in \mathscr{B}(B)$. Assume that α is an isomorphism of A onto the interval algebra $[0, b]$ of B, and that β is an isomorphism of B onto the interval algebra $[0, a]$ of A. Then A and B are isomorphic.*

For the sake of completeness, we recall the definition and some basic facts concerning the notion of pseudo MV-algebra. (Cf. [4], [5].)

Let $\mathcal{A} = (A; \oplus, ^-, ^\sim, 0, 1)$ be an algebraic structure of type $(2, 1, 1, 0, 0)$ where A is a nonempty set. Let $x, y \in A$. We put

$$y \odot x = (x^- \oplus y^-)^\sim .$$

The structure $\mathcal{A}$ is called a pseudo MV-algebra if the following identities (A1)–(A8) are satisfied for $x, y, z \in A$:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

Assume that $\mathcal{A}$ is a pseudo MV-algebra. For $x, y \in A$ we define $x \leqq y$ if $x^- \oplus y = 1$. Then $\leqq$ is a partial order on A. We put $L(\mathcal{A}) = (A; \leqq)$. The structure $L(\mathcal{A})$ is a distributive lattice with the least element 0 and with the greatest element 1; we have

$$x \vee y = x \oplus (x^{\sim} \odot y)\,, \quad x \wedge y = x \odot (x^{-} \oplus y)\,.$$

Now let G be a lattice ordered group with a strong unit u. The group operation in G is denoted additively; we do not assume the commutativity of this operation. The interval $[0,u]$ of G will be denoted by A. For $x, y \in A$ we put

$$x \oplus y = (x+y) \wedge u\,,$$

$$x^{-} = u - x\,, \quad x^{\sim} = -x + u\,.$$

2.1. Theorem. (Cf. [4], [5], [13].) *Let G be as above. The algebraic structure $\mathcal{A} = (A; \oplus, ^{-}, ^{\sim}, 0, u)$ is a pseudo MV-algebra.*

If $\mathcal{A}$ and G are as in 2.1, then we put $\Gamma(G,u) = \mathcal{A}$.

2.2. Theorem. (Cf. Dvurečenskij [3].) *For each pseudo MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G,u)$.*

For a pseudo MV-algebra $\mathcal{A}$ we shall always assume that G is as in 2.2.

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A is called orthogonal if, whenever $n(1)$ and $n(2)$ are distinct elements of $\mathbb{N}$, then

$$x_{n(1)} \wedge x_{n(2)} = 0\,.$$

A pseudo MV-algebra $\mathcal{A}$ is said to be orthogonally σ-complete if each orthogonal sequence of elements of A possesses the supremum in the lattice $L(\mathcal{A})$.

Example. Let Q be the additive group of all rationals with the natural linear order. Put $u = 1$ and $\mathcal{A} = \Gamma(G,u)$. Then $\mathcal{A}$ is a pseudo MV-algebra (in fact, it is an MV-algebra) which is orthogonally σ-complete, but it fails to be σ-complete.

2.3. Lemma. (Cf. [5].) *Let $\mathcal{A}$ be a pseudo MV-algebra, $x \in A$, $\emptyset \neq \{y_i\}_{i\in I} \subseteq A$. If $\bigvee\limits_{i\in I} y_i$ exists in $L(\mathcal{A})$, then*

$$x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I} (x \wedge y_i)\,.$$

3. On elements having a complement

Again, let $\mathcal{A}$ be a pseudo MV-algebra.

3.1. Lemma. *Let $a \in A$ such that $a \oplus a = a$. Then $a^{\sim}$ is the complement of a in the lattice $L(\mathcal{A})$ and $a^{-} = a^{\sim}$.*

P r o o f. We have $a \oplus a = (a+a) \wedge u$, whence

$$(a+a) \wedge u = a.$$

This yields

$$(-a) + \big((a+a) \wedge u\big) = 0,$$
$$a \wedge (-a+u) = 0,$$

thus $a \wedge a^{\sim} = 0$. Therefore

$$a + a^{\sim} = a \vee a^{\sim}.$$

Since $a + a^{\sim} = a + (-a+u) = u$, we obtain that $a^{\sim}$ is a complement of a in $L(\mathcal{A})$.

