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ABELIAN EXTENSIONS OF DIFFERENCE SETS

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. In analogy with the abelian extensions of effect algebras, extensions of D-sets by abelian groups are studied. It is shown that every cancellative, unital D-set can be considered as an extension of a D-poset by an abelian group. The notion of a cocycle on a D-set is introduced and relations between abelian extensions and cocycles are shown.

1. Introduction

The notion of a D-set (DS, in short) has been introduced in [N1]. A DS is a partial algebra with a partial binary operation $\ominus$, endowed with properties analogous to those of the usual subtraction. It can be considered as an $\ominus$-counterpart of a partial abelian semigroup (PAS) [FW], where the partial operation is a commutative and associative binary operation $\oplus$. In both structures usually the existence of a neutral element is assumed. With some appropriate cancellation properties on both sides, there is a one-to one correspondence between DS and PAS as well as their morphisms. Therefore, all known quantum logics — orthomodular lattices and posets, orthoalgebras, D-posets and effect algebras — may be regarded as cancellative, unital DS. On the other hand, every abelian group can be regarded as a DS.

In the present paper, we show that the set $I(0)$ of invertible elements in a DS with 0 forms an ideal, which bears the structure of an abelian group. The quotient with respect to this ideal is a DS with no nonzero invertible elements. If a DS is unital and cancellative, the quotient with respect to $I(0)$ is a D-poset. We show that every DS can be considered as an extension of a DS with trivial $I(0)$ by an abelian group. We introduce the notion of a cocycle, and show its relations to the extensions. In the last paragraph, some illustrating examples are shown.

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2. Difference sets and their invertible elements

Definition 2.1. [N1] A *D-set* is a non empty set L endowed with a partial binary operation $\ominus$ satisfying the following conditions:

(d1) for each $a \in L$, $a \ominus a \in L$;

(d2) if $a, b, a \ominus b \in L$, then $a \ominus (a \ominus b) \in L$ and, moreover, $a \ominus (a \ominus b) = b$;

(d3) if $a, b, c, a \ominus b$, $b \ominus c \in L$, then $a \ominus c \in L$ and, moreover, $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

Denote $a \ominus a =: 0_a$.

Lemma 2.1. [N1] *If L is a DS, the following properties are valid:*

(1) *for each $a \in L$, $a \ominus 0_a \in L$, and $a \ominus 0_a = a$;*

(2) *if $c \ominus a \in L$, then $0_a = 0_c = 0_{c \ominus a}$; if moreover $c \ominus a = 0$, then $c = a$;*

(3) *if $c \ominus a = d$, then $c \ominus d = a$;*

(4) *if $c \ominus b, (c \ominus b) \ominus a \in L$, then $c \ominus a, (c \ominus a) \ominus b \in L$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$;*

(5) *if $a, b, 0_a \ominus a \in L$, then $0_a = 0_b$ iff $b \ominus a \in L$;*

(6) *if $0_a \ominus a, a \ominus b \in L$, then $0_a \ominus b$, $b \ominus a \in L$;*

(7) *if $a \ominus b = a \ominus c$, then $b = c$.*

Lemma 2.2. *Every DS L can be written as a disjoint union of DS's L_α, each of L_α has a unique zero 0_α as a neutral element.*

Proof. Define a binary relation on L by $a \sim b$ if $0_a = 0_b$. This relation is an equivalence, and hence L can be written as a disjoint union of equivalence classes. □

Lemma 2.3. *Let L be a set with the properties* (d1), (d2). *Then* (d3) *is fulfilled iff the following condition holds:*

(d4) *if $a, b, a \ominus b, (a \ominus b) \ominus c \in L$, then $a \ominus c, (a \ominus c) \ominus b \in L$ and $(a \ominus b) \ominus c = (a \ominus c) \ominus b$.*

Proof. It is enough to show only that this law implies (d3), because the opposite implication follows by Lemma 2.1 (4). Let $a \ominus b, b \ominus c \in L$. Then $b = a \ominus (a \ominus b)$, hence

$$b \ominus c = \big[a \ominus (a \ominus b)\big] \ominus c\,.$$

This implies that

$$a \ominus c, (a \ominus c) \ominus (a \ominus b) \in L\,, \quad \text{and} \quad (a \ominus c) \ominus (a \ominus b) = b \ominus c\,.$$

□

In what follows, we will assume that L is a DS with only one 0. We will say that an element $a \in L$ is *invertible* if $0 \ominus a$ is defined in L. Define $I_L(0) := \{b \in L \colon 0 \ominus b \in L\}$, i.e., $I_L(0)$ is the set of all invertible elements in $I(0)$. In what follows we will use the symbol $I(0)$ instead $I_L(0)$ if no confusion may occur.

Let $a \in L$ and $k \in I(0)$. Define $a \oplus k := a \ominus (0 \ominus k)$.

Lemma 2.4. [N1] *Let L be a DS. Then the following properties hold:*

(1) *If $a \ominus b \in I(0)$, then $0 \ominus (a \ominus b) = (b \ominus a)$.*

(2) *For any $a, b \in I(0)$ $a \ominus b$ is defined and belongs to $I(0)$.*

(3) *The set $I(0)$ is an abelian group with the operation $x \oplus y = x \ominus (0 \ominus y)$.*

(4) *For any $a \in L$ and any $b \in I(0)$ the element $a \ominus b$ is defined.*

(5) *If $a \in L$, $b \in I(0)$ and $b \ominus a$ is defined, then $a \in I(0)$.*

Proof. (1) Let $a \ominus b \in I(0)$. From the definition of $I(0)$ it follows that $0 \ominus (a \ominus b)$ is defined and moreover

$$0 \ominus (a \ominus b) = (a \ominus a) \ominus (a \ominus b) = \big(a \ominus (a \ominus b)\big) \ominus a = b \ominus a\,.$$

(2) Let $a, b \in I(0)$. Then $a \ominus 0, 0 \ominus b \in L$ implies $a \ominus b \in L$. On the other hand $b \ominus 0, 0 \ominus a \in L$ implies $b \ominus a \in L$. And from (1) we get $a \ominus b \in I(0)$.

(3) Let $a, b \in I(0)$ then we define the operation $a \oplus b := a \ominus (0 \ominus b)$.

From (2) it follows that $a \ominus (0 \ominus b), b \ominus (0 \ominus a), (0 \ominus a) \ominus b, (0 \ominus b) \ominus a \in I(0)$, and moreover $(0 \ominus b) \ominus a = (0 \ominus a) \ominus b$. Then $0 \ominus \big((0 \ominus b) \ominus a\big) = 0 \ominus \big((0 \ominus a) \ominus b\big)$ and so $b \ominus (0 \ominus a) = a \ominus (0 \ominus b)$, hence $a \oplus b = b \oplus a$.

