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ON QUASI-CONTINUITY AND CLIQUISHNESS OF SECTIONS OF FUNCTIONS OF TWO VARIABLES

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ABSTRACT. We investigate quasi-continuity (cliquishness) of sections of quasi-continuous (respectively cliquish) functions of two variables.

The letters $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the real line, the set of rationals, and the set of positive integers, respectively. For each $A \subset \mathbb{R}$, the symbol $\operatorname{int} A$ denotes its interior.

Let $g\colon \mathbb{R}^n \to \mathbb{R}$. The *oscillation of g on a nonempty set $A \subset \mathbb{R}^n$* will be denoted by $\operatorname{osc}(g, A)$, i.e.,

$$\operatorname{osc}(g, A) = \sup\bigl\{\bigl|g(x) - g(y)\bigr| : x, y \in A\bigr\}.$$

We say that g is *quasi-continuous* in the sense of Kempisty [2] (*cliquish* [5]) at a point $z \in \mathbb{R}^n$, if for each $\varepsilon > 0$ and each open set $G \ni z$, we can find a nonempty open set $G' \subset G$ such that $g[G'] \subset \bigl(g(z) - \varepsilon, g(z) + \varepsilon\bigr)$ (respectively $\operatorname{osc}(g, G') < \varepsilon$). We say that g is *quasi-continuous* (*cliquish*), if it is quasi-continuous (cliquish) at each point $z \in \mathbb{R}^n$. Evidently quasi-continuous functions are cliquish.

Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}$. We define the *vertical section* $g_a\colon \mathbb{R} \to \mathbb{R}$ of g by $g_a(y) = g(a, y)$. Similarly, the function $g^a\colon \mathbb{R} \to \mathbb{R}$ given by $g^a(x) = g(x, a)$ is called a *horizontal section* of g.

The notion of quasi-continuity was introduced by Kempisty in 1932. Extending the results of Hahn and Banach, he proved the following theorem [2].

Theorem. *Assume that all sections (both vertical and horizontal) of a function $g\colon \mathbb{R}^2 \to \mathbb{R}$ are quasi-continuous. Then g is quasi-continuous as well.*

In this note we address an opposite question, which is closely related to the so-called double difference property. (See, e.g., [3] or [1] for the definition and the

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discussion of this property.) More precisely, we show that almost every (in the sense of category) section of a quasi-continuous (cliquish) function $g\colon \mathbb{R}^2 \to \mathbb{R}$ is quasi-continuous (respectively cliquish). On the other hand, for each meager set $F \subset \mathbb{R}$ there is a quasi-continuous function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that all horizontal sections of g are quasi-continuous, while the vertical section g_x is constant whenever $x \notin F$, and it is nowhere cliquish if $x \in F$.

Lemma 1. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be quasi-continuous. Assume that $G, H \subset \mathbb{R}$ are open. Then the set*

$$A = \{x \in \mathbb{R}\colon \operatorname{int}(G \cap g_x^{-1}(H)) = \emptyset \neq G \cap g_x^{-1}(H)\}$$

is nowhere dense.

Proof. Pick $x \in A$ and let $U \ni x$ be open. Take an arbitrary $y \in G \cap g_x^{-1}(H)$. Then $g(x,y) \in H$. Using the quasi-continuity of g at (x,y), we can find nonempty open sets $U' \subset U$ and $G' \subset G$ such that $g[U' \times G'] \subset H$. Observe that if $t \in U'$, then $G' \subset G \cap g_t^{-1}(H)$. Consequently, $U' \cap A = \emptyset$ and A is nowhere dense. ☐

Theorem 1. *Assume that $g\colon \mathbb{R}^2 \to \mathbb{R}$ is quasi-continuous. Then the set*

$$\widetilde{A} = \{x \in \mathbb{R}\colon \text{ the section } g_x \text{ is not quasi-continuous}\}$$

is meager.

Proof. Arrange all open intervals with rational end points in a sequence, (I_n). For all n and m define

$$A_{nm} = \{x \in \mathbb{R}\colon \operatorname{int}(I_n \cap g_x^{-1}(I_m)) = \emptyset \neq I_n \cap g_x^{-1}(I_m)\}\,.$$

By Lemma 1, each A_{nm} is nowhere dense. To complete the proof we will show that $\widetilde{A} \subset \bigcup_{n,m\in\mathbb{N}} A_{nm}$.