Similarly we verify that a^{-} is a complement of a in $L(\mathcal{A})$. Since the lattice $L(\mathcal{A})$ is distributive, we infer that $a^{-} = a^{\sim}$. □

3.2. Lemma. *Let $a \in A$. Assume that a has a complement b in $L(\mathcal{A})$. Then $a \oplus a = a$.*

P r o o f. We have $a \wedge b = 0$, $a \vee b = u$. Then

$$\begin{aligned} a \oplus a &= (a+a) \wedge u = (a+a) \wedge (a \vee b) = (a+a) \wedge (a+b) \\ &= a + (a \wedge b) = a + 0 = a. \end{aligned}$$

□

3.3. Corollary. *If a and b are as in 3.2, then $b = a^{\sim} = a^{-}$.*

3.4. Proposition. *Let (x_n) be an orthogonal sequence in A, $x \in A$, $x = \bigvee\limits_{n \in \mathbb{N}} x_n$. Assume that each element x_n has a complement in $L(\mathcal{A})$. Then x has a complement in $L(\mathcal{A})$.*

P r o o f. According to 3.2, we have $x_n \oplus x_n = x_n$ for each $n \in \mathbb{N}$. Hence we obtain

$$\begin{aligned} x \oplus x &= \left(\left(\bigvee_{n \in \mathbb{N}} x_n\right) + \left(\bigvee_{m \in \mathbb{N}} x_m\right)\right) \wedge u \\ &= \left(\bigvee_{n \in \mathbb{N}, m \in \mathbb{N}} (x_n + x_m)\right) \wedge u = \bigvee_{n \in \mathbb{N}, m \in \mathbb{N}} \big((x_n + x_m) \wedge u\big). \end{aligned}$$

If $m = n$, then

$$(x_n + x_m) \wedge u = (x_n + x_n) \wedge u = x_n \oplus x_n = x_n.$$

If $m \neq n$, then

$$(x_n + x_m) \wedge u = (x_n \vee x_m) \wedge u = x_n \vee x_m .$$

Therefore

$$\bigvee_{n,m \in \mathbb{N}} (x_n + x_m) \wedge u = \bigvee_{n \in \mathbb{N}} x_n = x .$$

Thus we have $x \oplus x = x$. Now it suffices to apply 3.1. □

3.5. Proposition. *Let $(x_n)_{n \in \mathbb{N}}$ and x be as in 3.4. Further, for $n \in \mathbb{N}$ let x'_n be the complement of x_n in $L(\mathcal{A})$. Then*

$$\bigwedge_{n \in \mathbb{N}} x'_n = u - x .$$

Proof. Let us remark that in view of 3.3 we have $u - x = -x + u$. Let $n \in \mathbb{N}$. Then $x \geqq x_n$, whence

$$u - x \leqq u - x_n = x'_n - 0 = x'_n.$$

Assume that $t \in A$, $t \leqq x'_n$ for each $n \in \mathbb{N}$. Then $x_n \wedge t = 0$, whence

$$x_n + t = x_n \vee t \leqq u$$

for each $n \in \mathbb{N}$. We obtain

$$x + t \leqq \left(\bigvee_{n \in \mathbb{N}} x_n \right) + t = \bigvee_{n \in \mathbb{N}} (x_n + t) \leqq u ,$$

thus $t \leqq -x + u = u - x$. Therefore $u - x$ is the greatest lower bound of $(x'_n)_{n \in \mathbb{N}}$ in $L(\mathcal{A})$. □

Assume that e is an element of A having a complement in the lattice $L(\mathcal{A})$. Consider the interval

$$X_1 = [0, e]$$

of $L(\mathcal{A})$. In view of Lemma 3.6 in [10], the set X_1 is closed with respect to the operation $\oplus$. For $x \in X_1$ we denote

$$P^1_x = \{p \in X_1 : p \oplus x = e\} ,$$
$$Q^1_x = \{q \in X_1 : x \oplus q = e\} ,$$

In Section 3 of [10] it has been proved that both sets P^1_x and Q^1_x have a minimum in X_1. Denote

$$x^{-(e)} = \min P^1_x , \quad x^{\sim(e)} = \min Q^1_x$$

and let us consider the structure

$$\mathcal{X} = (X_1 ; \oplus, {}^{-(e)}, {}^{\sim(e)}, 0, e)$$

of type $(2,1,1,0,0)$ (in fact, $\oplus$ is here reduced to the set X_1).

