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INSENSITIVITY REGIONS FOR ESTIMATORS OF MEAN VALUE PARAMETERS IN MIXED MODELS WITH CONSTRAINTS

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. Insensitivity regions for the variance of estimators of linear functions of the mean value parameter and for their confidence ellipsoids are investigated in a regular mixed linear model with constraints.

1. Introduction

Many experiments in natural science must be modelled by a linear regression model with inaccurate variance components $\boldsymbol{\vartheta}$, since they are known only approximately or they must be estimated. Therefore it is important to know whether and how much differences between true values of variance components $\boldsymbol{\vartheta}$ and approximate ones influence statistical characteristics. The problem is to find a region defined in the space of the parameter $\boldsymbol{\vartheta}$ such that a shift of these parameters inside this region does not cause any essential damage of chosen statistical characteristic, i.e., estimators of unknown parameters, the shape and the position of confidence ellipsoids, the level of statistical tests and their power function and threshold ellipsoids.

These problems are studied in [3], [4], [7], [8], [9] in the case of a regular mixed model without constraints. In [5], [6] a problem connected with estimators in the universal model with or without constraints is solved.

The aim of this paper is to generalize the solution for insensivity regions for the variance of estimators and estimates of linear functions of the mean value parameter and for confidence ellipsoids in the regular mixed linear model with constraints.

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2. Notations and auxiliary statements

Let

$$\begin{gathered}\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}(\boldsymbol{\vartheta})), \qquad \boldsymbol{\beta} \in \mathcal{B}, \ \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}} \subset \mathbb{R}^p, \\ \mathcal{B} = \{\boldsymbol{\beta} \colon \boldsymbol{\beta} \in \mathbb{R}^k, \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}\}\end{gathered} \tag{2.1}$$

where $\mathbf{Y}$ is an n-dimensional random vector (observation vector), $\mathbf{X}_{n\times k}$ a known matrix (design matrix), $\boldsymbol{\beta}$ an unknown vector (parameter of the first order), $\mathbf{B}_{q\times k}$ a known matrix, $\mathbf{b}$ a known q-dimensional vector such that $\mathbf{b} \in \mathcal{M}(\mathbf{B})$ and $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ a covariance matrix, where $\boldsymbol{\vartheta}$ is an unknown vector (parameter of the second order), $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}} = \{\boldsymbol{\vartheta} \colon \boldsymbol{\vartheta} \in \mathbb{R}^p, \ \vartheta_i > 0, \ i = 1, \dots, p\}$ and $\mathbf{V}_i$, $i = 1, \dots, p$, are known symmetric, positive semidefinite matrices.

In the following, the mixed linear model with constraints (2.1) will be supposed to be regular, i.e., the rank of the matrix $\mathbf{X}$ is $r(\mathbf{X}) = k < n$, $\boldsymbol{\Sigma}(\boldsymbol{\vartheta})$ is positive definite for all $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$ and the rank of the matrix $\mathbf{B}$ is $r(\mathbf{B}) = q < k$.

Let $\boldsymbol{\beta}^*$ and $\boldsymbol{\vartheta}^*$ be true value of the vector parameter $\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$, respectively.

Let $\mathbf{A}$ be an arbitrary matrix of the type $m \times n$, $\mathbf{W}$ a positive definite matrix of the type $m \times m$. Let $\mathcal{M}(\mathbf{A}) = \{\mathbf{Au} \colon \mathbf{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$ and $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{u} \in \mathbb{R}^n \colon \mathbf{Au} = \mathbf{0}\} \subset \mathbb{R}^n$. Then $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}^{-1}} = \mathbf{A}(\mathbf{A}'\mathbf{W}^{-1}\mathbf{A})^{-}\mathbf{A}'\mathbf{W}^{-1}$ is the $\mathbf{W}^{-1}$-projection matrix on $\mathcal{M}(\mathbf{A})$ ($^-$ means g-inverse). The symbol $\mathbf{M}_{\mathbf{A}}^{\mathbf{W}^{-1}}$ means $\mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\mathbf{W}^{-1}}$. If $\mathbf{W} = \mathbf{I}$, then symbols $\mathbf{P}_{\mathbf{A}}$ and $\mathbf{M}_{\mathbf{A}}$ are used instead of $\mathbf{P}_{\mathbf{A}}^{\mathbf{I}}$ and $\mathbf{M}_{\mathbf{A}}^{\mathbf{I}}$, respectively.