Let $a, b, c \in I(0)$. Then $(a \oplus b) \oplus c, (a \oplus c) \oplus b \in I(0)$ and $(a \oplus b) \oplus c = \big(a \ominus (0 \ominus b)\big) \ominus (0 \ominus c) = \big(a \ominus (0 \ominus c)\big) \ominus (0 \ominus b) = (a \oplus c) \oplus b$.

Let $c, d, a \in I(0)$ and $c \oplus a = d \oplus a$. Then $c \ominus (0 \ominus a) = d \ominus (0 \ominus a)$. From this we get

$$(0 \ominus a) \ominus c = (0 \ominus a) \ominus d\,.$$

By Lemma 2.1 (7), $c = d$.

For any $a \in I(0)$,

$$a \oplus (0 \ominus a) = a \ominus \big(0 \ominus (0 \ominus a)\big) = a \ominus a = 0\,.$$

It follows that $I(0)$ is an abelian group.

(4) Let $a \in L$ and $b \in I(0)$. Then $a \ominus 0, 0 \ominus b \in L$ implies $a \ominus b \in L$.

(5) Let $a \in L$, $b \in I(0)$ and $b \ominus a \in L$. Then $0 \ominus b, b \ominus a \in L$ implies $0 \ominus a \in L$. From this it follows that $a \in I(0)$. □

Lemma 2.5. [N2] *Let L be DS. For any $a \in L$ we define*

$$I(a) = \{b \in L \colon b \ominus a \in I(0)\}.$$

Then the following statements hold.

(1) *For any $b, c \in I(a)$ $c \ominus b \in I(0)$.*

(2) *For any $a, b \in L$, $b \in I(a)$ iff $I(b) = I(a)$.*

(3) *For any $a \in L$, $I(a) = \{a \ominus k \colon k \in I(0)\}$.*

(4) *For any $a \in L$, $I(a) = \{a \oplus k \colon k \in I(0)\}$.*

(5) *For any $a \in L$ and for any $p, q \in I(0)$ $a \ominus (p \oplus q) = (a \ominus p) \ominus q$.*

(6) *If $a \in I(b)$, $c, d \in I(0)$, then $(a \ominus c) \ominus (b \ominus d) \in L$, and $(a \ominus c) \ominus (b \ominus d) = (a \ominus b) \ominus (c \ominus d)$.*

(7) *Let $b \in I(a)$, $c \in I(d)$ and $a \ominus d \in L$. Then $b \ominus c \in L$, and $b \ominus c \in I(a \ominus d)$.*

Proof. (1) If $b, c \in I(a)$, then $b \ominus a, a \ominus c \in I(0)$. It implies $b \ominus c \in L$. On the other hand $c \ominus a, a \ominus b \in I(0)$ implies $c \ominus b \in L$. It follows that $c \ominus b \in I(0)$.

(2) Let $b \in I(a)$, then $a \ominus b \in I(0)$, hence also $b \ominus a \in I(0)$, which implies $a \in I(b)$. Moreover if $c \in I(a)$ then $c \ominus b \in I(0)$ by (1), hence from this it follows that $I(a) = I(b)$.

(3) Let $b \in I(a)$. Then $a \ominus b \in I(0)$. Put $a \ominus b = k$. Then $a \ominus k = b$, $k \in I(0)$. Hence $I(a) = \{a \ominus k \colon k \in I(0)\}$.

(4) It follows directly from the definition of $\oplus$ and (3).

(5) Let $a \in L$ and $p, q \in I(0)$. Then $a \ominus (p \oplus q), a \ominus p \in I(a)$, $I(a) = I(a \ominus p)$ and so $(a \ominus p) \ominus q \in I(a)$. From this it follows that $[(a \ominus p) \ominus q] \ominus [a \ominus (p \oplus q)]$ is defined. Moreover

$$\begin{aligned}
\big[(a \ominus p) \ominus q\big] \ominus \big[a \ominus (p \oplus q)\big] &= \big[(a \ominus p) \ominus \big[a \ominus (p \oplus q)\big]\big] \ominus q \\
&= \big[\big(a \ominus \big[a \ominus (p \oplus q)\big]\big) \ominus p\big] \ominus q \\
&= \big[(p \oplus q) \ominus p\big] \ominus q = \big[\big(p \ominus (0 \ominus q)\big) \ominus p\big] \ominus q \\
&= \big[(p \ominus p) \ominus (0 \ominus q)\big] \ominus q = \big(0 \ominus (0 \ominus q)\big) \ominus q \\
&= q \ominus q \ = \ 0.
\end{aligned}$$

Hence $(a \ominus p) \ominus q = a \ominus (p \oplus q)$.

(6) $a \in I(b)$, $c, d \in I(0)$ imply that $b \in I(a) = I(a \ominus c)$, $a \ominus c \in I(b) = I(b \ominus d)$. It follows that $(a \ominus c) \ominus (b \ominus d) \in L$. We then have

$$\begin{aligned}
\big((a \ominus c) \ominus (b \ominus d)\big) \ominus (a \ominus b) &= \big((a \ominus c) \ominus (a \ominus b)\big) \ominus (b \ominus d) \\
&= \big(\big(a \ominus (a \ominus b)\big) \ominus c\big) \ominus (b \ominus d) = (b \ominus c) \ominus (b \ominus d) \\
&= \big(b \ominus (b \ominus d)\big) \ominus c \ = \ d \ominus c,
\end{aligned}$$

which entails (6).

(7) Let $b \in I(a)$, $c \in I(d)$ and $a \ominus d \in L$. Then $b \ominus a$, $a \ominus d \in L$ imply $b \ominus d \in L$ and $b \ominus d$, $d \ominus c \in L$ imply $b \ominus c \in L$. Since $b \in I(a)$, $c \in I(d)$, there are $x, y \in I(0)$ with $b = a \ominus x$, $c = d \ominus y$. Then

$$b \ominus c = (a \ominus x) \ominus (d \ominus y) = (a \ominus d) \ominus (x \ominus y)$$

by (6). Hence $b \ominus c \in I(a \ominus d)$. □

Recall that a partially ordered set with 0 as the smallest element is a *poset with difference* [KCh] if it is endowed with a partial binary operation $\ominus$ such that $a \ominus b$ is defined iff $b \leq a$ and the following conditions are satisfied:

(1) $b \leq a$ implies $a \ominus b \leq a$ and $a \ominus (a \ominus b) = b$.

(2) $c \leq b \leq a$ imply $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

A difference set with a greatest element 1 is called a *D-poset*.

