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RECIPROCITY BETWEEN PRESENCE AND CONTENT FUNCTIONS ON A GESTALT COMPOSITION SPACE

CHANTAL BUTEAU

ABSTRACT. This paper is devoted to a qualitative and quantitative study of topological spaces, the *gestalt composition spaces*, built on premotif collections of musical scores, and to the modelling of the motivic analysis of music. Through *shape types*, *gestalts*, and *premotives distance* we obtain a motivic hierarchy of a score. By reason of the non-Hausdorff and asymmetric properties of the topologies, we consider 'reciprocal' functions, *presence* and *content*, quantifying the geometric information of these topological spaces. This model is completed by visualizing these non-intuitive topologies through motivic evolution trees (MET), graphical representation of an overall spectrum of a score's motivic structure.

1. Introduction

Mathematics is commonly used, at different levels, to inquire into the motivic structure of a musical score. It is well-known that Forte [7] used set theory to model the structure of atonal music which was later adapted and extended to study motivic structures in music by musicologists Morris [21], Rahn [23], Lewin [11], and the others (e.g., [9]). While in this musical set theory motif equivalence is a straightforward relation, motif similarity is a concept that remains difficult to manage (e.g., [22]). These approaches succeed in grouping equivalent motives and give numerical values for similarity between 'some' pairs of motives: It does not, however, restructure nor organize, in a hierarchy, the collection of a score's motives.

We propose a topological approach for which concepts of equivalence and similarity 'determine' a global hierarchical structure, called a *motivic topology*,

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on a collection of a score's motives. These topologies are, however, 'asymmetric', and this forces us to define 'presence' and 'content' functions in order to extract the geometric (motivic) information of the topologies. With these two 'reciprocal' functions we build the score's *motivic evolution tree*, a graphical representation of an overall spectrum of its motivic structure.

These motivic topologies, modeling motivic analysis of music, are part of the *Mathematical Music Theory* (MaMuTh) [13], [16]. More precisely, they are a particular case of the *Local Composition Theory.* A software implementation of our topological approach is available in the module MeloRubette® of the software RUBATO® [26]. Results from applications of our model (through RUBATO®) to different scores, such as Schumann's *Träumerei* [19] and Webern's *Variation für Klavier op. 27/2* [2], support the methodology's validity.

Following Reti [24], [25] and Kopfermann [20][1], the main goal of motivic analysis is to recognize the semantics of motivic units within a score, i.e., to find out which sequences of tones are the germs in the evolution of the motivic and thematic content. Reti's approach does not impose the germinal motives from outside, but constructs the germs from the thorough analysis of all possible motif structures and relations within a given score. In order to model this analysis we propose a topological approach for which our space is the set of all possible germinal motives, called *premotives* in our theory.

We first (Section 2) build a mathematical system whose objects are associated with the instances of motivic analysis of music. Two structural levels are distinguished: *premotives* and *abstract premotives* which are connected together by the *shape type mapping.* An equivalence relation leading to the concept of *gestalt*, and a distance function are defined on the space of abstract premotives and are lifted to the premotives. We define neighborhoods of premotives (and of gestalts) which, under certain conditions, form an open base (Theorem 1) for the *motivic topology.* At last we introduce the motivic space of a score called a *gestalt composition space.*

Because of the 'non Hausdorff' nature (Proposition 2) of the motivic topological spaces, we cannot directly extract the geometric information of these spaces. Therefore we introduce (Section 3) the *presence* and *content functions* in order to 'quantify' the motivic information from the topologies. These two functions are closely related to each other: by stressing the asymmetry in the topology they are *reciprocal* to each other (Lemma 2). In the particular situation when we consider a 'very small' neighborhood radius (corresponding to intolerance for premotif similarity cognition), we observe this reciprocity through a simple change of bases transformation (Theorem 2) linking the two functions. At last this reciprocity is stressed by the fact that, when considering both functions

[1] See the final remark in Kopfermann's translation of Reti's work on Schumann's *Kinderszenen.*

evaluated at changing epsilons, they each generate a (real) vector space with the same dimension (Conjecture 1).

Finally (Section 4) we bring together these two reciprocal functions into the *weight function*. In order to extract the 'essential' motivic information of our topologies, we 'reduce' this weight function to the *qualitative function*. We visualize the image of the latter function through a *motivic evolution tree* (MET) (Definition 14), a concept related to the systematic variation of the neighborhood radius (premotif similarity parameter). Such a MET was intuitively introduced in [5], and this last section is to present the mathematical structure behind this concept.

2. Construction of motivic topologies

The purpose of this section is to elaborate the basis of the mathematical model of structures that can be associated with musical objects handled in motivic analysis. Our objects, called premotives, are local compositions [13] with a specific restriction on the onset values. We map the premotives into different spaces according to concepts of shape types, t-spaces and abstract premotives, and we give examples of shape types, such as COM-matrix and elastic types. Then we let a paradigmatic group [13] act on a t-space, and thereby we introduce the gestalt of a premotif. We then define a distance $d_{t,n}$ between premotives of the same (abstract) cardinality n, and under certain conditions, we can deduce from $d_{t,n}$ distance functions $\mathrm{gd}_{t,n}$ between the gestalts of these premotives. Premotives, shape types, paradigmatic groups and pseudo-metrics on premotives are bricks for the basic framework.

Then, sometimes, the distance between premotives can be 'controlled' by the distance between their superpremotives: this is the "Inheritance Property". This property, as we shall see for Euclidean metric on the Dia-space, is not automatically satisfied. Using the distance functions $\mathrm{gd}_{t,n}$, we define Epsilon-neighborhoods of premotives and demonstrate that, whenever the Inheritance Property is satisfied, the collection of all Epsilon-neighborhoods forms a base for a topology $\mathcal{T}_t$ on the set of all premotives MOT. As an important result, we deduce from $\mathcal{T}_t$, a topology $\mathcal{T}_{\mathrm{Ges},t}$ on the (quotient) space of gestalts GES. Finally, we investigate some properties of the Epsilon-neighborhood topology for GES and for $\mathrm{Ges}^\star$, the space of gestalts of a given score.

For more details of this construction see [4], and for a detailed example of this whole construction in the context of the American Set Theory, see [5]. Through the construction, we include some remarks and examples on music in order to help the reader to associate the mathematical model with music. However, for more detailed considerations and justifications we refer the reader to [4] (Chapter 5), [5], [16], and [19].

2.1 Premotives and abstract premotives

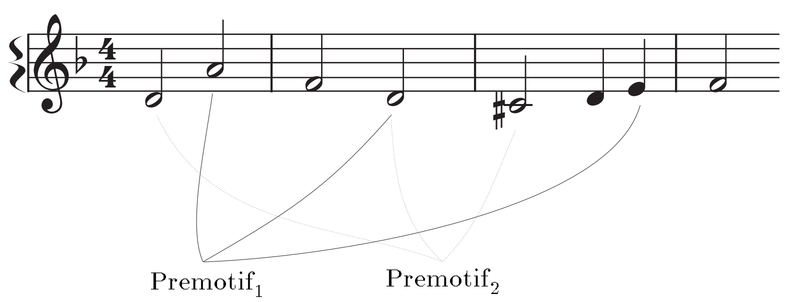
Given an arbitrary commutative ring R with unity we consider the *category* ${}_R\mathrm{Loc}$ [13] for which an object, called a *local composition*, is a couple (K, N), where K is a non-empty finite subset of a left R-module N, and a morphism $(K, N) \to (L, O)$ is a set map $f\colon K \to L$ which extends to an affine R-homomorphism $F\colon N \to O$. For our purposes the left R-modules are finite-dimensional vector spaces over $\mathbb{R}$ which are associated with the parametrization of tones in the classical score contexts. More specifically, using the tone parameters "onset" (O), "pitch" (P), "loudness" (L), "duration" (D), "glissando" (G), and "crescendo" (C), we consider the vector space $\mathbb{R}^{\{O,P,\dots\}} \cong \mathbb{R}^n$, where $\{O, P, \dots\}$ is a subset of $\{O, P, L, D, G, C\}$ containing at least O and P and $n \leq 6$ is the set cardinality of $\{O, P, \dots\}$.

Definition 1. A *premotif* is a local composition $\left(M, \mathbb{R}^{\{O,P,\dots\}}\right)$ such that the canonical projection $P_O\colon \mathbb{R}^{\{O,P,\dots\}} \to \mathbb{R}^{\{O\}}$ induces a bijective morphism $p_O\colon M \to P_O(M)$ of local compositions. A *subpremotif* (resp. a *superpremotif*) of a premotif $\left(M, \mathbb{R}^{\{O,P,\dots\}}\right)$ is a premotif $\left(M^*, \mathbb{R}^{\{O,P,\dots\}}\right)$ such that $M^* \subset M$ (resp. $M^* \supset M$). The *cardinality of a premotif* $\left(M, \mathbb{R}^{\{O,P,\dots\}}\right)$ is the set cardinality $|M|$ of M.

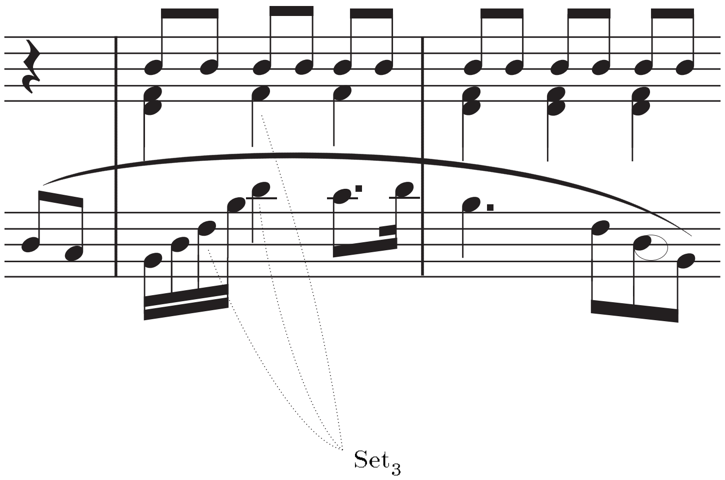
From now on we fix the space $\mathbb{R}^{\{O,P,\dots\}}$ and identify the premotif $\left(M, \mathbb{R}^{\{O,P,\dots\}}\right)$ with M. We set $\mathrm{MOT} = \{M\colon M \text{ is a premotif}\}$, and we have $\mathrm{MOT} = \coprod_n \mathrm{MOT}_n$ where $\mathrm{MOT}_n = \{M\colon M \text{ is a premotif such that } |M| = n\}$.

Example 1. Consider the space $\mathbb{R}^{\{O,P,D\}}$. We fix the parameters O, P, and D in a way which is standard in Mathematical Music Theory [13]: For the pitch values, we select the usual gauging with $C_4 = 0$, and the chromatic pitch set being parametrized by the integers, i.e., $C\sharp_4 = D\flat_4 = 1$, $D_4 = 2$, etc. Duration values are taken by the prescription that 1 in the O-coordinate corresponds to the literal mathematical value of $4/4$ duration. The first tone of a score is given onset value 0.

We consider the following sets of tones:



These two sets of tones form respectively premotives. However, the set containing the three tones



is not a premotif.

The two sets $\mathrm{Premotif}_1 = \{(\frac{1}{2}, 9, \frac{1}{2}), (\frac{3}{2}, 2, \frac{1}{2}), (\frac{11}{4}, 4, \frac{1}{4})\}$ and $\mathrm{Premotif}_2 = \{(0, 2, \frac{1}{2}),\ (2, 0, \frac{1}{2}), (\frac{3}{2}, 2, \frac{1}{2})\}$ are premotives but the set Set_3 is clearly not a premotif.

Intuitively, a premotif is a set of tones in which only one tone occurs at a given onset time, and in which tones are not necessarily consecutive in the given composition. Premotives are not necessarily germs of a composition, but only a priori candidates for carrying such a motivic meaning. The prefix "*pre*" was then introduced to make clear the difference between the formal structure of the mathematical theory and musically significant motivic germs, the "*real*" motives.