LEMMA 2.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric, positive semidefinite $n\times n$ matrices. Then $\mathcal{M}(\mathbf{A}, \mathbf{B}) = \mathcal{M}(\mathbf{A} + \mathbf{B})$.*

Proof. For the proof cf. [2], p. 445. □

LEMMA 2.2. *Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be elements of a finite dimension vector space $\mathscr{H}$ with the inner product $\langle \cdot, \cdot \rangle$. Then the Gram matrix $\mathbf{G}$ generated by the n-tuple $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, i.e., $\{\mathbf{G}\}_{i,j} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle$, $i, j = 1, \dots, n$, has the following properties*

(*i*) $\forall \mathbf{u} \in \mathscr{H} \colon \big(\langle \mathbf{u}, \mathbf{x}_1 \rangle, \dots, \langle \mathbf{u}, \mathbf{x}_n \rangle\big)' \in \mathcal{M}(\mathbf{G})$,

(*ii*) $\mathbf{G}$ *is at least positive semidefinite matrix.*

Proof. For the proof cf. [2], p. 442. □

LEMMA 2.3. *Let the regular mixed linear model with constraints (2.1) be under consideration. Then $\boldsymbol{\vartheta}^*$ –LBLUE (locally best linear unbiased estimator) of the parameter $\boldsymbol{\beta}$ is*

$$\begin{aligned}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) = {} & \big[\mathbf{I} - \mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big(\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big)^{-1}\mathbf{B}\big]\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{Y} \\ & - \mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big(\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big)^{-1}\mathbf{b}\end{aligned} \tag{2.2}$$

and its variance is

$$\operatorname{var}_{\vartheta^*}\left[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\right] = \left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+,$$

where

$$\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+ = \mathbf{C}^{-1}(\boldsymbol{\vartheta}^*) - \mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\left(\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\right)^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*),$$
$$\mathbf{C}(\boldsymbol{\vartheta}^*) = \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}.$$

P r o o f . The proof can be found in [1], p. 128. □

A small change of $\boldsymbol{\vartheta}^*$ to $\boldsymbol{\vartheta}^* + \boldsymbol{\Delta}$, $\boldsymbol{\Delta} \in \mathbb{R}^p$, causes a change of $\boldsymbol{\vartheta}^*$-LBLUE $\widehat{\mathbf{h}'\boldsymbol{\beta}}$ of the function $\mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$, $\mathbf{h} \in \mathbb{R}^k$. Since $\boldsymbol{\vartheta}^*$-LBLUE is unbiased for all $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$, the effect of change $\boldsymbol{\Delta}$ results in an enlargement of the variance of the estimator only. Hence, the problem is to find a region of those changes $\boldsymbol{\Delta}$ which do not cause a greater change in $\operatorname{var}_{\vartheta^*}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\right]$ than tolerable one.

DEFINITION 2.4. The set

$$\mathcal{O}_\varepsilon = \left\{\boldsymbol{\Delta}\colon \boldsymbol{\Delta} \in \mathbb{R}^p,\ \operatorname{var}_{\vartheta^*}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})\right] \le (1+\varepsilon^2)\operatorname{var}_{\vartheta^*}\left[\widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\right]\right\},$$

where $\varepsilon > 0$ is a number chosen by the user, is called the *insensitivity region of the variance of the estimator of the function* $\mathbf{h}'\boldsymbol{\beta}$.

3. Estimator of the function $\mathbf{h}(\cdot)$

DEFINITION 2.4. The random variable

$$\left.\frac{\partial \widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta})}{\partial \vartheta_j}\right|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}^*}, \qquad j = 1, \dots, p,$$

is called the *sensitiveness of a linear function* $\mathbf{h}'\boldsymbol{\beta}$ *with respect to* ϑ_j.