3.6. Lemma. (Cf. [10], Corollary 4.2.) *$\mathcal{X}$ is a pseudo MV-algebra.*

Under the assumptions as above, we will denote

$$\mathcal{X}_1 = \mathcal{X}[0,e].$$

3.7. Lemma. *Let e be as above and $\mathcal{X} = \mathcal{X}[0,e]$, $x \in [0,e]$. Then*

$$x^{-(e)} = e - x, \quad x^{\sim(e)} = -x + e.$$

P r o o f. We have

$$(e-x) \oplus x = \big((e-x)+x\big) \wedge u = e \wedge u = e,$$

thus $e - x \in P_x^1$. Let $p \in P_x^1$. Then $p \oplus x = e$; in other words, $(p+x) \wedge u = e$. Hence

$$e = (p+x) \wedge u \wedge e = (p+x) \wedge e,$$

and so $p + x \geqq e$, $p \geqq e - x$. Therefore $x^{-(e)} = e - x$. Similarly we can verify the relation $x^{\sim(e)} = -x + e$. □

3.8. Lemma. *Under the assumptions as in 3.7 we have*

$$x^{-(e)} = e \wedge x^{-}, \quad x^{\sim(e)} = e \wedge x^{\sim}.$$

P r o o f. In view of 3.7 we have

$$x^{-(e)} = e - x \leqq u - x = x^{-},$$

and clearly $x^{-(e)} \leqq e$; thus $x^{-(e)} \leqq e \wedge x^{-}$.

Let $t \in A$, $t \leqq e$ and $t \leqq x^{-}$. We denote by e' the complement of the element e in $L(\mathcal{A})$. Then

$$u - x = (e' \vee e) - x = (e' + e) - x = e' + (e - x).$$

Since $0 \leqq e - x \leqq e$, we get $e' \wedge (e-x) = 0$, whence

$$e' + (e-x) = e' \vee (e-x).$$

Thus we obtain

$$t = t \wedge (u-x) = t \wedge \big(e' \vee (e-x)\big) = (t \wedge e') \vee \big(t \wedge (e-x)\big) = t \wedge (e-x),$$

which yields $t \leqq e - x$. Therefore $x^{-(e)} = e - x$. Analogously we have $x^{\sim(e)} = -x + e$. □

The pseudo MV-algebra $\mathcal{A}$ turns out to be an MV-algebra if the operation $\oplus$ is commutative. In such case $x^{-} = x^{\sim}$ for each $x \in A$.

Let us now suppose that $\mathcal{A}$ is an MV-algebra and let $b \in \mathscr{B}(\mathcal{A})$. In [1] and [2], the notation $\neg x$ instead of x^- is applied for $x \in A$. For each $x \in [0, b]$ put

$$\neg_b x = x \wedge b$$

and consider the structure

$$\big([0, b]; \oplus, \neg_b, 0, b\big)$$

of type $(2, 1, 0, 0)$. (Of course, $\oplus$ is reduced to the set $[0, b]$; we have already remarked above that the set $[0, b]$ is closed with respect to the operation $\oplus$.)

In [2], the structure under consideration is called the interval algebra $[0, b]$. (Cf. Definition 2.2 and Proposition 2.3 in [2].)

Hence according to 3.8 we conclude that if $\mathcal{A}$ is an MV-algebra and $e \in A$, $e \oplus e = e$, then the structure $\mathcal{X}[0, e]$ coincides with the interval algebra $[0, e]$ if we write $\neg_e$ instead of ${}^{-(e)}$.