The idea of the next definition is that we want to compare premotives, not as such but via "simpler" images. In musicology this means that we compare some of their "relevant" shape properties.

Definition 2. A *shape type* t is a family $\{\Gamma_{t,n}\}_{n\in\mathbb{N}_+}$[2] of non-empty sets $\Gamma_{t,n}$ together with a mapping

$$\begin{aligned} t\colon\ \mathrm{MOT}(\mathbb{R}^{\{O,P,\dots\}}) &\longrightarrow \Gamma_t := \bigcup\nolimits_{n\in\mathbb{N}_+} \Gamma_{t,n}\,, \\ M &\longmapsto t(M) \end{aligned}$$

such that for each $n \in \mathbb{N}_+$ and $M \in \mathrm{MOT}_n$, we have $t(M) \in \Gamma_{t,n}$. And we have the following restriction map $t_n := t|_{\mathrm{MOT}_n}$:

$$\begin{aligned} t_n\colon\ \mathrm{MOT}_n &\longrightarrow\ \Gamma_{t,n}\,, \\ M &\longmapsto\ t_n(M)\,. \end{aligned}$$

[2] $\mathbb{N}_+ = \{1, 2, 3, \dots\}$.

The set Γ_t is called a t-space, and an element of Γ_t is called an *abstract premotif* (*of type* t). For an element $b \in \Gamma_t$ let $\operatorname{abcard}(b) := \min\{n \in \mathbb{N}\colon b \in \Gamma_{t,n}\}$ be the *abstract cardinality of* b, and for a premotif M we call $\operatorname{abcard}_t(M) := \operatorname{abcard}\big(t(M)\big)$ its *abstract cardinality* (*of type* t). We set $\Gamma_t|_k := \big\{b \in \Gamma_t\colon \operatorname{abcard}(b) = k\big\}$ and $\mathrm{MOT} := \coprod_k \mathrm{MOT}\,|_k$ where $\mathrm{MOT}\,|_k = \{M\colon M$ premotif s.t. $\operatorname{abcard}_t(M) = k\}$.

Here are some classical shape types in the MaMuTh:

1. *Rigid Shape Type:* We first consider the canonical projection $\mathrm{Pr}_{\{O,P\}}\colon \mathbb{R}^{\{O,P,\dots\}} \to \mathbb{R}^{\{O,P\}}$. For each $n \in \mathbb{N}$, we consider then the mapping
$$\begin{aligned} \mathrm{Rg}_n\colon\ \mathrm{MOT}_n &\longrightarrow \big(\mathbb{R}^{\{O,P\}}\big)^n, \\ M &\longmapsto \mathrm{Rg}_n(M) = (q_0, q_1, \dots, q_{n-1}), \end{aligned}$$
where $\mathrm{Rg}_n(M) = (q_0, q_1, \dots, q_{n-1})$ is the sequence of the elements $q_0 = (o_0, p_0), \dots, q_{n-1} = (o_{n-1}, p_{n-1})$ of $\mathrm{Pr}_{\{O,P\}}(M) \subset \mathbb{R}^{\{O,P\}}$ such that $o_0 < o_1 < \cdots < o_{n-1}$. The mapping $\mathrm{Rg} = \coprod_n \mathrm{Rg}_n$ together with the family $\big\{\Gamma_{\mathrm{Rg},n} := \big(\mathbb{R}^{\{O,P\}}\big)^n\big\}_{n\in\mathbb{N}}$ defines the *rigid type.*

   EXAMPLE 2. The abstract premotives of $\mathrm{Premotif}_1$ and $\mathrm{Premotif}_2$ of rigid type are respectively $\mathrm{Rg}_3(\mathrm{Premotif}_1) = \Big((\tfrac{1}{2}, 9), (\tfrac{3}{2}, 2), (\tfrac{11}{4}, 4)\Big)$ and $\mathrm{Rg}_3(\mathrm{Premotif}_2) = \Big((0, 2), (\tfrac{3}{2}, 2), (2, 0)\Big)$.

2. **COM**-*Matrix Shape Type:* Consider the mapping
$$\begin{aligned} \mathrm{COM}_n\colon\ \mathrm{Rg}_n(\mathrm{MOT}_n) &\longrightarrow \{-1, 0, 1\}^{n\times n}, \\ (q_0, q_1, \dots, q_{n-1}) &\longmapsto\ \mathrm{COM}_n\big((q_0, q_1, \dots, q_{n-1})\big), \end{aligned}$$
where $\mathrm{COM}_n\big((q_0, q_1, \dots, q_{n-1})\big) = (\delta_{i,j})_{i,j}$ for which (with notation $q_i = (o_i, p_i)$)
$$\delta_{i,j} = \begin{cases} 1 & \text{if } p_j - p_i > 0\,, \\ 0 & \text{if } p_j = p_i\,, \\ -1 & \text{if } p_j - p_i < 0\,. \end{cases}$$
This matrix $(\delta_{i,j})_{i,j}$ is antisymmetric and has zeros in its diagonal. For $n \geq 2$, we consider also
$$\begin{aligned} \mathrm{UTr}_n\colon \{-1, 0, 1\}^{n\times n} &\longrightarrow \{-1, 0, 1\}^{n(n-1)/2}, \\ (b_{i,j})_{i,j} &\longmapsto \big(b_{1,2}, b_{1,3}, \dots, b_{1,n}, b_{2,3}, \dots, b_{(n-1),n}\big) \end{aligned}$$
which means that the image $\mathrm{UTr}_n\big((b_{i,j})_{i,j}\big)$ is the the upper triangle values of the matrix $(b_{i,j})_{i,j}$.

Let Com_1 be defined as the unique mapping $\mathrm{MOT}_1 \to \{\infty\}$, and for $n \geq 2$ we define the mapping

$$\begin{aligned}\mathrm{Com}_n\colon \mathrm{MOT}_n &\longrightarrow \{-1,0,1\}^{n(n-1)/2},\\ M &\longmapsto \mathrm{Com}_n(M) := \mathrm{UTr}_n \circ \mathrm{COM}_n \circ \mathrm{Rg}_n(M).\end{aligned}$$

The mapping $\mathrm{Com} = \coprod\limits_n \mathrm{Com}_n$ together with the family $\Gamma_{\mathrm{Com}} = \{\infty\} \cup \left\{\{-1,0,1\}^{n(n-1)/2}\right\}_{n\geq 2}$ defines the COM-*matrix shape type*.

Example 3. The abstract premotives of $\mathrm{Premotif}_1$ and $\mathrm{Premotif}_2$ of COM-Matrix shape type are respectively $\mathrm{Com}_3(\mathrm{Premotif}_1) = (-1,-1,1)$ and $\mathrm{Com}_3(\mathrm{Premotif}_2) = (0,-1,-1)$.

3. *Diastematic Index Shape Type:* We first consider the mapping

$$\begin{aligned}\mathrm{UVec}_n\colon \{-1,0,1\}^{n\times n} &\longrightarrow \{-1,0,1\}^{(n-1)},\\ (b_{i,j})_{i,j} &\longmapsto \left(b_{1,2}, b_{2,3}, \ldots, b_{(n-1),n}\right).\end{aligned}$$

The *diastematic index shape type* Dia is defined through the mapping Dia_1, which is the unique mapping $\mathrm{MOT}_1 \to \{\infty\}$, and for $n \geq 2$, through the mappings

$$\begin{aligned}\mathrm{Dia}_n\colon \mathrm{MOT}_n &\longrightarrow \{-1,0,1\}^{(n-1)},\\ M &\longmapsto \mathrm{Com}_n(M) := \mathrm{UVec}_n \circ \mathrm{COM}_n \circ \mathrm{Rg}_n(M).\end{aligned}$$

Example 4. The abstract premotives of $\mathrm{Premotif}_1$ and $\mathrm{Premotif}_2$ of diastematic index type are respectively $\mathrm{Dia}_3(\mathrm{Premotif}_1) = (-1,1)$ and $\mathrm{Dia}_3(\mathrm{Premotif}_2) = (0,-1)$.

4. *Elastic Shape Type:* The mapping El_1 is defined to be the unique mapping $\mathrm{MOT}_1 \to \{\infty\}$. For $n \geq 2$, we define the mapping

$$\begin{aligned}\mathrm{El}_n\colon \mathrm{MOT}_n &\longrightarrow \mathbb{R}^{2(n-1)},\\ M &\longmapsto (\alpha_1, \ldots, \alpha_{n-1}, r_1, \ldots, r_{n-1}) \in \mathbb{R}^{2(n-1)},\end{aligned}$$

where, by considering $\mathrm{Rg}_n(M) = (q_0, q_1, \ldots, q_{n-1})$, the value α_i is the slope angle (radian) of $\overrightarrow{q_{(i-1)}q_i}$, $1 \leq i \leq n-1$, with respect to the O-axis, and $r_i = l_i/L(M)$ is the ratio of l_i, the Euclidean length of $\overline{q_{i-1}q_i}$ in the real vector space $\mathbb{R}^2$, over the length $L(M) := \sum\limits_{i=1}^{n-1} l_i$. The *elastic shape type* is then defined by $\mathrm{El} = \coprod\limits_n \mathrm{El}_n$ together with the family $\{\infty\} \cup \{\mathbb{R}^{2(n-1)}\}_{n\geq 2}$.

EXAMPLE 5. The abstract premotives of Premotif_1 and Premotif_2 of elastic type are respectively $\text{El}_3(\text{Premotif}_1) = (-1.429, 1.012, 0.688, 0.312)$ and $\text{El}_3(\text{Premotif}_2) = (0, -1.326, 0.421, 0.579)$.

In this work we meet t-spaces only of the form $\Gamma_t := \coprod_n \Gamma_{t,n}$, and therefore for any $n \in \mathbb{N}$ and for any $M \in \text{MOT}_n$, $\text{abcard}_t(M) = n$. The toroid shape type To [4] would be an example of such a space with $\Gamma_{\text{To}} \neq \coprod_n \Gamma_{\text{To},n}$.

2.2 Gestalts

So far some premotives are identified through the mapping t. In general this identification is not sufficient in the sense that we want to let a group act on abstract premotives (eventually on premotives) to yield a coarser classification of the premotives.

DEFINITION 3. Given a shape type t a *paradigmatic group* P for t is a group acting from the left on Γ_t:

$$\begin{aligned} P \times \Gamma_t &\longrightarrow \Gamma_t\,, \\ (p, b) &\longmapsto p \cdot b\,, \end{aligned}$$

and such that each space $\Gamma_{t,n}$ is P-invariant, i.e., $\forall p \in P$, $\forall b \in \Gamma_{t,n}$, $p \cdot b \in \Gamma_{t,n}$.

It is easy to show [4] that a paradigmatic group leaves invariant each $\Gamma_t|_k$, $k \in \mathbb{N}$.

DEFINITION 4. Given a shape type t and a paradigmatic group P the *gestalt of a premotif M (of type t)* is $\text{Ges}_t(M) := t^{-1}\big(P \cdot t(M)\big)$. For two premotives M and N if $\text{Ges}_t(M) = \text{Ges}_t(N)$, then we say that *$M$ and N have the same gestalt* and we write $M \sim_{\text{Ges}} N$, for which "$\sim_{\text{Ges}}$" is in fact an equivalence relation on MOT. We call the surjective mapping

$$\begin{aligned} \text{Ges}_t\colon\ \text{MOT} &\twoheadrightarrow \text{MOT} / \sim_{\text{Ges}}\,, \\ M &\mapsto \text{Ges}_t(M)\,, \end{aligned}$$

the *gestalt mapping (for type t and paradigmatic group P)*. We denote $\text{GES} := \text{Ges}_t(\text{MOT}) = \text{MOT} / \sim_{\text{Ges}}$ and $\text{GES}_k := \text{Ges}_t\big(\text{MOT}\,|_k\big)$. It is clear that $\text{GES} = \coprod_k \text{GES}_k$, and for each $k \in \mathbb{N}$ and each $G \in \text{GES}_k$ we call $\text{card}_t(G) = k$ the *(t)-cardinality of gestalt G*.