LEMMA 3.2. *Let the regular mixed linear model with constraints* (2.1) *be under consideration. Then*

$$\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta} = -\mathbf{h}'\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{\Sigma}(\boldsymbol{\Delta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{v}(\boldsymbol{\vartheta}^*),$$

where $\boldsymbol{v}(\boldsymbol{\vartheta}^*) = \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)$. *Further*

(i) $\forall\{\boldsymbol{\beta} \in \mathcal{B},\ \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}\}\quad E\left[\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta}\right] = 0,$

(ii) $\forall\{\boldsymbol{\beta} \in \mathcal{B}\}\quad \operatorname{cov}_{\vartheta^*}\left[\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta},\ \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\right] = \mathbf{0},$

(iii) $\forall\{\boldsymbol{\beta}\in\mathcal{B}\}\quad \operatorname{var}_{\vartheta^*}\Big[\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta}\Big]=\boldsymbol{\Delta}'\mathbf{W}_{\mathbf{h}}\boldsymbol{\Delta}$, *where*

$$\begin{aligned}\{\mathbf{W}_{\mathbf{h}}\}_{ij}=\mathbf{h}'\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\\ \times\Big(\big[\mathbf{M}_{\mathbf{X}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{X}}\big]^{+}+\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}\Big)\\ \times\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{h}\,,\qquad i,j=1,\dots,p,\end{aligned}$$

and

$$\begin{aligned}\big[\mathbf{M}_{\mathbf{X}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{X}}\big]^{+}&=\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)-\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X}\big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\big)^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\,,\\ \mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}&=\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big(\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'\big)^{-1}\mathbf{B}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\,.\end{aligned}$$

P r o o f . The first part of the proof is based on the expressions:

$$\begin{aligned}\frac{\partial\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}{\partial\vartheta_j}&=-\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\,,\\ \frac{\partial\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)}{\partial\vartheta_j}&=\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\,,\qquad j=1,\dots,p.\end{aligned}$$

Then statements (i) and (iii) are obvious and (ii) follows from the relation $\operatorname{cov}_{\vartheta^*}\big[\boldsymbol{v}(\boldsymbol{\vartheta}^*),\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]=\mathbf{0}$. □

Corollary 3.3. *With respect to Lemma 3.2 there holds*

$$\operatorname{var}_{\vartheta^*}\big[\widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*+\boldsymbol{\Delta})\big]\approx\operatorname{var}_{\vartheta^*}\big[\widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]+\operatorname{var}_{\vartheta^*}\left[\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta}\right].$$

Theorem 3.4. *In the regular mixed linear model with constraints* (2.1) *the insensitivity region of the variance of the estimator of the linear function* $\mathbf{h}'\boldsymbol{\beta}$ *is the set*

$$\mathcal{O}_\varepsilon=\big\{\boldsymbol{\Delta}\colon\boldsymbol{\Delta}\in\mathbb{R}^p,\ \boldsymbol{\Delta}'\mathbf{W}_{\mathbf{h}}\boldsymbol{\Delta}\le\varepsilon^2\sigma^2_{\mathbf{h}}\big\}\,,$$

where $\mathbf{W}_{\mathbf{h}}$ *is defined in Lemma 3.2 and* $\sigma^2_{\mathbf{h}}=\mathbf{h}'\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{h}$. *Further*

$$\boldsymbol{\Delta}\in\mathcal{O}_\varepsilon\quad\Longrightarrow\quad\sqrt{\operatorname{var}_{\vartheta^*}\left[\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta}\right]}\le\varepsilon\sigma_{\mathbf{h}}\,.$$

P r o o f . The first part follows from Corollary 3.3, Lemma 3.2 and Definition 2.4. Then the second part is obvious. □

Remark 3.5. The shift $\boldsymbol{\Delta}$ in the direction $\boldsymbol{\vartheta}^*$, i.e., $\boldsymbol{\Delta}=k\boldsymbol{\vartheta}^*$, $k\in\mathbb{R}^1$, does not cause a change of the estimator of a function $\mathbf{h}'\boldsymbol{\beta}$. Hence $\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\vartheta}^*=0$ and $(\boldsymbol{\vartheta}^*)'\mathbf{W}_{\mathbf{h}}\boldsymbol{\vartheta}^*=0$.