4. Direct product decompositions

The direct product of pseudo MV-algebras is defined in the usual way. We apply the notation $\mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n$ or

$$\prod_{i \in I} \mathcal{A}_i .$$

Consider an isomorphism

$$\varphi\colon \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i \,, \tag{1}$$

where $\mathcal{A}$ and $\mathcal{A}_i$ $(i \in I)$ are pseudo MV-algebras. We say that (1) is a direct product decomposition of $\mathcal{A}$. For $x \in A$ we denote

$$\varphi(a) = (a_i)_{i \in I} \,;$$

a_i is the component of a in $\mathcal{A}_i$.

An analogous notation will be applied for direct products of lattices.

In this section we assume that $\mathcal{A}$ is an orthogonally σ-complete pseudo MV-algebra. Further, G is as in 2.2. For $a_1, a_2 \in A$, the meaning of $a_1 - a_2$ is as in G.

Suppose that $(a_n)_{n \in \mathbb{N}}$ is an orthogonal sequence of elements of A such that

$$\bigvee_{n \in \mathbb{N}} a_n = u \,.$$

Then for each $n \in \mathbb{N}$ there exists $a_n^1 \in A$ with

$$a_n^1 = \bigvee_{m\in\mathbb{N},\ m\neq n} a_m \,.$$

We clearly have $a_n \vee a_n^1 = u$. Further, according to 2.3,

$$a_n \wedge a_n^1 = \bigvee_{m\in\mathbb{N},\ m\neq n} (a_n \wedge a_m) = 0 \,.$$

Hence a_n^1 is the complement of a_n in the lattice $L(\mathcal{A})$. Thus we can consider the pseudo MV-algebra

$$\mathcal{A}_n = \mathcal{X}[0, a_n]$$

defined as in Section 3. Denote $L_n = L(\mathcal{A}_n)$.

For each $x \in A$ we put

$$\varphi(x) = (x \wedge a_n)_{n\in\mathbb{N}} \,.$$

Then φ is a mapping of the lattice $L(\mathcal{A})$ into the lattice

$$L^1 = \prod_{n\in\mathbb{N}} L_n \,.$$

For each $n \in \mathbb{N}$, the mapping $x \to x \wedge a_n$ is a homomorphism of $L(\mathcal{A})$ into L_n; let us denote this mapping by φ_n. If $x \in L_n$, then $\varphi_n(x) = x$; hence φ_n is surjective. Further, φ is a homomorphism of $L(\mathcal{A})$ into L^1. We put $x \wedge a_n = x_n$.

Let $(y^n)_{n\in\mathbb{N}} \in L^1$. Then $y^{n(1)} \wedge y^{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct elements of $\mathbb{N}$. Thus there exists $y \in L(\mathcal{A})$ with

$$y = \bigvee_{n\in\mathbb{N}} y^n \,.$$

Let $m \in \mathbb{N}$. We have

$$y_m = a_m \wedge y = \bigvee_{n\in\mathbb{N}} (a_m \wedge y^n) = y^m \,.$$

Hence $\varphi(y) = (y^n)_{n\in\mathbb{N}}$; thus φ is injective and

$$y = \bigvee_{n\in\mathbb{N}} y_n \,.$$

Let $z \in L(\mathcal{A})$ and assume that $\varphi(y) = \varphi(z)$. Then $y_n = z_n$ for each $n \in \mathbb{N}$, whence

$$z = \bigvee_{n\in\mathbb{N}} z_n = \bigvee_{n\in\mathbb{N}} y_n = y \,.$$

Therefore φ is injective. Summarizing, we obtain

4.1. Lemma. *Under the assumptions as above, the mapping*

$$\varphi\colon L(\mathcal{A}) \to \prod_{n\in\mathbb{N}} L_n$$

is a direct product decomposition of the lattice $L(\mathcal{A})$.