Theorem 2.7. [N1] *Let L be a D-set with 0 such that $I(0) = \{0\}$. Define, for $a, b \in L$, $a \leq b$ if $b \ominus a \in L$. Then $\leq$ is a partial order on L and L with the operation $\ominus$ is a poset with difference in the sense of* [KCh], *and $0 \leq a$ for every $a \in L$.*

We will say that a D-set L with 0 has a unit $u \in L$, if for every $a \in L$ $u \ominus a \in L$. Observe that if u is a unit and $0 \ominus a \in L$, then $u \ominus a$ is a unit, too. Indeed, for any $b \in L$ $b \ominus a \in L$, hence $u \ominus (b \ominus a) = (u \ominus a) \ominus b \in L$.

If u is a unit, and $I(0) = \{0\}$, then u is uniquely defined. Indeed, let u, u_1 be units. Then $u_1 \ominus u, u \ominus u_1 \in L$ imply $(u_1 \ominus u_1) \ominus (u_1 \ominus u) = u \ominus u_1$, hence $0 \ominus (u_1 \ominus u)$ is defined, and since $I(0) = \{0\}$, we get $u = u_1$.

A D-set with a distinguished unit will be called a *unital DS*. We note that a unital DS with 0 in which $I(0) = \{0\}$ is a D-poset in the sense of [KCh].

We will say that a D-set with 0 is *cancellative*, if $a \ominus c = b \ominus c$ imply $a = b$. In a cancellative DS with 0, we can define the operation $\oplus$ by $c = a \oplus b$ iff there is $c \in L$ such that $c \ominus b = a$. By cancellativity, c is unique.

Observe that a unital DS with 0 is cancellative. Indeed, let $c \ominus a = b \ominus a$. Then by (d3), $1 \ominus c, c \ominus a \in L$ implies $(1 \ominus a) \ominus (1 \ominus c) = c \ominus a$. Similarly, $(1 \ominus a) \ominus (1 \ominus b) = b \ominus a$. This yields

$$\begin{aligned} 0 &= \big((1 \ominus a) \ominus (1 \ominus c)\big) \ominus \big((1 \ominus a) \ominus (1 \ominus b)\big) \\ &= \big((1 \ominus a) \ominus \big((1 \ominus a) \ominus (1 \ominus b)\big)\big) \ominus (1 \ominus c) = (1 \ominus b) \ominus (1 \ominus c) \\ &= \big(1 \ominus (1 \ominus c)\big) \ominus b \ = \ c \ominus b. \end{aligned}$$

Recall that a *partial abelian monoid* (PAM in short) (or a *partial abelian semigroup with* 0 [W1], [W2] [GuPu]) is a nonempty set with a partially defined binary operation $\oplus$ which is commutative and cancellative (in the sense that if one side exists then there exists also the other and they are equal) and with a neutral element 0. A PAM is

– *cancellative* if $a \oplus b = a \oplus c$ implies $b = c$.

– *positive* if $a \oplus b = 0$ iff $a = b = 0$.

– *unital* if there is an element u, called a *unit*, such that for every element a there is an element a' such that $a \oplus a' = u$.

We recall that a positive, cancellative PAM coincides with a generalized effect algebra and is in one-to one relation with a generalized D-poset ([HePu]), or with an abelian relative inverse set ([KaRi]). A positive, cancellative, unital PAS coincides with an effect algebra [FoBe] and is in one-to-one relation with a D-poset.

THEOREM 2.8. *A cancellative DS with 0, with operation $\oplus$ defined by $a \oplus b = c$ iff $c \ominus b = a$, is a cancellative* PAM *in the sense of* [W1,2]. *Conversely, a cancellative* PAM *is a DS with 0, if we define $a \ominus b = c$ iff $a = b \oplus c$.*

P r o o f. If $c \ominus b = a$, then $c \ominus a = b$ by Lemma 1(3). Therefore $a \oplus b = b \oplus a$ (in the sense that if one side exists so does the other and they are equal). Assume that $a \oplus b$, $(a \oplus b) \oplus c$ are defined, then there is $d \in L$ with $a \oplus b = d$, and $e \in L$ with $(a \oplus b) \oplus c = e$. Then $e \ominus d, d \ominus b \in L$ imply $e \ominus b \in L$, and $(e \ominus b) \ominus (e \ominus d) = d \ominus b$, hence $(e \ominus b) \ominus c = a$. This implies that $(e \ominus b) \ominus a = c$ and then $e \ominus b = a \oplus c$, $e = b \oplus (a \oplus c)$. This proves associativity of $\oplus$. Moreover, $a \ominus 0 = a$ implies $a \oplus 0 = a$ for every a.

If $a \oplus b = a \oplus c = d$, so that $d \ominus b = d \ominus c$, then $0 = (d \ominus b) \ominus (d \ominus c) = c \ominus b$, hence $c = b$. It follows that $(L, \oplus, 0)$ is a cancellative PAM. Moreover, if $0 \ominus b \in L$, then $c = a \ominus (0 \ominus b) \in L$ implies $a = c \oplus (0 \ominus b)$. This implies $a \ominus c = 0 \ominus b$. Then

$$\begin{aligned} 0 \ominus (a \ominus c) = 0 \ominus (0 \ominus b) &= b\,, \\ c \ominus a &= b\,, \\ c &= a \oplus b\,. \end{aligned}$$

Hence $a \oplus b = a \ominus (0 \ominus b)$ if $b \in I(0)$.

Conversely, let L be a cancellative PAM. Define

$$a \ominus c \quad \text{iff} \quad a = b \oplus c\,.$$

Owing to cancellativity, $b \oplus c = d \oplus c$ implies $b = d$, so that the operation $\ominus$ is well defined. Proof of commutativity and associativity is left to the reader. $\square$

COROLLARY 2.8.1. *Unital D-sets with 0 are in one-to-one correspondence with unital cancellative* PAMs.

3. Morphisms, ideals and congruences of D-sets

Definition 3.1. [KCh] A *morphism of D-sets* is a mapping $h\colon L \to K$ such that $a \ominus b \in L$ implies $h(a) \ominus h(b) \in K$ and $h(a \ominus b) = h(a) \ominus h(b)$.

If L and K are unital, we also require $h(1) = 1$.

A morphism h is *faithful* if $h(a) \ominus h(b)$ exists in K if and only if $a \ominus b$ exists in L.

A bijective morphism is an isomorphism if the inverse mapping is also a morphism.

Observe that an isomorphism is the same thing as a faithful surjective morphism.

Observe that if L and K have a 0, then $h(0) = h(a \ominus a) = h(a) \ominus h(a) = 0$.