Fix an $x \in \widetilde{A}$. Let $y \in \mathbb{R}$ be such that g_x is not quasi-continuous at y. There are an $\varepsilon > 0$ and an open set $G \ni y$ such that $g_x[G'] \not\subset (g_x(y) - \varepsilon,\, g_x(y) + \varepsilon)$ for no nonempty open set $G' \subset G$. Pick $n, m \in \mathbb{N}$ such that $y \in I_n \subset G$ and $g_x(y) \in I_m \subset (g_x(y) - \varepsilon,\, g_x(y) + \varepsilon)$. Clearly $x \in A_{nm}$. ☐

Lemma 2. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be cliquish. Assume that $G \subset \mathbb{R}$ is open and $\varepsilon > 0$. Then the set*

$$B = \{x \in \mathbb{R}\colon \operatorname{osc}(g_x, G') \geq \varepsilon \text{ for each nonempty open set } G' \subset G\}$$

is nowhere dense.

Proof. Pick $x \in B$ and let $U \ni x$ be open. Take an arbitrary $y \in G$. Using the cliquishness of g at (x,y), we can find nonempty open sets $U' \subset U$ and $G' \subset G$ such that $\operatorname{osc}(g, U' \times G') < \varepsilon$. Observe that if $t \in U'$, then $\operatorname{osc}(g_t, G') < \varepsilon$. Consequently, $U' \cap B = \emptyset$ and B is nowhere dense. ☐

Theorem 2. *Assume that* $g\colon \mathbb{R}^2 \to \mathbb{R}$ *is cliquish. Then the set*

$$\widetilde{B} = \{x \in \mathbb{R}\colon \text{ the section } g_x \text{ is not cliquish}\}$$

is meager.

Proof. Arrange all open intervals with rational end points in a sequence, (I_n). For all n and m define

$$B_{nm} = \{x \in \mathbb{R}\colon \operatorname{osc}(g_x, G') \geq m^{-1} \text{ for each nonempty open set } G' \subset I_n\}.$$

By Lemma 2, each B_{nm} is nowhere dense. To complete the proof we will show that $\widetilde{B} \subset \bigcup_{n,m\in\mathbb{N}} B_{nm}$.

Fix an $x \in \widetilde{B}$. Let $y \in \mathbb{R}$ be such that g_x is not cliquish at y. There are an $\varepsilon > 0$ and an open set $G \ni y$ such that $\operatorname{osc}(g_x, G') \geq \varepsilon$ for each nonempty open set $G' \subset G$. Pick $n, m \in \mathbb{N}$ such that $y \in I_n \subset G$ and $m^{-1} < \varepsilon$. Clearly $x \in B_{nm}$. □

Theorem 3. *Let* $F \subset \mathbb{R}$ *be meager. There is a function* $g\colon \mathbb{R}^2 \to \mathbb{R}$ *such that*

- *both* g *and all horizontal sections of* g *are quasi-continuous,*
- g_x *is constant, if* $x \notin F$,
- g_x *is nowhere cliquish, if* $x \in F$.

Proof. Let (F_n) be an ascending sequence of nonempty nowhere dense closed sets such that $F \subset \bigcup_{n\in\mathbb{N}} F_n$. Put $F_0 = \emptyset$. Fix an $n \in \mathbb{N}$. Define $\tilde{g}^{(n)}\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{g}^{(n)}(x,y) = \begin{cases} \sin\bigl(\operatorname{dist}(x, F_n)^{-1}\bigr) & \text{if } x \notin F_n\,,\\ 1/2 & \text{if } x \in F \cap F_n \setminus F_{n-1} \text{ and } y \in \mathbb{Q}\,,\\ 0 & \text{otherwise}\,, \end{cases}$$

where $\operatorname{dist}(x, F_n) = \min\{|x-t|\colon t \in F_n\}$, and let $g^{(n)} = \sum_{i=1}^{n} \tilde{g}^{(i)}/4^i$. Observe that

(i) $\tilde{g}^{(n)}_x$ is constant, if $x \notin F_n$ or $x \in F \cap F_{n-1}$,

(ii) $\tilde{g}^{(n)}_x$ is nowhere cliquish, if $x \in F \cap F_n \setminus F_{n-1}$,

(iii) $\bigl|\tilde{g}^{(n)}\bigr| \leq 1$ on $\mathbb{R}^2$,

(iv) $g^{(n)}$ and all horizontal sections of $g^{(n)}$ are quasi-continuous.

Define $g = \lim_{n\to\infty} g^{(n)}$. By (iii), the sequence $\bigl(g^{(n)}\bigr)$ is uniformly convergent. So by (iv), g and all horizontal sections of g are quasi-continuous. (Cf., e.g., [4].) By (i), g_x is constant, if $x \notin F$. Finally by (i) and (ii), g_x is nowhere cliquish, if $x \in F$. □

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