Then according to Theorem 6.4 in [10] we get

4.2. Proposition. *Under the assumptions as above we have a direct product decomposition of the pseudo* MV*-algebra* $\mathcal{A}$ *in the form*

$$\varphi\colon \mathcal{A} \to \prod_{n\in\mathbb{N}} \mathcal{A}_n\,.$$

We denote by $c(\mathcal{A})$ the set of all elements of $\mathcal{A}$ which have a complement in $L(\mathcal{A})$.

4.3. Lemma. *Let* $\mathcal{A}$ *be an orthogonally* σ*-complete pseudo* MV*-algebra. Assume that* $(a_n^0)_{n\in\mathbb{N}}$ *is an orthogonal sequence of elements of* $c(\mathcal{A})$. *Put*

$$d = u - \bigvee_{n\in\mathbb{N}} a_n^0\,.$$

Then $d \in c(\mathcal{A})$ *and we have a direct product decomposition*

$$\varphi\colon \mathcal{A} \to \left(\prod_{n\in\mathbb{N}} \mathcal{X}\left[0, a_n^0\right]\right) \times \mathcal{X}[0, d]\,,$$

where

$$\varphi(x) = \left((x \wedge a_n^0)_{n\in\mathbb{N}},\, x \wedge d\right)$$

for each $x \in X$.

P r o o f. In view of 3.3 and 3.4 we infer that the element d belongs to $c(\mathcal{A})$. Then the pseudo MV-algebras $\mathcal{X}\left[0, a_n^0\right]$ $(n \in \mathbb{N})$ and $\mathcal{X}[0, d]$ exist. Further,

$$d \wedge a_n^0 = 0 \qquad \text{for each } \ n \in \mathbb{N}\,.$$

Put $L_n = \left[0, a_n^0\right]$, $L_d = [0, d]$. In view of 4.1, we have a direct product decomposition

$$\varphi\colon L(\mathcal{A}) \to \left(\prod_{n\in\mathbb{N}} L_n\right) \times L_d\,.$$

Now it suffices to apply Proposition 4.2. $\square$

5. A Cantor-Bernstein type theorem

In the present section we prove the following result.

5.1. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be orthogonally σ-complete pseudo MV-algebras. Further, let $a \in c(\mathcal{A})$, $b \in c(\mathcal{B})$. Suppose that*

(i) *$\mathcal{A}$ is isomorphic to the pseudo MV-algebra $\mathcal{X}[0,b]$;*

(ii) *$\mathcal{B}$ is isomorphic to the pseudo MV-algebra $\mathcal{X}[0,a]$.*

Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Proof. Let us apply the following notation. Whenever $\mathcal{A}^1$ is a pseudo MV-algebra, $a \in c(\mathcal{A}^1)$ and $\mathcal{A}^2 = \mathcal{X}[0,a]$, then we denote

$$\mathcal{A}^1 \ominus \mathcal{A}^2 = \mathcal{X}[0,a^1],$$

where a^1 is the complement of a in the lattice $L(\mathcal{A}^1)$.

Thus, if u^1 is the greatest element of $\mathcal{A}^1$, then

$$u^1 = a \vee a^1, \quad a \wedge a^1 = 0, \tag{$*_1$}$$

whence

$$u^1 = a + a^1. \tag{$*_2$}$$

We also remark that if $b \in c(\mathcal{A}^2)$, then clearly $b \in c(\mathcal{A}^1)$.

In view of the assumption, there exists an isomorphism

$$\alpha\colon \mathcal{A} \to \mathcal{X}[0,b];$$

next, there exists an isomorphism

$$\beta\colon \mathcal{B} \to \mathcal{X}[0,a].$$

For each $x \in A$ we put

$$\varphi(x) = \beta\big(\alpha(x)\big).$$

Further, we denote

$$c^1 = a, \quad c^{n+1} = \varphi(c^n) \qquad \text{for each } n \in \mathbb{N}.$$