It can occur that a paradigmatic group also acts on premotives. We say that a shape type t is *P-equivariant* whenever the action of P on the space Γ_t is "induced" from an action on the premotives in the sense that for all $p \in P$ and for all $M \in \text{MOT}$ we have $p \cdot t(M) = t(p \cdot M)$. Usually, equivariant actions are defined when P is a subgroup of $\overrightarrow{\text{GL}}(\mathbb{R}^{\{O,P,\dots\}})$ (the group of affine $\mathbb{R}$-automorphisms of the fixed vector space $\mathbb{R}^{\{O,P,\dots\}}$), acting pointwise on the premotives $M \in \text{MOT}$, i.e., $p \cdot M := \{p \cdot x \colon x \in M\}$ for $p \in P$.

EXAMPLE 6. Let CP be the affine Klein group within $\overrightarrow{\mathrm{GL}}(\mathbb{R}^{\{O,P,\cdots\}})$. The group CP is defined as the subgroup of $\overrightarrow{\mathrm{GL}}(\mathbb{R}^{\{O,P,\cdots\}})$ generated by the subgroup of translations in onset and pitch directions, and by the linear Klein group $LCP = \langle U, K\rangle$ generated by the pitch inversion $U\colon U(o,p,\ldots) = (o,-p,\ldots)$, and the retrograde $K\colon K(o,p,\ldots) = (-o,p,\ldots)$. The group CP is a well-known transformation group (formed by transpositions, inversions, and retrograde) in European counterpoint. This is why it is called the *counterpoint group* in Mathematical Music Theory [13]. The group CP acts pointwise on $\mathrm{MOT}(\mathbb{R}^{\{O,P,\cdots\}})$, since translations, inversion, and retrograde transform premotives into premotives. In fact CP acts as a paradigmatic group for $t = \mathrm{Rg}$, Com, Dia and El, and t is then P-equivariant under the pointwise action of CP on $\mathrm{MOT}(\mathbb{R}^{\{O,P,\cdots\}})$.

For example, if we consider the COM-Matrix shape type and the CP group, the gestalt $\mathrm{Ges}_{\mathrm{Com}}(\mathrm{Premotif}_1)$ of $\mathrm{Premotif}_1$ is composed of all premotives $M \in \mathrm{MOT}$ such that $\mathrm{Com}_3(M) = (-1,-1,1)$, $(1,1,-1)$, $(-1,1,1)$, or $(1,-1,-1)$.

Since our goal is to construct a topology on the gestalt level we define a corresponding concept to "subpremotif" on these premotif classes.

DEFINITION 5. Given a shape type t and a paradigmatic group P we say that a gestalt G^* is a *small gestalt* of the gestalt G, denoted by $G^* \sqsubset G$, if there exist premotives $M^* \in G^*$ and $M \in G$ such that $M^* \subset M$.

If we have the property that for any triple M, M^*, and $M_1 \in \mathrm{MOT}$ such that $M^* \subset M$ and $M_1 \in \mathrm{Ges}_t(M)$, there exists a subpremotif M_1^* of M_1 such that $M_1^* \in \mathrm{Ges}_t(M^*)$, then we say that gestalts *behave well* (*for* t) and the small gestalt relation is a transitive relation.

2.3 Distance on premotives and on gestalts

Until now premotives as well as abstract premotives are regrouped by the equivalence relation "$\sim_{\mathrm{Ges}}$". The next step is to include the concept of *similarity* in our structure. Similarity is applied to premotives and then to gestalts (well-defined!).

DEFINITION 6. Given a shape type t let $d = (d_n)_{n\in\mathbb{N}_+}$ be a sequence of pseudo-metrics d_n where d_n is defined on the space $\Gamma_t|_n$ for each $n \in \mathbb{N}_+$. Let M and $N \in \mathrm{MOT}|_n$ where $n \in \mathbb{N}_+$. We set

$$d_{t,n}(M,N) := d_n\big(t(M), t(N)\big)$$

which defines a pseudo-metric on $\mathrm{MOT}|_n$. Let $d_t := (d_{t,n})_{n\in\mathbb{N}_+}$. Then we say that d_t (respectively d) is a pseudo-metric on MOT (resp. Γ_t) called the (t-)*distance on* MOT.

Moreover, for a given paradigmatic group P such that t is P-equivariant, if for all $n \in \mathbb{N}$, all $p \in P$ and all $a, b \in \Gamma_t|_n$, $d_n(p \cdot a, p \cdot b) = d_n(a, b)$, we say that P consists of *isometries with respect to* d *on space* Γ_t, and we have a pseudo-metric [4] on GES_k: for each $k \in \mathbb{N}$

$$d_{G,k}(G_1, G_2) := \inf_{p \in P} d_{t,n}(p \cdot M_1, M_2),$$

where $G_1, G_2 \in \mathrm{GES}_k$ and $M_1, M_2 \in \mathrm{MOT}\,|_k$ for which $M_1 \in G_1$ and $M_2 \in G_2$. For any two premotives $M_1, M_2 \in \mathrm{MOT}\,|_k$ we set the *gestalt distance between premotives* M_1 *and* M_2 as $\mathrm{gd}_{t,k}(M_1, M_2) := d_{G,k}\big(\mathrm{Ges}_t(M_1),\ \mathrm{Ges}_t(M_2)\big)$ (which is also a pseudo-metric on $\mathrm{MOT}\,|_k$ for any $k \in \mathbb{N}$). Then we say again that d_G (resp. gd_t) is a pseudo-metric on GES (resp. on MOT).

If there is no possible confusion we omit the abstract cardinality index n of d_n, $d_{t,n}$, $\mathrm{gd}_{t,n}$, and of $d_{G,n}$ in the notation.

Example 7. For two abstract premotives b_1 and b_2 in $\Gamma_{\mathrm{Rg},n} = \big(\mathbb{R}^{\{O,P\}}\big)^2 \overset{\sim}{\to} \mathbb{R}^{2n}$, we can for example use the Euclidean metric Ed_n on $\mathbb{R}^{2n}$. This defines a (Rg-)distance, called the Euclidean distance, on MOT. Similar for $\Gamma_{\mathrm{Com},n} = \{-1,0,1\}^{(n-1)n/2} \subset \mathbb{R}^{(n-1)n/2}$, $\Gamma_{\mathrm{Dia},n} = \{-1,0,1\}^{(n-1)} \subset \mathbb{R}^{(n-1)}$, and $\Gamma_{\mathrm{El},n} = \mathbb{R}^{2(n-1)}$, whenever $n \geq 2$, and for $n = 1$ we set $\mathrm{Ed}_1 = 0$. Similarly, given a shape type t and a $n \in \mathbb{N}$ we can define a pseudo-metric $\mathrm{REd}_n := \mathrm{Ed}_n / k(t,n)$, for which REd is called the relative Euclidean distance, on MOT_n where $k(t,n)$ is n if $t = \mathrm{Rg}$ or Com, is $n-1$ if $t = \mathrm{Dia}$, and is $2(n-1)$ if $t = \mathrm{El}$.

For example, the Com-relative Euclidean distance between $\mathrm{Premotif}_1$ and $\mathrm{Premotif}_2$ is $\mathrm{REd}_{\mathrm{Com}}(\mathrm{Premotif}_1, \mathrm{Premotif}_2) = \frac{\sqrt{5}}{3}$ and their gestalt distance is $g\,\mathrm{REd}_{\mathrm{Com}}(\mathrm{Premotif}_1,\ \mathrm{Premotif}_2) = \min\big\{\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}, 1, \frac{1}{3}\big\} = \frac{1}{3}$.

2.4 Motivic topologies

We note that until now we have restructured the set MOT of all premotives by means of an equivalence relation, identifying premotives with same abstract cardinality, and of a distance function on premotives again with same abstract cardinality. Basically the set MOT has been restructured on each of its layers $\mathrm{MOT}\,|_n$ (each $\mathrm{MOT}\,|_n$ is a pseudo-metric space!), but not as a whole. Here is now the step where premotives with different abstract cardinalities are linked:

Definition 7. Given a shape type t, a pseudo-metric d on Γ_t, and a paradigmatic group P of isometries with respect to d, and such that t is P-equivariant, let $\varepsilon \in \mathbb{R}_+ = \big\{x \in \mathbb{R} \colon x > 0\big\}$. Then

$$V_\varepsilon(M) := \big\{N \in \mathrm{MOT} \colon \exists N^* \subset N \text{ s.t. } \mathrm{gd}_t(M, N^*) < \varepsilon\big\}$$

is called the *ε-neighborhood of the premotif $M \in \mathrm{MOT}$ (with respect to t, P, and d).*

However such neighborhoods do not necessary form an open base for a topology on MOT since it is not true in general that the intersection of ε-neighborhoods is a union of such neighborhoods. We need then to impose a condition on all premotives 'living' together within an ε-neighborhood.

2.4.1 Inheritance property

Definition 8. Given a shape type t and a pseudo-metric d_t on MOT, if for any $M \in \mathrm{MOT}|_n$, $n \in \mathbb{N}_+$, any subpremotif $M^* \subset M$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any $N \in \mathrm{MOT}|_n$,

$$\begin{aligned} d_t(M,N) < \delta \Rightarrow \ \exists \text{ subpremotif } N^* \subset N \text{ s.t. } & \operatorname{abcard}_t(N^*) = \operatorname{abcard}_t(M^*) \\ & \text{and } d_t(M^*, N^*) < \varepsilon, \end{aligned}$$

then we say that *d_t is (t-)inherited.*

If a paradigmatic group P consists of isometries with respect to d and t is P-equivariant, then, if d_t is inherited, gd_t is also inherited [4].

Example 8. For the rigid, Com-Matrix, and elastic shape types the Euclidean and the relative Euclidean distances on MOT are inherited. However, the distances $\mathrm{Ed}_{\mathrm{Dia}}$ and $\mathrm{REd}_{\mathrm{Dia}}$ for the diastematic index shape type are not inherited.

Intuitively the inheritance property insures us that two similar premotives are similar *in their subpremotives.* For example, a metric on Γ_{Dia} cannot be Dia-inherited since Dia disregards the subpremotif information with respect to the vector's (Dia-abstract premotif) total information: What could we say, for example, of the (abstract) subpremotif composed with the first and last tone of the premotif, for which its Dia-abstract premotif is $(1,-1)$? That could be (1), (0) or even (-1)!

The inheritance property links premotives on different layers $\mathrm{MOT}|_n$, and it insures us also of a stability on the gestalt level:

Lemma 1. *Given a shape type t, a pseudo-metric d on Γ_t, and a paradigmatic group P of isometries with respect to d, and such that t is P-equivariant, if the pseudo-metric d_G is a metric on GES and d_t is inherited, then gestalts behave well for t.*

Proof. Let M, M_1, and $M^* \in \mathrm{MOT}$. Suppose that $M^* \subset M$ and $M_1 \in \mathrm{Ges}_t(M)$, hence $\mathrm{gd}_t(M_1, M) = 0$. Let $r := \min\{\mathrm{gd}_t(M_1', M^*)$ where $M_1' \subset M_1\}$, and suppose $r > 0$. Since d_t is inherited, then for $\varepsilon = r/2 > 0$, there exists $\delta > 0$ such that any premotif M_1 with $\mathrm{gd}_t(M, M_1) < \delta$ contains a subpremotif $M_1^* \subset M_1$ s.t. $\mathrm{gd}_t(M^*, M_1^*) < \varepsilon$. However $\mathrm{gd}_t(M, M_1) = 0 < \delta$

which implies that $\mathrm{gd}_t(M^*, M_1^*) < r$ for a $M_1^* \subset M_1$: contradiction. Therefore, $r = 0$ and M^* and M_1^* have same gestalt since d_G is a metric on GES. Hence, gestalts behave well for t. □

It is important to notice that the inheritance condition is sufficient to insure us that the ε-neighborhoods form a base for a topology on MOT.