Definition 3.2. Let L be a DS with 0. A subset I of L is an *ideal* if for any $a, b \in L$ such that $b \ominus a$ is defined in L,

(i1) if $b \in I$, $b \ominus a \in I$, then $a \in I$,

(i2) if $a \in I$, $b \ominus a \in I$, then $b \in I$.

If I is an ideal, define a binary relation $\sim_I$ on L as follows: $a \sim_1 b$ if there are $i, j \in I$ such that $a \ominus i$, $b \ominus j \in L$ and $a \ominus i = b \ominus j$.

Definition 3.3. A *binary relation* on a DS is a congruence if

(c1) $\sim$ is an equivalence relation;

(c2) $a \sim a_1$, $b \sim b_1$, $a \ominus b$, $a_1 \ominus b_1 \in L$ imply $a \ominus b \sim a_1 \ominus b_1$.

If $\sim$ is a congruence, we define a partial binary operation $\ominus$ on the quotient $L/{\sim}$ by $[a] \ominus [b]$ is defined if there are a_1, b_1, $a \sim a_1$, $b \sim b_1$ such that $a_1 \ominus b_1 \in L$, and then put $[a] \ominus [b] = [a_1 \ominus b_1]$. Then (c2) implies that $\ominus$ is well-defined. It is easy to see that (d1) and (d2) are satisfied, (d3) may be not satisfied.

Theorem 3.1. *Let L be a DS with 0. Then $I(0)$ is an ideal.*

Proof. (i1) Even a stronger property, namely "*If $b \in I$, $b \ominus a \in L$, then $a \in I$*", follows by Lemma 2.4 (5).

(i2) Let a, $b \ominus a \in I(0)$. By Lemma 2.4 (1), $a \ominus b \in I(0)$ and by Lemma 2.4 (3), (2), $b = a \ominus (a \ominus b) \in I(0)$. □

Lemma 3.2. *$a \sim b$ if and only if $a \ominus b$, $b \ominus a \in L$ and belong to $I(0)$. In other words, $a \sim b$ if $a \in I(b)$.*

Proof. If $a \ominus b$, $b \ominus a \in I(0)$, then $a \ominus (a \ominus b) = b \ominus 0$, hence $a \sim b$.

Conversely, assume that $a \sim b$. Then there are $j, k \in I(0)$ with $a \ominus j = b \ominus k$. Then $(a \ominus j) \ominus (b \ominus k) = 0$, hence $(a \ominus b) \ominus (j \ominus k) = 0$ so that $a \ominus b = j \ominus k \in I(0)$. □

It is clear that $[a] = I(a)$.

THEOREM 3.3. *Let L be a DS with 0. Then*

(i) $\sim := \sim_{I(0)}$ *is a congruence.*

(ii) *The quotient $L/I(0) := L/\sim_{I(0)}$ is a DS.*

P r o o f . (i) It is clear that $\sim$ is reflexive and symmetric. It remains to prove transitivity. Assume that $a \sim_{I(0)} b$, $b \sim_{I(0)} c$. Then there are $i, j_1, j_2, k \in I(0)$ such that $a \ominus i = b \ominus j_1$, $b \ominus j_2 = c \ominus k$. By Lemma 4 (4), every element in $I(0)$ can be subtracted from every element in L. So we may write

$$(a \ominus i) \ominus j_2 = (b \ominus j_1) \ominus j_2 = (b \ominus j_2) \ominus j_1 = (c \ominus k) \ominus j_1 .$$

By Lemma 5 (5),

$$(a \ominus i) \ominus j_2 = a \ominus (i \oplus j_2) = (c \ominus k) \ominus j_1 = c \ominus (k \oplus j_1),$$

$i \oplus j_2, k \oplus j_1 \in I(0)$, hence $a \sim c$. This concludes the proof of (c1).

To prove (c2), assume $a \ominus a_1$, $b \ominus b_1 \in L$, $a \sim b$, $a_1 \sim b_1$. By Lemma 6, $a \in I(b)$, $a_1 \in I(b_1)$, and by Lemma 5 (7), $a \ominus a_1 \in I(b \ominus b_1)$, hence by Lemma 6, $a \ominus a_1 \sim b \ominus b_1$.

(ii) It remains to prove that the quotient $L/I(0)$ is a DS. We see from Lemma 6 that $a \sim b$ iff $a \ominus b \in I(0)$, and we also know that $a \ominus b \in I(0)$ implies $b \ominus a \in I(0)$. Therefore $L/\sim$ coincides with the set $\{I(a) \colon a \in L\}$ where $I(a) = \{b \in L \colon b \ominus a \in I(0)\}$ (in the sense than $b \ominus a$ is defined in L and belongs to $I(0)$). We have to prove properties (d1), (d2) and (d3). Since (d1) and (d2) are satisfied for every congruence, it suffices to prove (d3). Assume that $I(a) \ominus I(b), I(b) \ominus I(c)$ are defined in $L/I(0)$. It follows that there are $a_1 \sim a$, $b_1 \sim b$, $b_2 \sim b$, $c_1 \sim c$ and $a_1 \ominus b_1, b_2 \ominus c_1 \in L$, and $b_1 \ominus b_2 \in I(0)$.Then $a_1 \ominus b_1$, $b_1 \ominus b_2 \in L$ imply $a_1 \ominus b_2 \in L$, and $a_1 \ominus b_2$, $b_2 \ominus c_1 \in L$ imply $a_1 \ominus c_1 \in L$, and

$$(a_1 \ominus c_1) \ominus (a_1 \ominus b_2) = b_2 \ominus c_1 ,$$

hence

$$\big(I(a) \ominus I(c)\big) \ominus \big(I(a) \ominus I(b)\big) = I(b) \ominus I(c) .$$

□

As a consequence of Theorem 7 we obtain the following statements.

COROLLARY 3.7.1. *Let L be a D-set with 0. Then the quotient $L/I(0)$ is a poset with difference such that $I(0) \leq I(a)$, for every $I(a) \in L/I(0)$.*

COROLLARY 3.7.2. *Let L be a unital D-set with 0. Then $L/I(0)$ is a D-poset.*

4. Abelian extensions of D-sets

In [FW] the notion of abelian extensions from the theory of groups was generalized to extensions of effect algebras by abelian groups. It was shown there that unital cancellative PAMs are extensions of effect algebras by abelian groups. An analogy of the latter result in the category of D-sets is given in what follows. We start with some definitions.

A *short exact sequence* in the category of D-sets is the following diagram

$$0 \to H \xrightarrow{\tau} G \xrightarrow{\pi} K \to 0\,,$$

where H, G, K are D-sets with 0, τ, π are morphisms, τ is injective, π is surjective and $\tau(H) = \ker(\pi)$.