P r o o f . Let $\mathbf{\Delta} = k\boldsymbol{\vartheta}^*$, $k \in \mathbb{R}^1$. Then

$$\mathbf{\Sigma}(\boldsymbol{\vartheta}^* + \mathbf{\Delta}) = \mathbf{\Sigma}(\boldsymbol{\vartheta}^* + k\boldsymbol{\vartheta}^*) = (k+1)\mathbf{\Sigma}(\boldsymbol{\vartheta}^*)$$

and from Lemma 2.3 it follows $\widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) = \widehat{\mathbf{h}'\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \mathbf{\Delta})$. □

Obviously, for a different function $\mathbf{h}'\boldsymbol{\beta}$ we get a different region $\mathcal{O}_\varepsilon$ since the matrix $\mathbf{W}_\mathbf{h}$ depends on $\mathbf{h}$. If we want to find a region $\mathcal{O}_\varepsilon$ for all functions $\mathbf{h}'\boldsymbol{\beta}$ simultaneously, we must eliminate this dependence.

Lemma 3.6. *Let $\mathbf{E}_{i,j}$ be a matrix of the type $p \times p$ with the (i,j)th entry equal to 1 and with others entries equal to 0 and let $\mathbf{L}_{i,j}$ be a matrix of the type $k \times k$ given by the relation*

$$\begin{aligned}\mathbf{L}_{i,j} = {} & \left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^{+}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i \\ & \times \left(\left[\mathbf{M}_\mathbf{X}\mathbf{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M}_\mathbf{X}\right]^{+} + \mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}\right) \\ & \times \mathbf{V}_j\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^{+}, \qquad i,j = 1,\dots,p.\end{aligned}$$

Then in the regular mixed linear model with constraints (2.1) *there holds*

$$\forall\{\mathbf{h} \in \mathbb{R}^k\}\ \forall\{\mathbf{\Delta} \in \mathbb{R}^p\}\colon \mathbf{\Delta}'\mathbf{W}_\mathbf{h}\mathbf{\Delta} \le \mathbf{h}'\mathbf{h}\mathbf{\Delta}'\sum_{i=1}^{pk}\gamma_i\mathbf{G}_i\mathbf{G}_i'\mathbf{\Delta}\,,$$

where

$$\mathbf{G}_r = \begin{pmatrix}\mathbf{g}'_{r,1}\\ \vdots \\ \mathbf{g}'_{r,p}\end{pmatrix}, \qquad r = 1,\dots,pk\,,$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{E}_{i,j} \otimes \mathbf{L}_{i,j}) = \sum_{r=1}^{pk}\gamma_r\,\mathbf{g}_r\,\mathbf{g}_r' \quad (\textit{spectral decomposition}),$$

where $\mathbf{g}_r \in \mathbb{R}^{pk}$, $\mathbf{g}_r'\mathbf{g}_s = \begin{cases}1 & \textit{if } r = s\,,\\ 0 & \textit{otherwise}\,,\end{cases}$ $\mathbf{g}_{r,i} \in \mathbb{R}^k$, $\mathbf{g}_r = \left(\mathbf{g}'_{r,1},\dots,\mathbf{g}'_{r,p}\right)'$, $i = 1,\dots,p$ *and* $r = 1,\dots,pk$ *and* $\otimes$ *means Kronecker product.*

P r o o f . Here an analogous procedure is used as in [6]. Obviously

$$\forall\mathbf{h} \in \mathbb{R}^k\colon\ \{\mathbf{W}_\mathbf{h}\}_{i,j} = \mathbf{h}'\mathbf{L}_{i,j}\mathbf{h}\,, \qquad i,j = 1,\dots,p\,.$$