2.4.2 Motivic spaces

PROPOSITION 1. *Given a shape type t, a pseudo-metric d on Γ_t, and a paradigmatic group P of isometries with respect to d, and such that t is P-equivariant, if d_t is inherited, then the collection $\{V_\varepsilon(M) \colon M \in \mathrm{MOT}, \varepsilon > 0\}$ of all ε-neighborhoods forms a base for a topology $\mathcal{T}_t$ on* MOT.

We call $(\mathrm{MOT}, \mathcal{T}_t)$ *a motivic space and $\mathcal{T}_t$ a motivic topology (with respect to t, P, and d) for* MOT.

Proof. It is sufficient to show that for all $O, M_1 \in \mathrm{MOT}$ and any $\varepsilon_1 > 0$ such that $O \in V_{\varepsilon_1}(M_1)$ there exists an $\varepsilon_3 > 0$ such that $V_{\varepsilon_3}(O) \subset V_{\varepsilon_1}(M_1)$. Since $O \in V_{\varepsilon_1}(M_1)$, there exists a premotif $O^* \subset O$ such that $\mathrm{gd}_t(O^*, M_1) < \varepsilon_1$. Using the inheritance property, given this $O^* \subset O$ and $\varepsilon = \varepsilon_1 - \mathrm{gd}_t(O^*, M_1) > 0$, there exists a $\delta > 0$ such that we have

$$\mathrm{gd}_t(O, N^*) < \delta \Rightarrow \exists \text{ subpremotif } N^{**} \subset N^* \text{ s.t. } \mathrm{gd}_t(N^{**}, O^*) < \varepsilon .$$

Let $\varepsilon_3 = \delta$. Let $N \in V_{\varepsilon_3}(O)$. This means that we have a subpremotif $N^* \subset N$ such that $\mathrm{gd}_t(O^*, N^*) < \varepsilon_3$. Therefore, we find a subpremotif $N^{**} \subset N^*$ as required by the above implication. But then:

$$\begin{aligned} \mathrm{gd}_t(N^{**}, M_1) &\leq \mathrm{gd}_t(N^{**}, O^*) + \mathrm{gd}_t(O^*, M_1) \\ &< \varepsilon + \mathrm{gd}_t(O^*, M_1) \\ &= \varepsilon_1 - \mathrm{gd}_t(O^*, M_1) + \mathrm{gd}_t(O^*, M_1) \\ &= \varepsilon_1 . \end{aligned}$$

Therefore, there exists $\delta > 0$ such that if, $N \in V_\delta(O)$, then $N \in V_{\varepsilon_1}(M_1)$ and hence $V_{\varepsilon_3}(O) \subset V_{\varepsilon_1}(M_1)$. □

Since the ε-neighborhoods V in MOT are stable with respect to gestalts, i.e., that all open sets in MOT are unions of gestalts, we also have a (quotient) topological structure on the gestalt level.

THEOREM 1. *Given a motivic topology $\mathcal{T}_t$ on* MOT *as described in Proposition 1, consider on* GES *the quotient topology $\mathcal{T}_{t,\mathrm{Ges}}$ relative to the gestalt mapping Ges_t and to $\mathcal{T}_t$. We suppose that d_G is a metric on* GES *and d_t is inherited. Then Ges_t is an open mapping. Moreover the collection of all sets*

$$U_\varepsilon(H) := \{G \in \mathrm{MOT} / \sim_{\mathrm{Ges}} \colon \exists G^* \sqsubset G \text{ s.t. } d_G(G^*, H) < \varepsilon\} ,$$

where $H \in \mathrm{GES}$ *and* $\varepsilon > 0$, *forms a base for* $\mathcal{T}_{t,\mathrm{Ges}}$. *We call* $(GES, \mathcal{T}_{t,\mathrm{Ges}})$ *a motivic gestalt space and* $\mathcal{T}_{t,\mathrm{Ges}}$ *a motivic gestalt topology* (*with respect to* t, P, *and* d) *for* GES.

P r o o f . We first observe that for a given $M' \in \mathrm{MOT}$ and $\varepsilon > 0$, by the definition of the relation "$\sqsubset$" and the U's, we have $\mathrm{Ges}_t(V_\varepsilon(M_1)) = U_\varepsilon(\mathrm{Ges}_t(M_1))$. We now claim that $\mathrm{Ges}_t^{-1}(\mathrm{Ges}_t(V_\varepsilon(M_1))) = V_\varepsilon(M_1)$ for all $\varepsilon > 0$ and $M_1 \in \mathrm{MOT}$. It is clear that $V_\varepsilon(M_1) \subset \mathrm{Ges}_t^{-1}(\mathrm{Ges}_t(V_\varepsilon(M_1)))$. For the other inclusion, suppose that $M_2 \in \mathrm{Ges}_t^{-1}(\mathrm{Ges}_t(V_\varepsilon(M_1)))$. Then we have $\mathrm{Ges}_t(M_2) = \mathrm{Ges}_t(M)$ where $M \in V_\varepsilon(M_1)$. By definition, this means that there exists $M^* \subset M$ such that $\mathrm{gd}_t(M^*, M_1) < \varepsilon$. Since gestalts behave well, there exists $M_2^* \subset M_2$ such that $\mathrm{Ges}_t(M_2^*) = \mathrm{Ges}_t(M^*)$, and $\mathrm{gd}_t(M_2^*, M_1) = \mathrm{gd}_t(M^*, M_1) < \varepsilon$. Hence, $M_2 \in V_\varepsilon(M_1)$. □

2.4.3 Motivic topology of a score

We have now a topological structure on MOT (and on GES), the set of all possible premotives. We want to model the motivic analysis of a score and if we follow Reti's [24] approach, we should exclusively deal with premotives taken in the score. Obviously we have (infinitely) too many premotives in MOT. And this motivates the last step of our topological construction.

Definition 9. Given a motivic space $(\mathrm{MOT}, \mathcal{T}_t)$, a *motivic composition space* $(\mathrm{MOT}^\star, \mathcal{T}_t^\star)$ (with respect to $(\mathrm{MOT}, \mathcal{T}_t)$) is a finite non-empty subset $\mathrm{MOT}^\star$ of MOT, satisfying the *Subpremotif Existence Axiom*: This means that for a given $n_{\min} \in \mathbb{N}_+$, every subpremotif $M^* \subset M$ of any premotif $M \in \mathrm{MOT}^\star$ with $|M^*| \geq n_{\min}$ is a premotif within $\mathrm{MOT}^\star$. The relative topology of $\mathcal{T}_t$ on $\mathrm{MOT}^\star$ is denoted by $\mathcal{T}_t^\star$.

A *gestalt composition space* $(\mathrm{Ges}^\star, \mathcal{T}_{t,\mathrm{Ges}}^\star)$ (with respect to a motivic composition space $(\mathrm{MOT}^\star, \mathcal{T}_t^\star)$) is the space $\mathrm{Ges}^\star := \mathrm{Ges}_t(\mathrm{MOT}^\star)$ together with the quotient topology $\mathcal{T}_{t,\mathrm{Ges}}^\star$ relative to the gestalt mapping Ges_t (restricted to $\mathrm{MOT}^\star$) and to $\mathcal{T}_t^\star$.

For a given score, one could think of constructing a motivic topology directly on $\mathrm{Ges}^\star$. However this would yield another structure since the d_G distance between gestalts is (and must be) defined on GES. Intuitively, the calculation of d_G for two gestalts *within* a given composition is determined by first taking the representatives (premotives within the composition) of each gestalt, by looking at their shapes, and then by comparing not only all these abstract premotives (from premotives within the score) but also all the a priori imaginable (abstract) premotives with same gestalt, as fully used in musicology when comparing sequences of tones together.

The following definition will be useful in the next section.

Definition 10. Given a gestalt composition space $(\mathrm{Ges}^\star, \mathcal{T}_t^\star)$ (with respect to $(\mathrm{MOT}^\star, \mathcal{T}_t^\star)$) and a gestalt $G \in \mathrm{Ges}^\star$, the *multiplicity of* G, denoted by $\mathrm{mult}_{t,P,\mathrm{MOT}^\star}(G)$ (or simply $\mathrm{mult}(G)$), is the set cardinality of $\{M \in \mathrm{MOT}^\star \colon M \in G\}$. We denote $\mathrm{Ges}_k^\star := \mathrm{Ges}_k \cap \mathrm{Ges}^\star$, and for each $G \in \mathrm{Ges}^\star$ and $\varepsilon > 0$,

$$U_\varepsilon^\star(G) := U_\varepsilon(G) \cap \mathrm{Ges}^\star .$$

2.5 First Properties of motivic topologies

We have a look at some properties of our motivic spaces, for which we omit the proofs [4]. For calculation reason in the next sections we choose to work on GES and on $\mathrm{Ges}^\star$. It is however clear that all the following can be easily 'translated' into MOT and $\mathrm{MOT}^\star$.

Proposition 2. *A motivic gestalt space* $(\mathrm{GES}, \mathcal{T}_{t,\mathrm{Ges}})$ *is a* T_0*-space and "almost* T_1*-space", in the senses that given two gestalts* $G, H \in \mathrm{GES}$ *such that* $G \not\sqsubset H$, *there exists an open neighborhood of* G *which does not contain* H.

Proposition 3. *Given a motivic gestalt space* $(\mathrm{GES}, \mathcal{T}_{t,\mathrm{Ges}})$, *and suppose that all translations in time are elements of the paradigmatic group* P, *then the topological space* $(\mathrm{GES}, \mathcal{T}_{t,\mathrm{Ges}})$ *is irreducible* [8], *i.e., every non-empty open set in* GES *is dense.*

Proposition 4. *Given a motivic gestalt space* $(\mathrm{GES}, \mathcal{T}_{t,\mathrm{Ges}})$, *then the following holds:*

1. $\overline{\{G\}} = \{H \in \mathrm{GES} \colon H \sqsubset G\}$;
2. $\overline{\{G\}} = G \iff G \in \mathrm{GES}_1$.

GES satisfies the first axiom of countability, and if GES_1 is composed of a finite collection of gestalts then it is clear that GES is compact.

We consider now a gestalt composition space $(\mathrm{Ges}^\star, \mathcal{T}_t^\star)$. The space $\mathrm{Ges}^\star$ is T_0 and "almost T_1", compact, and satisfies the second axiom of countability since $\mathrm{Ges}^\star$ is finite by hypothesis.

Proposition 5. *Given a gestalt composition space* $(\mathrm{Ges}^\star, \mathcal{T}_t^\star)$, *let the gestalt* $G \in \mathrm{Ges}^\star_{n_{\max}}$ *where* $n_{\max} = \max\{\mathrm{card}_t(G) \colon G \in \mathrm{Ges}^\star\}$. *Then* $\overline{\{G\}}^{\mathcal{T}_t^\star}$ *is an irreducible component of* $\mathrm{Ges}^\star$. *Moreover* $\mathrm{Ges}^\star$ *is sober* [8], *i.e., each irreducible component contains one and only one generic gestalt.*

3. Quantifying a gestalt composition space through Presence and Content

Given a score and a selection $\mathrm{MOT}^\star$ of premotives, we would like to extract the motivic information of its gestalt composition space $\mathrm{Ges}^\star$ (for t, P, and d) and this brings us to try to visualize $\mathrm{Ges}^\star$. As we saw in the previous section, the topological space $\mathrm{Ges}^\star$ is not an intuitive Haussdorff space. It is only "almost T_1" and therefore does not have a 'standard' representation. This motivates us to define functions ("Presence" and "Content" [19]) in order to extract some geometric information from $\mathrm{Ges}^\star$. The MeloRubette®, simulating a motivic analysis of a score in the software RUBATO® [26],[18], uses the Mazzola presence and content function [19] for quantifying the motivic topologies. Generalized *associated* presence and content functions, stressing the asymmetry in a motivic topology, are *reciprocal* to each other, and, when evaluated at the *changing epsilons*, generate vector spaces with same dimensions.