Definition 4.1. Let K be a D-set with 0, H an abelian group. An *extension of K by H* is any short exact sequence

$$0 \to H \to G \to K \to 0\,.$$

It is convenient to identify the extension with the middle element G.

If L is a DS with 0, $I(0)$ the ideal of invertible elements in L, then $I(0)$ is an abelian group, and the quotient $L/I(0)$ is a D-poset. Both abelian groups and D-posets belong to the category of DS with 0. The diagram

$$0 \to I(0) \xrightarrow{i} L \xrightarrow{\pi} L/I(0) \to 0$$

is a short exact sequence, so that it is the extension of the D-poset $L/I(0)$ by the abelian group $I(0)$.

If D is a D-set, A an abelian group, then $D \times A$, endowed with operation $\ominus$ defined coordinatewise, is a DS. The diagram

$$0 \to A \to D \times A \to D \to 0$$

is a short exact sequence, hence an extension of the D-poset D by the abelian group A.

Let L be a DS, $I(0)$ the ideal of invertible elements in L, $\pi\colon L \to D := L/I(0)$ the canonical mapping $\pi(a) = I(a)$. Let $\sigma\colon D \to L$ be any section of π with $\sigma\big(\pi(0)\big) = 0$. That is, $\sigma\big(\pi(a)\big) = a \oplus k_a$ with $k_a \in I(0)$, $k_0 = 0$. Set $\Delta(L) := \big\{(a,b) \in L \times L\colon a \ominus b \in L\big\}$. For $(a,b) \in \Delta(L)$ define $\beta(a,b)$ to be the (unique) element $k \in I(0)$ such that $\sigma\big(\pi(a \ominus b)\big) = \sigma\big(\pi(a)\big) \ominus \sigma\big(\pi(b)\big) \ominus k$.

Lemma 4.1. *The mapping $\beta\colon \Delta(L) \to I(0)$ has the following properties:*

($\beta 1$) $\beta(a,a) = 0$,

($\beta 2$) $\beta(a,b) = \beta(a, a \ominus b)$,

($\beta 3$) $\beta(a,b) + \beta(a \ominus b, c) = \beta(a,c) + \beta(a \ominus c, b)$.

Proof. $(\beta 1)$ $\sigma(\pi(a)) \ominus \sigma(\pi(a)) \ominus \beta(a,a) = \sigma(\pi(a \ominus a)) = \sigma(\pi(0)) = 0$.
$(\beta 2)$ We have

$$\begin{aligned}\sigma(\pi(b)) &= \sigma\Big(\pi\big(a \ominus (a \ominus b)\big)\Big) = \big[\sigma(\pi(a)) \ominus \sigma(\pi(a \ominus b))\big] \ominus \beta(a, a \ominus b) \\ &= \Big[\sigma(\pi(a)) \ominus \Big(\big(\sigma(\pi(a)) \ominus \sigma(\pi(b))\big) \ominus \beta(a,b)\Big)\Big] \ominus \beta(a, a \ominus b) \\ &= \sigma(\pi(b)) \ominus \big(\beta(a, a \ominus b) \ominus \beta(a,b)\big),\end{aligned}$$

which implies $\beta(a,b) = \beta(a, a \ominus b)$.
$(\beta 3)$ Calculate

$$\begin{aligned}\sigma\Big(\pi\big((a \ominus b) \ominus c\big)\Big) &= \big[\sigma(\pi(a \ominus b)) \ominus \sigma(\pi(c))\big] \ominus \beta(a \ominus b, c) \\ &= \Big(\big[\sigma(\pi(a)) \ominus \sigma(\pi(b)) \ominus \beta(a,b)\big] \ominus \sigma(\pi(c))\Big) \ominus \beta(a \ominus b, c) \\ &= \big[\big(\sigma(\pi(a)) \ominus \sigma(\pi(b))\big) \ominus \sigma(\pi(c))\big] \ominus \big[\beta(a,b) \oplus \beta(a \ominus b, c)\big].\end{aligned}$$

Analogously,

$$\sigma\Big(\pi\big((a \ominus c) \ominus b\big)\Big) = \big[\sigma(\pi(a)) \ominus \sigma(\pi(c)) \ominus (\pi(b))\big] \ominus \big[\beta(a,c) \oplus \beta(a \ominus c, b)\big].$$

Applying (d4) to both sides, we obtain $(\beta 3)$. □

Definition 4.2. Let L be a DS, A an abelian group. A mapping $\beta\colon \Delta(L) \to A$ satisfying conditions $(\beta 1)$, $(\beta 2)$ and $(\beta 3)$ will be called an *A-valued cocycle on L*.

Let L be a DS, A an abelian group. Let $g\colon L \to A$ be any mapping with $g(0) = 0$. It is easy to check that $\beta_g\colon \Delta(L) \to A$ defined by

$$\beta_g(a,b) := \big(g(a) \ominus g(b)\big) \ominus g(a \ominus b)$$

is an A-valued cocycle on L.

The set of all A-valued cocycles on L forms a group with the pointwise addition. We denote it by $\mathscr{B}(L,A)$. The set of cocycles of the form β_g is its subgroup, we denote it by $\mathscr{B}^2(L,A)$.

Definition 4.3. Let D be a DS with $I_D(0) = \{0\}$, and A an abelian group, $\beta \in \mathscr{B}(D,A)$. Then

$$D \times_\beta A := (D \times A, \ominus_\beta),$$

where

$$(a,x) \ominus_\beta (b,y) = \big(a \ominus b,\ x - y - \beta(a,b)\big), \qquad (a,b) \in \Delta(D).$$

THEOREM 4.2. *Let D be a DS with $I(0) = 0$, A an abelian group, β an A-valued cocycle on D. The $D \times_\beta A$ is an extension of D by A.*

Conversely, if $D = L/I_D(0)$, $A = I_D(0)$ for some DS L, then for any section $\sigma\colon D \to L$ with $\sigma(0) = 0$, L is isomorphic with $D \times_\beta A$, where β is the cocycle arising from σ.

Proof. Set $L := D \times_\beta A$. First we have to prove that L is a DS. We will check properties (d1), (d2) and (d4).