Then $\mathbf{\Delta}'\mathbf{W}_\mathbf{h}\mathbf{\Delta}$ can be rewritten in the form

$$\mathbf{\Delta}'\mathbf{W}_\mathbf{h}\mathbf{\Delta} = (\mathbf{\Delta}' \otimes \mathbf{h}')\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mathbf{E}_{i,j} \otimes \mathbf{L}_{i,j}\right)(\mathbf{\Delta} \otimes \mathbf{h})\,.$$

Using the spectral decomposition of the matrix $\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{E}_{i,j}\otimes\mathbf{L}_{i,j})$ and the Schwarz inequality with the Euclidean norm $\|\mathbf{x}\|=\sqrt{\mathbf{x}'\mathbf{x}}$ we get

$$\begin{aligned}\boldsymbol{\Delta}'\mathbf{W}_{\mathbf{h}}\boldsymbol{\Delta} &= (\boldsymbol{\Delta}'\otimes\mathbf{h}')\sum_{r=1}^{pk}\gamma_r\,\mathbf{g}_r\,\mathbf{g}_r'(\boldsymbol{\Delta}\otimes\mathbf{h}) = \sum_{r=1}^{pk}\gamma_r(\boldsymbol{\Delta}'\mathbf{G}_r\mathbf{h})^2\\ &\le \mathbf{h}'\mathbf{h}\boldsymbol{\Delta}'\left(\sum_{r=1}^{pk}\gamma_r\mathbf{G}_r\mathbf{G}_r'\right)\boldsymbol{\Delta}\,.\end{aligned}$$

□

THEOREM 3.7. *Let assumptions of Lemma 3.6 be fulfilled, $\sum_{i=1}^{p}\kappa_i\mathbf{l}_i\mathbf{l}_i'$ be the spectral decomposition of the matrix $\sum_{i=1}^{pk}\gamma_i\mathbf{G}_i\mathbf{G}_i'$ and let $\kappa_1\ge\kappa_2\ge\dots$. Let ω_1 be the maximum eigenvalue of the matrix $\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}$. Then*

$$\forall\boldsymbol{\Delta}\in\mathcal{O}_\varepsilon\,,\ \forall\mathbf{h}\in\mathbb{R}^k\colon\ \sqrt{\operatorname{var}_{\vartheta^*}\left[\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta}\right]}\le\varepsilon\sigma_{\mathbf{h}}\,,$$

where

$$\mathcal{O}_\varepsilon=\left\{\boldsymbol{\Delta}\colon\|\boldsymbol{\Delta}\|\le\frac{\varepsilon}{\sqrt{\kappa_1\omega_1}}\right\}.$$

Proof. With respect to Lemma 3.6 we have

$$\forall\mathbf{h}\in\mathbb{R}^k\colon\ \boldsymbol{\Delta}'\mathbf{W}_{\mathbf{h}}\boldsymbol{\Delta}\le\mathbf{h}'\mathbf{h}\boldsymbol{\Delta}'\left(\sum_{r=1}^{pk}\gamma_r\mathbf{G}_r\mathbf{G}_r'\right)\boldsymbol{\Delta}=\mathbf{h}'\mathbf{h}\sum_{i=1}^{p}\kappa_i(\mathbf{l}_i'\boldsymbol{\Delta})^2.$$

Since

$$\max\left\{\sum_{i=1}^{p}\kappa_i(\mathbf{l}_i'\boldsymbol{\Delta})^2\colon\ \|\boldsymbol{\Delta}\|=c,\ c>0\right\}=c^2\kappa_1\,,$$

then $\boldsymbol{\Delta}'\mathbf{W}_{\mathbf{h}}\boldsymbol{\Delta}\le\|\mathbf{h}\|^2\|\boldsymbol{\Delta}\|^2\kappa_1$. Analogously

$$\sigma_{\mathbf{h}}^2=\mathbf{h}'\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{h}\ge\|\mathbf{h}\|^2\frac{1}{\omega_1}\,,$$

where ω_1 is the maximum eigenvalue of the matrix $\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}$, i.e., $\frac{1}{\omega_1}$ is the minimum non-zero eigenvalue of the matrix $\left[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\right]^+$. Now, the proof follows from Definition 2.4. □

Till now the problem was considered in sense of estimators, i.e., before the realization of the observation vector $\mathbf{Y}$. Let us suppose that an experiment was already realized, i.e., we have a realization $\mathbf{y}$ of the observation vector $\mathbf{Y}$ at our disposal.