3.1 Presence and Content functions

We first try to extract the motivic information of a gestalt composition space. The topological property "almost T_1" (Proposition 2) of any $\mathrm{Ges}^\star$ means that given a gestalt G_1 and a second gestalt $G_2 \in \mathrm{Ges}^\star$ such that $G_1 \not\sqsubset G_2$, there exists an open neighborhood around G_1 that does not contain G_2. If $\mathrm{card}_t(G_1) > \mathrm{card}_t(G_2)$, then for all $\varepsilon > 0$, $U^\star_\varepsilon(G_1) \not\ni G_2$. In addition, G_1 and G_2 cannot always be separated by two disjoint open neighborhoods. These 'separation' facts stress an asymmetric topological property between gestalts, which one can also easily observe by looking at elements of the open base $\{U_\varepsilon(G)\}_{G\in\mathrm{Ges}^\star,\varepsilon>0}$. This motivates our construction of the two following functions in order to 'quantify' a gestalt composition space: First consider two gestalts G and $H \in \mathrm{Ges}^\star$ and a neighborhood radius $\varepsilon > 0$. If $H \in U^\star_\varepsilon(G)$, then one measures the *presence of gestalt G in gestalt H (or, with inversed roles: H being "contained" in G)* by the intensity number

$$\mathrm{Int}_{\varepsilon,G}(H) := \mathrm{card}\big\{H^* \sqsubset H:\ \mathrm{card}(H^*) = \mathrm{card}(G) \wedge H^* \in U^\star_\varepsilon(G)\big\} \cdot \mathrm{mult}(H).$$

Remark that the gestalt H is not necessarily in $\mathrm{Ges}^\star_{\mathrm{card}_t(G)}$ and that is why we consider small gestalts $H^* \sqsubset H$ in $\mathrm{Ges}^\star_{\mathrm{card}_t(G)}$ for which "$H^* \in U^\star_\varepsilon(G)$" means that $d_G(H^*, G) < \varepsilon$. Since the higher cardinality difference between G and H, the higher probability that small gestalts $H^* \sqsubset H$ have with G the distance $d_G(H^*, G) < \varepsilon$, we weight the intensity by $\frac{1}{2^{(\mathrm{card}_t(H)-\mathrm{card}_t(G))}}$. The presence and the content of gestalt G in the whole topological space $\mathrm{Ges}^\star$, at neighborhood radius $\varepsilon > 0$, is then defined by summing up its presence (resp. content) in every gestalt of $\mathrm{Ges}^\star$:

$$\mathrm{Presence}_{\mathrm{Maz}}(G,\varepsilon) := \sum_{H\in U_\varepsilon(G)} \frac{1}{2^{(\mathrm{card}_t(H)-\mathrm{card}_t(G))}} \cdot \mathrm{Int}_{\varepsilon,G}(H)\,,$$

$$\mathrm{Content}_{\mathrm{Maz}}(G,\varepsilon) := \sum_{G\in U_\varepsilon(H)} \frac{1}{2^{(\mathrm{card}_t(G)-\mathrm{card}_t(H))}} \cdot \mathrm{Int}_{\varepsilon,H}(G)\,.$$

These two functions are called *Mazzola presence and content functions.* Clearly, this quantification is not unique, and we give a generalized definition of these two functions: We consider a gestalt composition space $\mathrm{Ges}^\star$, i.e., we fix a shape type t, a paradigmatic group P, and a pseudo-metric d such that the hypothesis of Theorem 1 holds, together with a selection $\mathrm{MOT}^\star$ of premotives as described in Definition 9. Let $n_{\min} := \min\{\mathrm{card}_t(G)\colon G \in \mathrm{Ges}^\star\}$ and $n_{\max} := \max\{\mathrm{card}_t(G)\colon G \in \mathrm{Ges}^\star\}$. For $G, H \in \mathrm{Ges}^\star$ and $\varepsilon \in \mathbb{R}_+$, we define the *neighborhood relation function*:

$$\mathrm{NeiRel}_\varepsilon(G,H) := \begin{cases} 1 & \text{if } H \in U^\star_\varepsilon(G)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $f, g, g_{n_{\min},n_{\max}}\colon \mathbb{N}_+ \to \mathbb{R}_+$ and $h, l\colon \mathbb{N} \to \mathbb{R}_+$ be positive real functions. In the following we will see that the Mazzola presence and content functions are defined through $f(n) = n$ and $g, g_{n_{\min},n_{\max}}(n) = 1$ for $n \in \mathbb{N}_+$, and $l(n) = n$ and $h(n) = \frac{1}{2^n}$ for $n \in \mathbb{N}$. We define first

$$\begin{aligned} \mathrm{Status} : \mathrm{Ges}^\star &\longrightarrow \mathbb{R}_+\,, \\ G &\longmapsto \mathrm{Status}(G) := (g_{n_{\min},n_{\max}} \circ \mathrm{card}_t)(G) \cdot (g \circ \mathrm{mult})(G)\,. \end{aligned}$$

Status is a gauging function weighting the combinatorial preference of the gestalt cardinalities (with respect to $n_{\min}$ and $n_{\max}$) and certain multiplicities in the topological space. We define also

$$\begin{aligned} \mathrm{Evol}\colon \mathbb{R} &\longrightarrow \mathbb{R}_+\,, \\ \varepsilon &\longmapsto \mathrm{Evol}(\varepsilon)\,, \end{aligned}$$

where Evol is one of these forms:

$$\mathrm{Evol}(\varepsilon) := \begin{cases} \varepsilon^k, \\ (1+\varepsilon)^k \end{cases}$$

for a $k \in \mathbb{Z}$. In the Mazzola functions we have $\mathrm{Evol}(r) = 1$ for $r \in \mathbb{R}_+$. This function weights the cognitive tolerance for premotif similarity. It is essential, for example, when observing the image of presence and content functions on radius intervals (and not only at fixed neighborhood radii). We define also

$$\begin{aligned} \Delta\,\mathrm{card}_h\colon \mathrm{Ges}^\star \times \mathrm{Ges}^\star &\longrightarrow \mathbb{R}_+\,, \\ (G,H) &\longmapsto \Delta\,\mathrm{card}_h(G,H) := (h \circ \Delta\,\mathrm{card})(G,H)\,, \end{aligned}$$

where $\Delta\,\mathrm{card}(H,G) := \left|\mathrm{card}_t(G) - \mathrm{card}_t(H)\right|$. This function adjusts the combinatoric effect of the small gestalt relation with respect to the difference of the gestalt cardinalities: the higher the difference, the higher the probability that there is a small gestalt relation. Let

$$\begin{aligned} \#_l\colon \mathrm{Ges}^\star \times \mathrm{Ges}^\star \times \mathbb{R}_+ &\longrightarrow \mathbb{R}_+\,, \\ (G,H,\varepsilon) &\longmapsto \#_l(G,H,\varepsilon) := (l \circ \#)(G,H,\varepsilon)\,, \end{aligned}$$

where $\#(G,H,\varepsilon) := \mathrm{card}\{H^* \sqsubset H \,|\, \mathrm{card}(H^*) = \mathrm{card}(G) \wedge H^* \in U^\star_\varepsilon(G)\}$, and we set $\mathrm{rel}_\varepsilon(G,H) = \#_l(G,H,\varepsilon)\cdot \mathrm{NeiRel}_\varepsilon(G,H)$. Then we define

$$\begin{aligned} \mathrm{mult}_f\colon \mathrm{Ges}^\star &\longrightarrow \mathbb{R}_+\,, \\ G &\longmapsto \mathrm{mult}_f(G) := (f \circ \mathrm{mult})(G)\,. \end{aligned}$$

This function mult_f adjusts the weight of a gestalt with respect to its 'size as a point' in the topology. Finally let $\mathrm{coef}\colon \mathrm{Ges}^\star \to \mathbb{R}$ be a real function. In the Mazzola presence and content functions we have $\mathrm{coef}(G) = 1$ for $G \in \mathrm{Ges}^\star$. The function coef is introduced for an extern weighting on the gestalts, i.e., when introducing datas coming from an extern analysis to our topology.

Definition 11. A *presence function*, denoted by Presence, is a real function on a gestalt composition space:

$$\begin{aligned} \mathrm{Presence} : \mathrm{Ges}^\star \times \mathbb{R}_+ &\longrightarrow \mathbb{R}\,, \\ (G,\varepsilon) &\longmapsto \mathrm{Presence}(G,\varepsilon)\,, \end{aligned}$$

where $\mathrm{Presence}(G,\varepsilon) := \mathrm{Evol}(\varepsilon)\cdot \mathrm{Status}(G)\cdot \sum_{H\in \mathrm{Ges}^\star} \mathrm{coef}(H)\cdot \mathrm{pr}_{G,\varepsilon}(H)$, with

$$\mathrm{pr}_{G,\varepsilon}(H) := \mathrm{rel}_\varepsilon(G,H)\cdot \Delta\,\mathrm{card}_h(G,H)\cdot \mathrm{mult}_f(H)$$

for given functions Status, Evol, mult_f, $\Delta\,\mathrm{card}_h$, coef, and rel as described above. Similarly a *content function*, denoted by Content, is a real function on a gestalt composition space:

$$\begin{aligned} \mathrm{Content} : \mathrm{Ges}^\star \times \mathbb{R}_+ &\longrightarrow \mathbb{R}\,, \\ (G,\varepsilon) &\longmapsto \mathrm{Content}(G,\varepsilon)\,, \end{aligned}$$

where $\mathrm{Content}(G,\varepsilon) := \mathrm{Evol}(\varepsilon)\cdot \mathrm{Status}(G)\cdot \sum_{H\in \mathrm{Ges}^\star} \mathrm{coef}(H)\cdot \mathrm{ct}_{G,\varepsilon}(H)$, for which

$$\mathrm{ct}_{G,\varepsilon}(H) := \mathrm{rel}_\varepsilon(H,G)\cdot \Delta\,\mathrm{card}_h(H,G)\cdot \mathrm{mult}_f(H)$$

for given functions Status, Evol, mult_f, $\Delta\,\mathrm{card}_h$, coef, and rel as described above.

Whenever Presence and Content are defined through the same functions Status, Evol, mult_f, rel, $\Delta\,\mathrm{card}_h$, and coef, we say that Content *and* Presence *are associated with each other.*

3.2 Reciprocity between associated presence and content functions

Here are two results underlying the asymmetric character of a motivic topology expressed through associated presence and content functions. The first result is actually the exact statement about the asymmetry in $\mathrm{Ges}^\star$.

Lemma 2. *Given a gestalt composition space* $\mathrm{Ges}^\star$, *consider a presence function* Presence *and its associated content function* Content. *The Reciprocity between* Presence *and* Content *is expressed through*

$$\mathrm{mult}_f(H)\cdot \mathrm{ct}_{H,\varepsilon}(G) = \mathrm{pr}_{G,\varepsilon}(H)\cdot \mathrm{mult}_f(G)\,.$$

The function Content *can then be redefined as a 'reciprocal' function of* Presence *(and conversely!):*

$$\mathrm{Content}(G,\varepsilon) := \mathrm{Evol}(\varepsilon)\cdot \mathrm{Status}(G)\cdot \sum_{H\in\mathrm{Ges}^\star} \mathrm{coef}(H)\cdot\left(\frac{\mathrm{mult}_f(H)}{\mathrm{mult}_f(G)}\cdot\big(\mathrm{pr}_{H,\varepsilon}(G)\big)\right).$$

Proof. From the definition of associated presence and content functions we have $\mathrm{rel}_\varepsilon(G,H) = \frac{\mathrm{pr}_{G,\varepsilon}(H)}{\mathrm{mult}_f(H)\cdot\Delta\,\mathrm{card}_h(G,H)} = \frac{\mathrm{ct}_{H,\varepsilon}(G)}{\mathrm{mult}_f(G)\cdot\Delta\,\mathrm{card}_h(H,G)}$, where $\Delta\,\mathrm{card}_h(G,H) = \Delta\,\mathrm{card}_h(H,G)$. □

We should observe in the 'redefinition of Content' that in comparison with Presence the function Content contains also the function pr in the sum, but the latter, in addition to the factor $\mathrm{mult}_f(H)/\,\mathrm{mult}_f(G)$, has its gestalt variables interchanged. In consequence, the sum is evaluated by varying the index variable, a variable which is fixed in the Presence's sum.