(d1) Owing to (β1),

$$(a,x) \ominus_\beta (a,x) = \big(a \ominus a,\ x - x - \beta(a,a)\big) = (0,0)\,.$$

(d2) Owing to (β2),

$$\begin{aligned}(a,x) \ominus_\beta \big((a,x) \ominus_\beta (b,y)\big) &= (a,x) \ominus_\beta \big(a \ominus b,\ x - y - \beta(a,b)\big)\\ &= \big(a \ominus (a \ominus b),\ x - \big(x - y - \beta(a,b)\big) - \beta(a, a \ominus b)\big) = (b,y)\,.\end{aligned}$$

(d4) Owing to (β3),

$$\begin{aligned}\big((a,x) \ominus_\beta (b,y)\big) \ominus_\beta (c,z) &= \big(a \ominus b, x - y - \beta(a,b)\big) \ominus_\beta (c,z)\\ &= \big((a \ominus b) \ominus c,\ x - y - \beta(a,b) - z - \beta(a \ominus b, c)\big)\\ &= \big((a \ominus c) \ominus b,\ x - z - y - \beta(a,c) - \beta(a \ominus c, b)\big)\\ &= \big((a,x) \ominus_\beta (c,z)\big) \ominus_\beta (b,y)\,.\end{aligned}$$

Since $I_D(0) = \{0\}$, the set of invertible elements in L is $I_L(0) = \big\{(0,x)\colon x \in A\big\}$, which is isomorphic with A. Moreover, since $(a,x) \sim_{I_L(0)} (b,y)$ iff $(a,x) \ominus_\beta (b,y) \in I_L(0)$, it follows that $(a,x) \sim_{I_L(0)} (b,y)$ iff $a = b$. Therefore the mapping $\phi\colon D \to L/I_L(0)$ defined by $\phi(a) = \big\{(a,x)\colon x \in A\big\}$ is an isomorphism. It follows that L is an extension of D by A.

Conversely, let L be a DS, $A = I_L(0) = I(0)$, $D = L/I(0)$. Let further $\pi\colon L \to D$ be the canonical mapping $a \mapsto I(a)$. Choose any section σ of π with $\sigma\big(I(0)\big) = 0$ and let β be the corresponding cocycle. The section σ gives rise to a mapping $g\colon L \to I(0)$ via $\sigma\big(I(a)\big) = a \oplus g(a)$. We have

$$\begin{aligned}\sigma\big(I(a \ominus b)\big) &= (a \ominus b) \oplus g(a \ominus b)\\ &= \sigma\big(I(a)\big) \ominus \sigma\big(I(b)\big) \ominus \beta(a,b)\\ &= \big(a \oplus g(a)\big) \ominus \big(b \oplus g(b)\big) \ominus \beta(a,b)\,,\end{aligned}$$

which entails $g(a \ominus b) = g(a) \ominus g(b) \ominus \beta(a,b)$.

Now define a mapping $\Phi\colon L \to D \times_\beta A$ by setting

$$\Phi(a) = \big(I(a), g(a)\big)\,.$$

It then follows, for $(a,b) \in \Delta(L)$, that

$$\begin{aligned}\Phi(a \ominus b) &= \big(I(a \ominus b),\, g(a \ominus b)\big) = \big(I(a) \ominus I(b), \big(g(a) \ominus g(b)\big) \ominus \beta(a,b)\big)\\ &= \big(I(a), g(a)\big) \ominus_\beta \big(I(b), g(b)\big) = \Phi(a) \ominus_\beta \Phi(b)\,.\end{aligned}$$

Clearly, $\Phi(0) = (0,0)$. If L happens to be unital, then $\Phi(1) = \big(I(1), g(1)\big)$, and for any $\Phi(a) = \big(I(a),\, g(a)\big)$, $\big(I(1 \ominus a),\, g(1 \ominus a)\big) = \big(I(1) \ominus I(a), \big(g(1) \ominus g(a)\big) \ominus \beta(1,a)\big) = \Phi(1) \ominus_\beta \Phi(a)$, hence $\Phi(1)$ is a unit in $D \times_\beta A$.

This proves that Φ is a morphism.

Now assume that $\Phi(a) \ominus_\beta \Phi(b)$ is defined in $D \times_\beta A$. Then

$$\begin{aligned}\Phi(a) \ominus_\beta \Phi(b) &= \big(I(a), g(a)\big) \ominus_\beta \big(I(b), g(b)\big)\\ &= \big(I(a) \ominus I(b), \big(g(a) \ominus g(b)\big) \ominus \beta(a,b)\big)\\ &= \big(I(a \ominus b),\, g(a \ominus b)\big)\,,\end{aligned}$$

which implies that $a \ominus b$ is defined in L. This proves that Φ is a faithful morphism.

It remains to prove that Φ is surjective. Choose an arbitrary $\big(I(a), x\big) \in D \times A$. We have $\Phi(a) = \big(I(a),\, g(a)\big)$. Then $a \oplus g(a) = a \oplus x = \oplus g(a) \oplus x$, hence $\big(I(a), x\big) = \Phi\big(\big(a \oplus g(a)\big) \ominus x\big)$. ◻

We will say that two extensions $D \times_{\beta'} A$ and $D \times_\beta A$ of D by A are *equivalent* if $\beta - \beta' \in \mathscr{B}^2(D,A)$.

THEOREM 4.3. *Two extensions $D \times_\beta A$, $D \times_{\beta'} A$ of a D-set D with a trivial set of invertible elements by an abelian group A are equivalent if and only if there is an isomorphism $\Phi\colon D \times_\beta A \to D \times_{\beta'} A$ defined by $\Phi(a,x) = \big(a, x - g(a)\big)$, where $g\colon D \to A$ is any mapping with $g(0) = 0$.*

P r o o f. Assume that $D \times_\beta A$ and $D \times_{\beta'} A$ are equivalent. Then

$$\begin{aligned}\Phi\big((a,x) \ominus_\beta (b,y)\big) &= \Phi\big(a \ominus b,\, x - y - \beta(a,b)\big)\\ &= \big(a \ominus b,\, x - y - \beta(a,b) - g(a \ominus b)\big)\\ &= \big(a \ominus b,\, x - y - \beta'(a,b) - g(a) + g(b) + g(a \ominus b) - g(a \ominus b)\big)\\ &= \big(a, x - g(a)\big) \ominus_{\beta'} \big(b, y - g(b)\big) = \Phi(a,x) \ominus_{\beta'} \Phi(b,y)\,.\end{aligned}$$

This shows that Φ is a morphism. In the same way we prove that the inverse mapping $\Phi^{-1}(a,x) = \big(a, x + g(a)\big)$ is a morphism.