Lemma 3.8. *Let the regular mixed linear model with constraints* (2.1) *be under consideration. Let* $\mathbf{K_h}$ *be a* p*-dimensional row vector given by the relationship*

$$\mathbf{K_h} = \mathbf{h}'\big[\mathbf{M_{B'}C}(\boldsymbol{\vartheta}^*)\mathbf{M_{B'}}\big]^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \\ \times \big[\mathbf{V}_1\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{v}(\mathbf{y},\boldsymbol{\vartheta}^*),\dots,\mathbf{V}_p\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{v}(\mathbf{y},\boldsymbol{\vartheta}^*)\big]\,,$$

where $\boldsymbol{v}(\mathbf{y},\boldsymbol{\vartheta}^*) = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}(\mathbf{y},\boldsymbol{\vartheta}^*)$. *An insensivity region of the variance of the estimate* $\mathbf{h}'\boldsymbol{\beta}(\mathbf{y},\boldsymbol{\vartheta}^*)$ *is the set*

$$\big\{\boldsymbol{\Delta}\colon |\mathbf{K_h}\boldsymbol{\Delta}| \le \varepsilon\sigma_{\mathbf{h}}\big\}\,.$$

P r o o f . Obviously

$$\widehat{\mathbf{h}'\boldsymbol{\beta}}(\mathbf{y},\boldsymbol{\vartheta}^*+\boldsymbol{\Delta}) \approx \widehat{\mathbf{h}'\boldsymbol{\beta}}(\mathbf{y},\boldsymbol{\vartheta}^*) + \mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\mathbf{y},\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta}\,.$$

With respect to Lemma (3.2) we obtain $\mathbf{h}'\frac{\partial\hat{\boldsymbol{\beta}}(\mathbf{y},\boldsymbol{\vartheta}^*)}{\partial\boldsymbol{\vartheta}'}\boldsymbol{\Delta} = \mathbf{K_h}\boldsymbol{\Delta}$. Now the statement follows from Definition 2.4 and from the fact that the maximum tolerable movement $\big|\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{y},\boldsymbol{\vartheta}^*+\boldsymbol{\Delta}) - \mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{y},\boldsymbol{\vartheta}^*)\big|$ is not greater than $\varepsilon\sigma_{\mathbf{h}}$. □

4. Confidence ellipsoids of a vector function of the first order parameter

Let ε be a chosen probability expressing the maximum tolerable reduction of the confidence level caused by the fact that the true value $\boldsymbol{\vartheta}^*$ of the second order parameter for estimating the parameter $\boldsymbol{\beta}$ is not known precisely. The problem is to determine for a given confidence level $1-\alpha$ and a given probability ε a region $\mathcal{K}_\varepsilon$ of all shifts $\boldsymbol{\Delta}$ of the parameter $\boldsymbol{\vartheta}^*$ such that they do not cause decrease of the confidence level larger than ε.

Let $\mathbf{G}_{s\times k}$ be a given matrix of the rank $r(\mathbf{G}) = s \le k$ such that

$$\mathbf{G}\,\mathrm{var}_{\vartheta^*}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]\mathbf{G}'$$

is a regular matrix.