For the next result we fix an $\varepsilon > 0$, and we can write the image of $\mathrm{Ges}^\star$ under the Presence function as

$$P_\varepsilon = \begin{pmatrix}\mathrm{Presence}(G_0,\varepsilon)\\ \vdots\\ \mathrm{Presence}(G_k,\varepsilon)\end{pmatrix} = \mathrm{Evol}(\varepsilon)\cdot S\cdot M_\varepsilon\cdot\Delta\,,$$

where S is the diagonal matrix with diagonal $\mathrm{Status}(G_0),\dots,\mathrm{Status}(G_k)$, $\Delta^T = \big(\mathrm{coef}(G_0),\dots,\mathrm{coef}(G_k)\big)$, $M_\varepsilon = \big(\mathrm{pr}_{G_i,\varepsilon}(G_j)\big)_{0\le i,j\le k}$, and $k+1 = \mathrm{card}(\mathrm{Ges}^\star)$.

Theorem 2. *Given a gestalt composition space* $\mathrm{Ges}^\star$, *consider a presence function* Presence *and its associated content function* Content. *Denote*

$$C_\varepsilon = \begin{pmatrix}\mathrm{Content}(G_0,\varepsilon)\\ \vdots\\ \mathrm{Content}(G_k,\varepsilon)\end{pmatrix}.$$

With the above notation, given an $\varepsilon > 0$, *if the matrix* M_ε *is invertible, then there is an invertible* $(k+1) \times (k+1)$*-matrix* X_ε *such that*

$$X_\varepsilon^T S^{-1} C_\varepsilon = X_\varepsilon S^{-1} P_\varepsilon .$$

In particular, if Status $= 1$, *then* $X_\varepsilon^T C_\varepsilon = X_\varepsilon P_\varepsilon$.

P r o o f. By Lemma 2 we can write C_ε as

$$C_\varepsilon^T = \mathrm{Evol}(\varepsilon) \cdot \Delta^T \cdot \mu \cdot M_\varepsilon \cdot \mu^{-1} \cdot S ,$$

where μ is the diagonal matrix with diagonal $\mathrm{mult}_f(G_0), \dots, \mathrm{mult}_f(G_k)$. Therefore, if $X_\varepsilon = \mu M_\varepsilon^{-1}$, we have

$$\begin{aligned}
\frac{1}{\mathrm{Evol}(\varepsilon)} S^{-1} C_\varepsilon &= {\mu^{-1}}^T M_\varepsilon^T \mu^T {\Delta^T}^T \\
&= \mu^{-1} M_\varepsilon^T \mu \Delta \\
&= \mu^{-1} M_\varepsilon^T \mu M_\varepsilon^{-1} M_\varepsilon \Delta \\
&= \mu^{-1} M_\varepsilon^T \mu M_\varepsilon^{-1} (M_\varepsilon \Delta) \\
&= \mu^{-1} M_\varepsilon^T \mu M_\varepsilon^{-1} \Big(\frac{1}{\mathrm{Evol}(\varepsilon)} S^{-1} P_\varepsilon \Big) \\
&= (M_\varepsilon \mu^{-1})^T \mu M_\varepsilon^{-1} \Big(\frac{1}{\mathrm{Evol}(\varepsilon)} S^{-1} P_\varepsilon \Big) \\
&= \big((\mu M_\varepsilon^{-1})^{-1} \big)^T \mu M_\varepsilon^{-1} \Big(\frac{1}{\mathrm{Evol}(\varepsilon)} S^{-1} P_\varepsilon \Big) \\
&= \big((\mu M_\varepsilon^{-1})^T \big)^{-1} (\mu M_\varepsilon^{-1}) \Big(\frac{1}{\mathrm{Evol}(\varepsilon)} S^{-1} P_\varepsilon \Big) \\
&= (X_\varepsilon^T)^{-1} X_\varepsilon \Big(\frac{1}{\mathrm{Evol}(\varepsilon)} S^{-1} P_\varepsilon \Big) ,
\end{aligned}$$

and this implies $X_\varepsilon^T S^{-1} C_\varepsilon = X_\varepsilon S^{-1} P_\varepsilon$. ☐

We are then interested in the invertibility of each $M_\varepsilon = \big(\mathrm{pr}_{G_i,\varepsilon}(G_j)\big)_{0 \le i,j \le k}$, $\varepsilon > 0$, and this brings us to have a closer look at the function NeiRel since that latter is a factor in the function pr. Recall that

$$\mathrm{NeiRel}_\varepsilon(G, H) := \begin{cases} 1 & \text{if } H \in U_\varepsilon^\star(G) , \\ 0 & \text{otherwise} . \end{cases}$$

This 'logical' function has properties for an axiomatic system behind premotif similarity cognition (recall that the "ε" value corresponds to the tolerance in premotif similarity cognition):

PROPOSITION 6. *Given a gestalt composition space* $\mathrm{Ges}^\star$ *let* $G, H \in \mathrm{Ges}^\star$ *and* $\varepsilon > 0$. *Then*

1. *If there exists an* $\varepsilon > 0$ *such that* $\mathrm{NeiRel}_\varepsilon(G,H) = \mathrm{NeiRel}_\varepsilon(H,G) = 1$, *then* $\mathrm{card}_t(G) = \mathrm{card}_t(H)$, *and if for all* $\varepsilon > 0$ $\mathrm{NeiRel}_\varepsilon(G,H) = \mathrm{NeiRel}_\varepsilon(H,G) = 1$, *then* $G = H$;
2. *If for all* $\varepsilon > 0$, $\mathrm{NeiRel}_\varepsilon(G,H) = 0$, *then* $\mathrm{card}_t(G) > \mathrm{card}_t(H)$. *If there exists an* $\varepsilon > 0$ *such that* $\mathrm{NeiRel}_\varepsilon(G,H) = 1$, *then* $\mathrm{card}_t(G) \leq \mathrm{card}_t(H)$. *If for all* $\varepsilon > 0$ $\mathrm{NeiRel}_\varepsilon(G,H) = 1$, *then* $G \sqsubset H$;
3. *If* $\mathrm{NeiRel}_\varepsilon(G,H) = 1$, *then for all* $\varepsilon' \geq \varepsilon > 0$ $\mathrm{NeiRel}_{\varepsilon'}(G,H) = 1$, *and if* $\mathrm{NeiRel}_\varepsilon(G,H) = 0$, *then for all* $0 < \varepsilon' \leq \varepsilon$ $\mathrm{NeiRel}_{\varepsilon'}(G,H) = 0$.
4. *If* $\mathrm{card}_t(G) \neq \mathrm{card}_t(H)$, *then*
$$\mathrm{NeiRel}_\varepsilon(G,H) = 1 \implies \mathrm{NeiRel}_\varepsilon(H,G) = 0\,.$$

The inverse implication to the above property 4 is however false:

$$\mathrm{NeiRel}_\varepsilon(G,H) = 0 \;\not\Longrightarrow\; \mathrm{NeiRel}_\varepsilon(H,G) = 1\,,$$

since $\mathrm{NeiRel}_\varepsilon(G,H) = 0$ means that $H \notin U^\star_\varepsilon(G)$, but it could happen that $H \in U^\star_{\varepsilon'}(G)$ for an $\varepsilon' > \varepsilon$, and in this case, if $\mathrm{card}(G) \neq \mathrm{card}(H)$, then we necessarily have $\mathrm{NeiRel}_\varepsilon(H,G) = 0$.

Most of all the above properties of NeiRel reveal how the matrices M_ε, $\varepsilon > 0$, look like: Given a gestalt composition space $\mathrm{Ges}^\star$ (with maximal resp. minimal abstract cardinality $n'_{\max}$ resp. $n'_{\min}$) and for each $\varepsilon > 0$, the matrix M_ε has the shape

$$M_\varepsilon = \begin{pmatrix} B^{\varepsilon,0} & * & * & * \\ 0 & B^{\varepsilon,1} & * & * \\ \vdots & 0 & \ddots & * \\ 0 & \dots & 0 & B^{\varepsilon,n} \end{pmatrix}$$

with zeros below the diagonal of block matrices $B^{\varepsilon,i}$, where $n = n'_{\max} - n'_{\min}$. By the above proposition, each block matrix $(B^{\varepsilon,k})_{i,j} = \big(\mathrm{pr}_{G_i,\varepsilon}(G_j)\big)_{0 \leq i,j \leq p_k}$, where $p_k =$ set cardinality of $\mathrm{Ges}^\star_{k+1}$, is 'almost symmetric' in the sense that ${B^{\varepsilon,k}}_{i,i} > 0$, and if ${B^{\varepsilon,k}}_{i,j} = 0$ then ${B^{\varepsilon,k}}_{j,i} = 0$, and if ${B^{\varepsilon,k}}_{i,j} \neq 0$ then ${B^{\varepsilon,k}}_{j,i} \neq 0$. The matrix M_ε is therefore invertible if and only if each block matrix $({B^{\varepsilon,k}}_{i,j})_{0 \leq i,j \leq p_k}$ is invertible and this is not always the case. However, for each gestalt composition space there exists an $\varepsilon^* > 0$ such that M_{ε^*} is invertible: since $\mathrm{Ges}^\star$ is finite, for all $k \in \{n'_{\min}, \dots, n'_{\max}\}$, there exists an $\varepsilon^* > 0$ such that for all $G_1 \neq G_2 \in \mathrm{Ges}^\star_k$, $\mathrm{NeiRel}_{\varepsilon^*}(G_1, G_2) = 0$. Each block matrix $({B^{\varepsilon^*,k}}_{i,j})_{0 \leq i,j \leq p_k}$ is therefore diagonal and M_{ε^*} is invertible.

At this small radius $\varepsilon^* > 0$, the ε-neighborhood of a gestalt G contains only G and all gestalts for which G is a small gestalt. When considering presence and content at this radius, this means that no variation of a premotif is allowed in order to 'group' premotives: only imitation and subpremotif relations are tolerated. We can conclude that the invertibility of a matrix M_ε could be associated with the intolerance for premotif similarity cognition.

Theorem 2 could then state that, in the case of premotif similarity cognitive intolerance, associated Presence *and* Content *can be reduced to each other by transformation on a combinatorial level.*

3.3 Vector spaces: Reciprocal functions — same dimensions

We have expressed (and quantified) the asymmetric property of motivic topologies through associated presence and content functions, functions that are then reciprocal from each other. As a consequence of their reciprocity, the two vector spaces, one generated by the 'ε_i*-evaluated presence functions*' and the other by the 'ε_i*-evaluated content functions*', have the same dimensions.

Consider a gestalt composition space $\mathrm{Ges}^\star$. Recall that $\mathrm{Ges}^\star$ is finite, and that neighborhoods of gestalts stay constant for 'more or less small' intervals. That is to say that there is a finite sequence of neighborhood radii ε_i where neighborhoods change: We say that the radius ε is a *changing epsilon* (*for* $\mathrm{Ges}^\star$) if there exist gestalts $G, H \in \mathrm{Ges}^\star$ such that $\mathrm{NeiRel}_\varepsilon(G,H) < \mathrm{NeiRel}_{\varepsilon'}(G,H)$ for any $\varepsilon' > \varepsilon$. Let $\mathrm{seq} = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n\}$ be the set of all n changing epsilons, $\varepsilon_i < \varepsilon_{i+1}$ for all $0 \le i \le n-2$ and setting $\varepsilon_n := \varepsilon_{n-1} + 1$. We add ε_n in order to have a complete representation of the neighborhoods' evolution. We observe that for all $G \in \mathrm{Ges}^\star$, $H \in U^\star_{\varepsilon_0}(G)$ is equivalent to $H \sqsupset G$, and for all gestalts $H, G \in \mathrm{Ges}^\star$, $H \in U^\star_{\varepsilon_n}(G)$ or $G \in U^\star_{\varepsilon_n}(H)$.