Then we have $c^n \in c(\mathcal{A})$ for each $n \in \mathbb{N}$. Hence we can construct the pseudo MV-algebra

$$C^n = \mathcal{X}[0,c^n].$$

Now we can apply the same method as in [12], Section 2. We put $C^0 = \mathcal{A}$; in view of the isomorphisms α and β we obtain (where $\simeq$ stands for isomorphisms of pseudo MV-algebras)

$$C^0 \simeq C^2,\, C^1 \simeq C^3, \dots, C^{n-2} \simeq C^n, \dots$$

and, moreover,

$$C^0 \ominus C^1 \simeq C^2 \ominus C^3\,,\ C^1 \ominus C^2 \simeq C^3 \ominus C^4,\ \dots$$

We denote

$$C^{n-1} \ominus C^n = \mathcal{A}^n \qquad \text{for each } n \in \mathbb{N}\,.$$

Hence we have

$$\mathcal{A}^n \simeq \mathcal{A}^{n+2} \qquad \text{for each } n \in \mathbb{N}\,. \tag{1}$$

The definition of the pseudo MV-algebras $\mathcal{A}^n$ yields that

$$\mathcal{A}^{n(1)} \cap \mathcal{A}^{n(2)} = \{0\} \tag{2}$$

whenever $n(1)$ and $n(2)$ are distinct positive integers.

For each $n \in \mathbb{N}$ let a_n be the greatest element of $L(\mathcal{A}^n) = L^n$. According to (2), the sequence $(a_n)_{n\in\mathbb{N}}$ is orthogonal. Hence there exist the elements

$$x = \bigvee_{n\in\mathbb{N}} a_n\,, \quad y = \bigvee_{n\geqq 2} a_n$$

in A.

According to 4.3, the element x has a complement in $L(\mathcal{A})$ (we denote this complement by d) and there exists an isomorphism

$$\varphi\colon \mathcal{A} \to \left(\prod_{n\in\mathbb{N}} \mathcal{A}^n\right) \times D\,, \tag{3}$$

where $D = \mathcal{X}[0,d]$.

Let us consider the MV-algebras

$$\mathcal{A} = C^0, \quad C^1, \quad C^0 \ominus C^1 = \mathcal{A}^1$$

and the elements

$$u\,, \quad c^1 = a\,, \quad a_1\,.$$

In view of $(*_1)$ and $(*_2)$ we obtain

$$u = c^1 + a_1 = c^1 \vee a_1\,, \quad c^1 \wedge a_1 = 0\,. \tag{$*$}$$

Let $n \in \mathbb{N}$, $n \geqq 2$. Then $a_1 \wedge a_n = 0$, whence

$$a_n = a_n \wedge u = a_n \wedge (c^1 \vee a_1) = (a_n \wedge c^1) \vee (a_n \wedge a_1) = a_n \vee c^1\,,$$

yielding that $a_n \leqq c^1$. Then a_n has a complement in the interval $[0, c^1]$ (cf. the remark at the beginning of the present proof). Moreover, $y \leqq c^1$ and in view of 3.4, y has a complement in $[0, c^1]$ as well. According to 3.3, this complement is equal to $c^1 - y$.

We have

$$a_1 \wedge y = a_1 \wedge \bigvee_{n \geqq 2} a_n = \bigvee_{n \geqq 2} (a_1 \wedge a_n) = 0\,,$$

thus

$$a_1 + y = a_1 \vee y = a_1 \vee \bigvee_{n \geqq 2} a_n = x\,.$$

Since $x + d = u$ we obtain (cf. $(*)$)

$$a_1 + y + d = a_1 + c^1\,,$$

whence $y + d = c^1$. Therefore the complement of y in the interval $[0, c^1]$ is the element d. Thus by applying 4.3 again we infer that there exists and isomorphism

$$\varphi_1 \colon C^1 \to \left(\prod_{n \geqq 2} \mathcal{A}^n \right) \times D\,. \tag{4}$$

From (1) (applied for all odd $n \in \mathbb{N}$), (3) and (4) we conclude that $\mathcal{A}$ is isomorphic to C^1. In view of the isomorphism β, C^1 is isomorphic to $\mathcal{B}$. Hence $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. □

In view of 3.1 and 3.2 we infer that 5.1 is a generalization of Theorem (A).

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