Conversely, assume that $\Phi\colon D \times_\beta A \to D \times_{\beta'} A$, $\Phi(a,x) = \big(a, x + g(a)\big)$ is an isomorphism. Then

$$\begin{aligned}\Phi\big((a,x) \ominus_\beta \Phi(b,y)\big) &= \Phi\big(a \ominus b,\, x - y - \beta(a,b)\big)\\ &= \big(a \ominus b,\, x - y - \beta(a,b) - g(a \ominus b)\big)\\ &= \Phi(a,x) \ominus_{\beta'} \Phi(b,y) = \big(a, x - g(a)\big) \ominus_{\beta'} \big(b, y - g(b)\big)\\ &= \big(a \ominus b,\, x - g(a) - \big(y - g(b)\big) - \beta'(a,b)\big)\,,\end{aligned}$$

which entails $\beta(a,b)+g(a\ominus b)=\beta'(a,b)+g(a)-g(b)$, hence $\beta(a,b)-\beta'(a,b)\in\mathscr{B}^2(D,A)$. $\square$

5. Examples

EXAMPLE 5.1. The following example is inspired by the definition of erosion for computer graphics.

Let $\tau=\{[a,b]\colon a,b\in\mathbb{R}\}$. Let us define a partial operation $\ominus$ for $A,B\in\tau$ by

$$A\ominus B=\bigcap_{b\in B}A_b \quad \text{is defined iff} \quad \bigcap_{b\in B}A_b\neq\emptyset,$$

where $A_b=\{y\in\mathbb{R}\colon y=d-b, \text{ for } d\in A\}$.

LEMMA 5.1. *Let $A,B\in\tau$ and $A=[a,a_2]$, $B=[b,b_2]$. Then $A\ominus B\in\tau$ iff $b_2-b\leq a_2-a$.*

Proof. If $A,B\in\tau$ then there exist $x,y\in\mathbb{R}$, such that $x,y\geq 0$ and $A=[a,a+x]$, $B=[b,b+y]$. Since

$$\begin{aligned}&b_2-b\leq a_2-a,\\ &\text{iff}\quad b+y-b\leq a+x-a,\\ &\text{iff}\quad y\leq x,\end{aligned}$$

it is enough to prove $A\ominus B\in\tau$ iff $y\leq x$. From the definition of the operation $\ominus$ it follows that

$$\begin{aligned}A\ominus B&=[a,a_2]\ominus[b,b_2]\\ &=[a-b,a_2-b]\cap[a-b_2,a_2-b_2]\\ &=\big[(a-b),(a-b)+x\big]\cap\big[(a-b)-y,(a-b)+(x-y)\big]\\ &=[a-b,a_2-b_2].\end{aligned}$$

Since $x,y\geq 0$, we have the following possibilities

(1) $x=y\geq 0$;

(2) $x>y\geq 0$;

(3) $y>x\geq 0$;

(1) Let $x=y\geq 0$. Then

$$(a-b)-x\leq(a-b)+x$$

and, moreover,

$$\begin{aligned}A\ominus B&=\big[(a-b),(a-b)+x\big]\cap\big[(a-b)-x,(a-b)+(x-x)\big]\\ &=[a-b,a-b]\in\tau.\end{aligned}$$

(2) Let $x > y \geq 0$. Then

$$(a-b) - y \leq (a-b) < (a-b) + (x-y) \leq (a-b) + x\,.$$

From this it follows that

$$\begin{aligned} A \ominus B &= \big[(a-b),\, (a-b)+x\big] \cap \big[(a-b)-y,\, (a-b)+(x-y)\big] \\ &= \big[(a-b)\,,\, (a-b)+(x-y)\big] \in \tau\,. \end{aligned}$$

(3) Let $y > x \geq 0$. Then

$$(a-b) - y \leq (a-b) - (y-x) < (a-b) \leq (a-b) + x\,.$$

From this it follows that

$$A \ominus B = \big[(a-b),\, (a-b)+x\big] \cap \big[(a-b)-y,\, (a-b)-(y-x)\big] = \emptyset \notin \tau\,.$$

□

PROPOSITION 5.2. *The set τ with partial binary operation $\ominus$ a is D-set with zero element equal to $[0,0]$.*

Proof.

(1) Let $A \in \tau$. Then there exist $a, x \in \mathbb{R}$ and $x \geq 0$, such that $A = [a, a+x]$ and

$$A \ominus A = [a, a+x] \ominus [a, a+x] = [0,0] \in \tau\,.$$

(2) Let $A \ominus B \in \tau$ and let $A = [a, a+x]$, $B = [b, b+y]$, where $x, y \geq 0$. From Lemma 5.1. we have $y \leq x$ and moreover $A \ominus B = \big[(a-b), (a-b)+(x-y)\big]$. Since $x \geq x - y$, we get $A \ominus (A \ominus B) \in \tau$ and

$$\begin{aligned} A \ominus (A \ominus B) &= [a, a+x] \ominus \big[(a-b),\, (a-b)+(x-y)\big] \\ &= \big[a-(a-b),\, (a+x) - \big((a-b)+(x-y)\big)\big] \\ &= [b,\, b+y] = B\,. \end{aligned}$$

(3) Let $A \ominus B$, $B \ominus C \in \tau$. Let a, b, c, x, y, z be real numbers with $x, y, z \geq 0$. Let us suppose that

$$A = [a, a+x],\ B = [b, b+y],\ C = [c, c+z]\,.$$

From Lemma 5.1. it follows

$$x \geq y \geq 0 \qquad \text{and} \qquad y \geq z \geq 0\,.$$

Hence

$$x \geq z \geq 0$$

which implies

$$A \ominus C \in \tau\,.$$

From the definition of the operation $\ominus$ we get

$$\begin{aligned} A \ominus B &= \big[(a-b),\, (a-b)+(x-y)\big]\,, \\ A \ominus C &= \big[(a-c),\, (a-c)+(x-z)\big]\,, \\ B \ominus C &= \big[(b-c),\, (b-c)+(y-z)\big] \end{aligned}$$

and $x-z \geq x-y \geq 0$.

From Lemma 5.1 we have $(A\ominus C)\ominus(A\ominus B) \in \tau$ and

$$\begin{aligned} &(A\ominus C)\ominus(A\ominus B) \\ &= \big[(a-c)-(a-b),\, \big((a-c)+(x-z)\big)-\big((a-b)+(x-y)\big)\big] \\ &= \big[(b-c),\, (b-c)+(y-z)\big] = B\ominus C\,. \end{aligned}$$

$\square$

Let $\tau = \big\{[a,b]\colon a,b\in\mathbb{R},\ a\leq b\big\}$. Let us define a partial operation $\ominus$ such that $[a,a+p]\ominus[b,b+q] := \big[a-b,(a-b)+(p-q)\big]$ iff $0\leq q\leq p$ for any $a,b\in\mathbb{R}$. The set τ is DS. From Lemma 5.1 one can easily see that an element $[a,a+x]$ is invertible iff $x=0$, hence $I(0)=\big\{[a,a]\colon a\in\mathbb{R}\big\}$, and we may identify $I(0)$ with the group of real numbers $\mathbb{R}$. Lemma 5.1 also implies that $[a,a+x]\sim_{I(0)}[b,b+y]$ iff $x=y$. Hence we may identify equivalence classes $I\big([a,a+x]\big)\equiv[x]$, with nonnegative numbers $[x]\in\mathbb{R}^+$. Therefore τ can be considered as the direct product $\mathbb{R}^+\times\mathbb{R}$, the trivial extension of the D-set $\mathbb{R}^+$ by the group $\mathbb{R}$. All other extensions are encoded in the group of cocycles $\beta(x,y)$ via $\mathbb{R}^+\times_\beta\mathbb{R}$. In particular, any function $a\colon\mathbb{R}^+\to\mathbb{R}$ with $a(0)=0$ induces a cocycle, as can be seen from the following examples.