To maintain the finite (topological) character within the presence and content functions, we make the hypothesis that the function Evol is locally constant, i.e., constant on the intervals $(0, \varepsilon_0]$, $(\varepsilon_{i-1}, \varepsilon_i]$ for $i = 1, \ldots, n$, and $(\varepsilon_n, \varepsilon_n + 1]$. Recall that the factor function Evol was introduced in order to weight the presence and the content of a gestalt with respect to the considered radius, the latter corresponding to the cognitive tolerance to link two similar premotives (or gestalts). The above hypothesis on Evol means that we take into account only those epsilons where changes appear in the neighborhoods of gestalts. In other words, it means that the presence (and content) of a gestalt should stay constant until new gestalts appear in some neighborhoods.

We next consider the collection of ε_i*-evaluated presence functions* p_i:

$$\{p_i\colon \mathrm{Ges}^\star \to \mathbb{R}\}_{i \in \{0,1,\ldots,n\}},$$

where for each $\varepsilon_i \in \mathrm{seq}$, $p_i(G) := \mathrm{Presence}(G, \varepsilon_i)$ for all $G \in \mathrm{Ges}^\star$. Similarly, given a function Content, we have the collection of ε_i*-evaluated content functions* c_i

$$\{c_i\colon \mathrm{Ges}^\star \to \mathbb{R}\}_{i \in \{0,1,\ldots,n\}},$$

where for each $\varepsilon_i \in \mathrm{Seq}$, $c_i(G) := \mathrm{Content}(G, \varepsilon_i)$ for all $G \in \mathrm{Ges}^\star$.

Our calculations for different scores suggest that this following conjecture holds.

CONJECTURE 1. Given a gestalt composition space $\mathrm{Ges}^\star$ with changing epsilons $\varepsilon_0, \ldots, \varepsilon_{n-1}$ plus $\varepsilon_n = \varepsilon_{n-1} + 1$, let Presence be a presence function with its associated content function Content in which the factor function Evol is locally constant as described above, and coef $= 1$. Let A be an open set in $\mathrm{Ges}^\star$ and denote $Ap_i = p_i|_A$ and $Ac_i = c_i|_A$ where p_i and c_i are respectively the ε_i-evaluated Presence and Content functions. Then

$$\dim\big(\mathbb{R}(Ap_0, \ldots, Ap_n)\big) = \dim\big(\mathbb{R}(Ac_0, \ldots, Ac_n)\big)\,.$$

In particular it holds for $A = \mathrm{Ges}^\star$.

EXAMPLE 9. Consider the gestalt composition space of the eight-tone main theme of Bach's *"Kunst der Fuge"*, for which the motivic topology is defined by the COM-matrix shape type, the paradigmatic group $P = \{\mathrm{id}\}$, and the relative Euclidean metric REd on Γ_{Com} (see next section for details on the motivic topology). There are 119 gestalts and 107 changing epsilons. If we consider the presence and content functions $\mathrm{Presence}_{\mathrm{Maz}}$ and $\mathrm{Content}_{\mathrm{Maz}}$ we obtain

$$\mathrm{Dim}\big(\mathbb{R}(p_0, \ldots, p_{107})\big) = \mathrm{Dim}\big(\mathbb{R}(c_0, \ldots, c_{107})\big) = 96\,,$$

and for $A = U_{\varepsilon_0}(\text{Gestalt 4–14})$ (see Figure 3 for a representation of gestalt '4–14'),

$$\mathrm{Dim}\big(\mathbb{R}(Ap_0, \ldots, Ap_{107})\big) = \mathrm{Dim}\big(\mathbb{R}(Ac_0, \ldots, Ac_{107})\big) = 18\,.$$

We observe that the ε_i-evaluated presence functions $p_0, p_1, \ldots$, and p_n describe all together a whole picture of motivic structure in a score since they model each level of motivic similarity cognition. We can then interpret the dimension of such a vector space $\mathbb{R}(p_0, \ldots, p_n)$ as a measure of the motivic richness of a score. With a 'high' dimension this would mean that 'many' gestalts generate the score. Conjecture 1 says that both functions, Presence and Content, i.e., when considering similarity as 'being almost contained in' or as 'containing almost', express the same level of richness of motivic structure.

4. Construction of motivic evolution trees

This section gives the mathematical construction of a *motivic evolution tree* (MET), a concept that is related to the systematic variation of the similarity parameter (neighborhood radius) of motivic analysis (gestalt motivic space), and that was intuitively introduced in [5].

We combine associated presence and content functions together as a weight function in order to obtain a 'global' topological information, and then by defining the Qual function we obtain a simpler 'global' topological information: *which gestalts are the heaviest?* The Qual function could be seen as a "coarse" weight function for which an essential motivic information is in the prominent position. Then it can be 'visualized' through the intuitive representation, a motivic evolution tree (MET), of the Qual function. As an application of these METs we addressed [5] the still debated question on the length of Bach's *Kunst der Fuge* main theme.

4.1 Weight function

In the last section we have built two functions, Presence and (associated) Content, which are reciprocal from each other, in the sense that they quantify an 'opposite' motivic information from $\mathrm{Ges}^\star$. Now we combine these two 'partial' quantifications into a global one through a weight function.

DEFINITION 12. Given a gestalt composition space $\mathrm{Ges}^\star$, a *weight function on* $\mathrm{Ges}^\star$ is a real function

$$\begin{aligned} \mathrm{Weight} : \mathrm{Ges}^\star \times \mathbb{R}_+ &\longrightarrow \mathbb{R}\,, \\ (G,\varepsilon) &\longmapsto \mathrm{Weight}(G,\varepsilon)\,, \end{aligned}$$

where $\mathrm{Weight}(G,\varepsilon)$ is of one of these forms:

$$\mathrm{Weight}(G,\varepsilon) := \begin{cases} (\mathrm{Presence}^k \cdot \mathrm{Content}^l)(G,\varepsilon) & \text{for a } k \text{ and } l \in \mathbb{Z}\,, \\ (\mathrm{Presence}^k + \mathrm{Content}^l)(G,\varepsilon) & \text{for a } k \text{ and } l \in \mathbb{Z}\,, \end{cases}$$

and where Presence and Content are associated presence and content functions.

EXAMPLE 10. The Mazzola weight function is

$$\mathrm{Weight}_{\mathrm{Maz}} := \mathrm{Presence}_{\mathrm{Maz}} \cdot \mathrm{Content}_{\mathrm{Maz}}\,.$$

4.1.1 Mazzola weight function on Bach's Kunst der Fuge

This example was programmed in Mathematica®. The set $\mathrm{MOT}^\star$ of premotives to be analysed contains all the possible 2- to 8-tone premotives in the 8-tone main theme of *Kunst der Fuge* (see Example 1). We choose the COM-Matrix shape type, the paradigmatic group is $P = \{\text{identity}\}$, and the distance $d_{t,n} = \sqrt{2}\cdot\mathrm{REd}_n$. The minimal gestalt distance between two premotives of different gestalts is 0.202, and the maximal one is 1.732.

In our first attempt to visualize the motivic topology (and hierarchy) of the 8-tone main theme of *Kunst der Fuge*, we consider graphs of $\mathrm{Weight}_{\mathrm{Maz}}$ interpolations at fixed neighborhood radii: we order the premotives according to their cardinality, and for fixed cardinality, we order them lexicographically with respect to the temporal order of their tones. We then arrange the premotives on a grid where the first coordinate indicates the premotives' cardinalities, whereas the second coordinate indicates their lexicographic order. Of course, this premotif order is a strictly graphical device and does not influence the topology nor the premotif weights. Finally $\mathrm{Weight}_{\mathrm{Maz}}$, which is defined on the discrete set of premotives, is evaluated at a fixed neighborhood radius and then is interpolated to a continuous surface in order to make evident the overall weight distribution.

For example, when fixing the neighborhood radius at 0.85 we obtain the weight interpolation surface as shown in Figure 1.

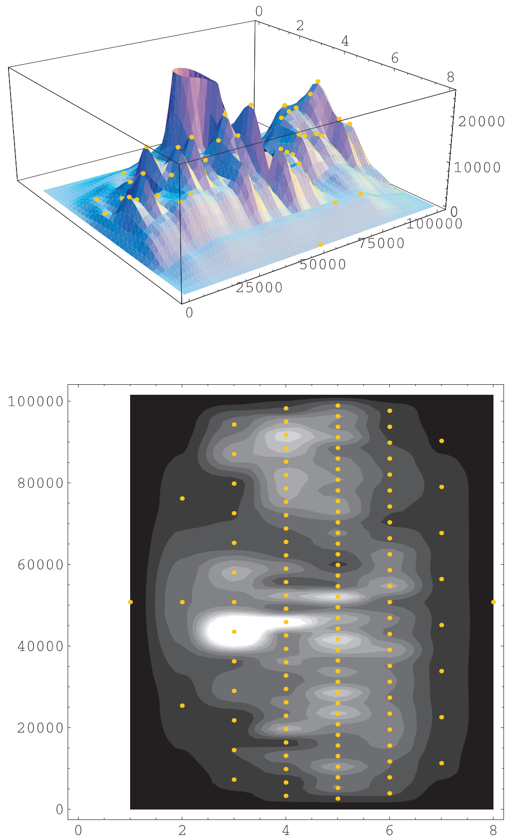


FIGURE 1. This figure shows the interpolation of the function $\mathrm{Weight}_{\mathrm{Maz}}$ evaluated at neighborhood radius 0.85 and at the (discrete) set of premotives in the 8-tone main theme of Bach's *Kunst der Fuge*. For both graphics premotives are arranged in a grid, with horizontal axis for cardinality, and vertical axis for the lexicographic order (the lexicographic scale numbers are not relevant). The graphic at the top is the graph of the interpolation for which peaks show premotives with high weights. In the bottom graphic, the premotives are represented as a set of small white points and the surface grey levels indicate relative weights: High weights are shown in light grey, low weights appear in dark grey.

If we come back to the initial goal of this motivic model we would like to say that the 'significant premotives' ('the' motives in musicology) are those represented by peaks in Figure 1. More precisely, these two above graphics give us a picture of the motivic structure (with respect to $t = \mathrm{Com}$, $P = \{\text{identity}\}$, and $d_{\mathrm{Com}} = \sqrt{2} \cdot \mathrm{REd}$) *at neighborhood radius 0.85* of the 8-tone main theme of *Kunst der Fuge*. This brings then the problem of choosing an 'adequate' neighborhood radius before determining the 'significant premotives'. But with which criteria? And would not a neighborhood radius interval be more adequate? These non-trivial questions are in the domain of cognitive science and cannot be answered in this paper. Therefore, we cannot use this visualization for extracting the motivic information from our motivic topologies. However, if we do not pick a neighborhood radius, while considering all neighborhood radii, then we reach our goal and have an *overall* motivic picture of a score.

4.2 Qual **function**

We would like to describe the entire 'spectrum' of 'simple' weight landscapes as it is produced when one moves along the axis of neighborhood radii. This motivates the construction of the qualitative function (Qual) of Weight and of its intuitive visualization through a *motivic evolution tree.*

Let A be a finite set of real numbers. Consider the function

$$\begin{aligned} \mathrm{Order} : \mathbb{R}^A &\longrightarrow \mathbb{N}^A, \\ (x_a)_{a \in A} &\longmapsto (y_a)_{a \in A}, \end{aligned}$$

where y_a is the rank of x_a in the ordered (with respect to the relation $\leq$) sequence of the x_a. If $x_a = x_b$, then $y_a = y_b$.

Fix $t \in [0,1]$. Consider now the function

$$\begin{aligned} \mathrm{TrOr}_t \colon \mathbb{N}^A &\longrightarrow [0,1]^A, \\ (n_a)_{a \in A} &\longmapsto \left(\mathrm{tror}_t(n_a)\right)_{a \in A}, \end{aligned}$$

where

$$\mathrm{tror}_t = \begin{cases} 1/n_a & \text{if } 1/n_a \geq t\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We can now combine these two definitions to the weight function:

Definition 13. Given a gestalt composition space $\mathrm{Ges}^\star$ and a Weight function, consider the function $\mathrm{Qual}_{\mathrm{Weight},\mathrm{Ges}^\star} = \mathrm{Qual}$:

$$\begin{aligned} \mathrm{Qual} \colon \mathbb{R}_+ \times [0,1] &\longrightarrow [0,1]^{\mathrm{Ges}^\star}, \\ (\varepsilon, t) &\longmapsto \mathrm{TrOr}_t \circ \mathrm{Order}\big(\left(\mathrm{Weight}(\varepsilon, G_H)\right)_{H \in \mathrm{Ges}^\star}\big)\,. \end{aligned}$$

This function is called the *qualitative function* (*of* Weight *to* $\mathrm{Ges}^\star$). We call the inversed parameter $\mathrm{tol} := 1/t$ the *tolerance for* Qual.

We suppose that the function Evol is locally constant (as described in Section 3.3) and that $\mathrm{Ges}^\star$ has n changing epsilons (i.e., $|\,\mathrm{seq}\,| = n+1$), and we fix a tolerance $\mathrm{tol} = 1/t$. The qualitative function can then be simplified in this case to

$$\begin{aligned} \mathrm{Qual}_t \colon \{0,1,\ldots,n\} &\longrightarrow [0,1]^{\mathrm{Ges}^\star}, \\ i &\longmapsto \mathrm{Qual}(\varepsilon_i, t), \end{aligned}$$

where $\varepsilon_i \in \mathrm{seq}$. By construction of the qualitative function we observe that such an image $\mathrm{Qual}_t(i)$ is a vector with $|\,\mathrm{Ges}^\star|$ coordinates whose values are $1, 1/2, \ldots\ , 1/t$ and 0 (but not necessarily in this order). Therefore the Qual_t's images are vectors positioned on faces of the multidimensional unit cube $[0,1]^{\mathrm{Ges}^\star}$. In order to visualize the image of Qual_t, or more specifically, the information carried by the image of Qual_t, we build a *motivic evolution tree* in the following subsection.

4.2.1 Motivic evolution tree (MET)

We give a short description of a motivic evolution tree. For more details refer to [5].

We are interested in an 'intuitive' visualization of these Qual_t's images, representation that we call a *motivic evolution tree*: Fix a tolerance tol. We construct the coordinate system for which in the vertical axis we consider the radius variable (growing from top to bottom), and in the horizontal axis the gestalt cardinality variable. For each $\mathrm{Qual}_t(i)$, we consider only gestalts $G_0, \ldots, G_k$ for which $\mathrm{Qual}_t(i)_{G_j} \neq 0$, for $j = 0, \ldots, k$. We represent all these gestalts G_j at coordinate $(\mathrm{card}_t(G_j), \varepsilon_i)$. These gestalts G_j are ordered by their $\mathrm{Qual}_t(i)_{G_j}$ values, and this order is represented by the grey intensity (value 1: black; value 0: white) of their gestalt representation. Then we link gestalts which are small gestalts from each other. Since this relation is transitive we link only gestalts with consecutive cardinalities. We do not repeat gestalts at consecutive $\mathrm{Qual}_t(i)$s.

Definition 14. Given a score $\mathcal{S}$ together with a motivic gestalt space $\mathrm{Ges}^\star$ and a tolerance tol, the (above described) intuitive representation of the Qual_t function is called the *motivic evolution tree* (MET) *of* $\mathcal{S}$ *with respect to* $\mathrm{Ges}^\star$.

Example 11. We consider again Bach's *Kunst der Fuge* with motivic topology as described in Section 4.1.1. We fix $\mathrm{tol} = 2$. Because of calculation constraints we evaluated the $\mathrm{Qual}_{1/2}$ at selected radii $r = 0.2, 0.25, \ldots 1.65$. The black gestalt representatives have value 1 in their respective vector $\mathrm{Qual}_{1/2}$, the grey ones value $1/2$. Recall that the vertical axis corresponds to the neighborhood radius variable and the horizontal one, to the gestalt cardinality variable. At each neighborhood radius corresponds gestalts (recall that this representation of gestalts is for the COM-Matrix shape type and for the paradigmatic group $P = \{\text{identity}\}$) for which the above flag corresponds to its 'identity' number (e.g., "3-13" stands for the 13th gestalt in the lexicographically ordered list of gestalts with cardinality 3) and the below flag, to its multiplicity. Dashed boxes around a gestalt means that the latter has already appeared in the MET at smaller neighborhood radius.

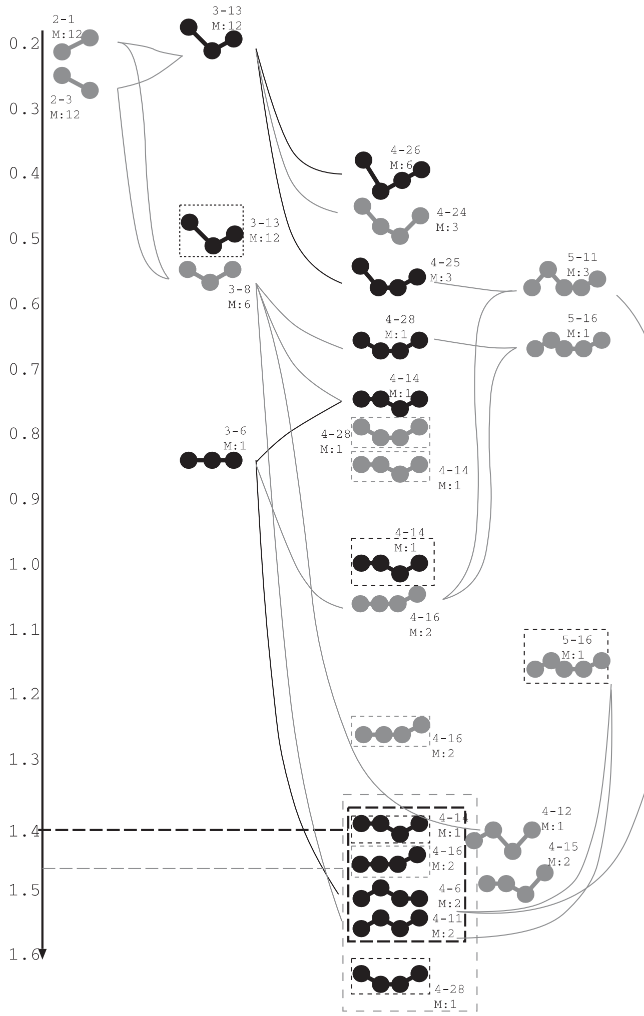


FIGURE 2. This graph shows the motivic evolution tree (MET), an intuitive visualization of the Qual_t's (for $\text{Weight}_{\text{Maz}}$) image, at tolerance 2, of the 8-tone main theme of Bach's *Kunst der Fuge*. As a global information from this tree, we should understand: When looking from top to bottom, i.e., when the neighborhood radius (similarity allowance) is growing, we view (in black) the evolution of the score's motif gestalt obtained from our motivic analysis.

As an application of this intuitive representation of a motivic space of a score we compared [5] Bach's *Kunst der Fuge* 8-tone and 12-tone main themes together in order to address the still debated question concerning the length of the main theme. We concluded that "*the significant contours* (*gestalts*) *of the 8-tone theme* (*subspace of the 12-tone theme*) *are part of the significant contours of the 12-tone theme, but the last four notes do not generate a proper extension to the set of significant premotives. However, the last four tones are all related to the significant contours of the 12-tone theme. In other words, the extension to twelve tones is 'substantial', but it is not a proper extension* [5]".[3]

REFERENCES

[1] ARTIN, M.: *Algebra*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[2] BERAN, J.—MAZZOLA, G.: *Timing Microstructure in Schumann's "Träumerei" as an Expression of Harmony, Rhythm, and Motivic Structure in Music Performance*, Comput. Math. Appl. **39** (2000), 99–130.

[3] BOURBAKI, N.: *Éléments de mathématique*, Algèbre commutative, Éditions Scientifiques Hermann, Paris, 1961.

[4] BUTEAU, C.: *Motivic Topologies and Their Meaning in the Motivic Analysis of Music*, Master Thesis, Université Laval, Québec, Canada; `http://www.math.ethz.ch/~cbuteau`.

[5] BUTEAU, C.—MAZZOLA, G.: *From Contour Similarity to Motivic Topologies*, in: Musicae Scientiae, European Society for Cognitive Sciences of Music (ESCOM), Vol IV, Fall 2000, pp. 125–149, 2000.

[6] EHRENFELS, C. VON: *Über Gestaltqualitäten*, Vierteljahresschrift für wissenschaftliche Philosophie XIV, 1890.

[7] FORTE, A.: *The Structure of Atonal Music*, Yale university Press, New Haven, 1973.

[8] GROTHENDIECK, A.—DIEUDONNÉ, J. A.: *Éléments de géométrie algébrique I*, Springer-Verlag, Berlin, 1971.

[9] HEWLETT, W. B.—SELFRIDGE-FIELD, E.: *Melodic Similarity-Concepts, Procedures, and Applications*, Computing in Musicology, 11, MIT Press, Cambridge, 1998.

[10] KELLEY, J. L.: *General Topology*, Springer-Verlag, Berlin, 1955.

[11] LEWIN, D.: *Some New Construct Involving Posets, and Probabilistics Applications*, in: PNM, 18/2, pp. 433–44, 1980.

[12] McLEAN, S.: *Categories for the Working Mathematician*, Springer-Verlag, Berlin, 1971.

[13] MAZZOLA, G.: *Geometrie der Töne*, Birkhäuser, Basel, 1990.

[14] MAZZOLA, G.: *Gruppen und Kategorien in der Musik*, Heldermann, Berlin, 1985.

[15] MAZZOLA, G.: *Mathematische Musiktheorie: Status quo 1990*, Jber. d.Dt. Math.-Verein. 93, pp. 6–29, 1991.

[16] MAZZOLA, G.: *The Topos of Music*, Birkhäuser, Basel, 2001.

[17] MAZZOLA, G.—BERAN, J.: *Rational Composition of Performance. Appears*, in: Proceedings of the Conference "Regulation of Creative Processes in Music" (Kopiez R. and Auhagen W., eds.), Lang, Frankfurt and New York, 1997.

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[18] MAZZOLA, G.—ZAHORKA, O.: *The RUBATO Performance Workstation on NEXT-STEP*, in: Proceedings of the 1994 International Computer Music Conference, International Computer Music Association, San Francisco, 1994.

[19] MAZZOLA, G.—ZAHORKA, O.—STANGE-ELBE, J.: *Analysis and Performance of a Dream*, Grammars for Music Performance, (Anders Friberg and John Sundberg, eds.), Stockholm, 2000, pp. 59–68.

[20] METZGER, H.-K.—RIEHN, R.: *Robert Schumann II*, Musik-Konzepte, München, 1982.

[21] MORRIS, R.: *Composition with Pitch-classes*, Yale University Press, New Haven, 1987.

[22] QUINN, J.: *Fuzzy Extensions to the Theory of Contour*, Music Theory Spectrum **19/2** (1997), 232–63.

[23] RAHN, J.: *Basic Atonal Music*, Schimer Books, New York, 1980.

[24] RETI, R.: *The Thematic Process in Music*, Greenwood Press, Connecticut, USA, 1951.

[25] RETI, R.: *Thematic Patterns in Sonatas of Beethoven*, Faber and Faber, London, 1967.

[26] `http://www.ifi.unizh.ch/groups/mml/musicmedia/rubato/rubato.html`.

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Mathematik Department
ETH Zürich & Universität Zürich
i2musics
(*Internet Institut for Musics Science*)
SWITZERLAND

E-mail: cbuteau@math.ethz.ch