(1) Let $a(x)=kx$, for $k\in\mathbb{R}$. Then

$$\beta\big([x],[y]\big) = a(x)-a(y)-a(x-y) = kx-ky-k(x-y) = 0$$

and the corresponding section is

$$\sigma\big([x]\big) = \big[kx,(k+1)x\big]\,.$$

(2) Let $a(x)=kx^2$, where $k\in\mathbb{R}$. Then

$$\begin{aligned} \beta\big([x],[y]\big) &= kx^2-ky^2-k(x-y)^2 \\ &= k(x^2-y^2-x^2+2xy-y^2) = k(2xy-2y^2) \\ &= 2ky(x-y) \end{aligned}$$

and

$$\sigma\big([x]\big) = [kx^2,kx^2+x]\,.$$

Both the above cocycles belong to $\mathscr{B}^2$, and so they induce equivalent extensions.

EXAMPLE 5.2. Let $L = \mathbb{Z} - \{0\}$, where $\mathbb{Z}$ is the set of integers. Let the partial binary operation $\ominus$ be the division given on the real numbers. Then the couple $(L, \ominus)$ is a D-set with zero element 1, $I(0) = \{-1, 1\}$ and for any $a \in L$, $I(a) = \{-a, a\}$. The proof is given below.

(1) Let $a \in L$. Then
$$a \ominus a = \frac{a}{a} = 1 \in L\,.$$
From this it follows that for any $a \in L$, $a \ominus a \in L$ and $0_a = 1$ for any $a \in L$.

(2) Let $a \ominus b \in L$. Then
$$a \ominus (a \ominus b) = \frac{a}{\frac{a}{b}} = b \in L\,.$$
From this it follows that the second condition from the definition of D-set is fulfilled.

(3) Let $a \ominus b$, $b \ominus c \in L$. It means that a is divisible by b and b is divisible by c. We get that a is divisible by c, and so $a \ominus c \in L$. Moreover
$$(a \ominus b) \ominus (a \ominus c) = \frac{\frac{a}{b}}{\frac{a}{c}} = \frac{ac}{ba} = \frac{c}{b} = c \ominus b \in L\,,$$
i.e., we have proved that the couple $(L, \ominus)$ is a D-set.

Clearly, an element $a \in L$ is invertible iff $a \in \{-1, 1\}$. So $I(0) = \{-1, 1\} = \mathbb{Z}_2$, and equivalence classes with respect to $I(0)$ can be identified with positive integers $\mathbb{Z}^+ \setminus \{0\}$. Therefore we may put $L \equiv \mathbb{Z}^+ \setminus \{0\} \times \mathbb{Z}_2$, i.e., L is a trivial extension of $\mathbb{Z}^+ \setminus \{0\}$ by $\mathbb{Z}_2$. All other extensions can be found using cocycles. We will restrict the set L as follows. Let $L(12) = \{a \in L\colon 12 \ominus a \in L\}$. Then
$$L(12) = \{-1, 1, -2, 2, -3, 3, -4, 4, -6, 6, -12, 12\}\,.$$
Then 12 is a unit in $L(12)$. As before, $I(0) = \{-1, 1\}$ and the equivalence classes are
$$\pi(-1) = \pi(1) = \{-1, 1\} = I(0)\,,$$
$$\pi(-2) = \pi(2) = \{-2, 2\} = I(2) = I(-2) \ldots$$

Define $L(12)^+ = \{1, 2, 3, 4, 6, 12\}$. Then $L(12) \equiv L(12)^+ \times \mathbb{Z}_2$.

$\ominus$	$I(0)$	$I(2)$	$I(3)$	$I(4)$	$I(6)$	$I(12)$
$I(0)$	$I(0)$					
$I(2)$	$I(2)$	$I(0)$				
$I(3)$	$I(3)$		$I(0)$			
$I(4)$	$I(4)$	$I(2)$		$I(0)$		
$I(6)$	$I(6)$	$I(3)$	$I(2)$		$I(0)$	
$I(12)$	$I(12)$	$I(6)$	$I(4)$	$I(3)$	$I(2)$	$I(0)$

Let σ_i, for $i = 1, 2$ be a map from $\pi\big(L(12)\big)$ to $L(12)$. Let us denote $L_i(12) := \big\{\sigma_i\big([a]\big) \colon a \in L(12)\big\}$. Let

$$L_1(12) = \{1, -2, -3, 4, 6, -12\}$$

and

$$L_2(12) = \{1, -2, -3, -4, 6, 12\}.$$

Hence if $L_1(12)$ is DS, then $\beta_1(a, b) = 1$ whenever $a \ominus b \in L(12)$. This yields $L(12)^+ \times_{\beta_1} \mathbb{Z}_2 = L \times \mathbb{Z}_2$, the trivial extension.

Now we will analyze the map σ_2. For example

$$\begin{aligned} -2 = \sigma\big(I(2)\big) &= \sigma\big(I(4) \ominus I(2)\big) \\ &= \Big(\sigma\big(I(4)\big) \ominus \sigma\big(I(2)\big)\Big) \ominus \beta_2\big(I(4), I(2)\big) \\ &= \frac{\frac{-4}{-2}}{\beta\big(I(4), I(2)\big)} = \frac{2}{\beta\big(I(4), I(2)\big)}\,. \end{aligned}$$

Hence

$$-2 = \frac{2}{\beta\big(I(4), I(2)\big)}$$

and

$$\beta\big(I(4), I(2)\big) = -1\,.$$

The values of β_2 are in the next table.

β_2	$I(0)$	$I(2)$	$I(3)$	$I(4)$	$I(6)$	$I(12)$
$I(0)$	1					
$I(2)$	1	1				
$I(3)$	1		1			
$I(4)$	1	-1		1		
$I(6)$	1		-1		1	
$I(12)$	1	1	1	1	1	1

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