Lemma 4.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be arbitrary matrices of the type* $m\times k$ *and* $n\times k$. *Then*

$$r\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} = r\big[\mathbf{A}\,\mathrm{Ker}(\mathbf{B})\big] + r(\mathbf{B}) = r\big[\mathbf{B}\,\mathrm{Ker}(\mathbf{A})\big] + r(\mathbf{A})\,.$$

P r o o f . For the proof see [2], p. 440. □

LEMMA 4.2. *Let the regular mixed linear model with constraints* (2.1) *be under consideration and let* $\mathbf{G}_{s\times k}$, $r(\mathbf{G}) = s \le k$. *The matrix* $\mathbf{G}\,\mathrm{var}_{\vartheta^*}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]\mathbf{G}'$ *is regular iff*

$$r\begin{pmatrix}\mathbf{G}\\ \mathbf{B}\end{pmatrix} = s + q\,.$$

P r o o f . Obviously $\mathcal{M}\Big(\mathrm{var}_{\vartheta^*}\Big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\Big]\Big) = \mathrm{Ker}(\mathbf{B})$. Let $\mathbf{K_B}$ be such a matrix that $\mathrm{Ker}(\mathbf{B}) = \mathcal{M}(\mathbf{K_B})$. Firstly we prove the sufficiency. Let $r(\mathbf{B}',\,\mathbf{G}') = q + s$. Thus $\mathcal{M}(\mathbf{G}')\cap\mathcal{M}(\mathbf{B}') = \{\mathbf{0}\}$. Further by using Lemma 4.1 we get

$$q + s = r(\mathbf{B}',\,\mathbf{G}') = r(\mathbf{G}\mathbf{K_B}) + r(\mathbf{B}) = r(\mathbf{G}\mathbf{K_B}) + q \qquad\Longrightarrow\qquad r(\mathbf{G}\mathbf{K_B}) = s\,.$$

Hence $r\big(\mathbf{G}\mathbf{K_B}(\mathbf{K_B})'\mathbf{G}'\big) = s$ and $\mathbf{G}\mathbf{K_B}(\mathbf{K_B})'\mathbf{G}'$ is a matrix of the type $s \times s$ and thus $\mathbf{G}\,\mathrm{var}_{\vartheta^*}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]\mathbf{G}'$ is a regular matrix.

Now we prove the necessity. Let a matrix $\mathbf{G}\,\mathrm{var}_{\vartheta^*}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]\mathbf{G}'$ be regular, i.e., $r\big(\mathbf{G}\,\mathrm{var}_{\vartheta^*}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big]\mathbf{G}'\big) = s$. Then

$$s = r(\mathbf{G}\mathbf{K_B}) \qquad\Longrightarrow\qquad r\begin{pmatrix}\mathbf{B}\\ \mathbf{G}\end{pmatrix} = r(\mathbf{G}\mathbf{K_B}) + r(\mathbf{B}) = s + q$$

and the proof is finished. □

Denote by $k_{\mathbf{G}}\,(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta})$ the random variable

$$\begin{aligned} k_{\mathbf{G}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) = {} & \big[\boldsymbol{\beta}^* - \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})\big]' \\ & \times \mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{G}'\big)^{-1}\mathbf{G}\big[\boldsymbol{\beta}^* - \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})\big]\,. \end{aligned}$$

Obviously $k_{\mathbf{G}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) \sim \chi^2_s(0)$ and the set

$$\begin{aligned} \mathcal{E}_{1-\alpha}(\mathbf{u}) = \Big\{\mathbf{u}\colon\, & \big(\mathbf{u} - \mathbf{G}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big)' \\ & \times \big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{G}'\big)^{-1}\big(\mathbf{u} - \mathbf{G}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*)\big) \le \chi^2_s(0, 1-\alpha)\Big\}\,, \end{aligned}$$

where $\chi^2_s(0, 1-\alpha)$ denotes the $(1-\alpha)$-quantile of chi-square distribution with s degrees of freedom, represents the $(1-\alpha)$-confidence ellipsoid of a function $\mathbf{G}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{B}$.

LEMMA 4.3. *Let us consider the random variable*

$$\zeta_{k_G} = \boldsymbol{\Delta}'\frac{\partial k_{\mathbf{G}}(\mathbf{Y}, \boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}}\Big|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}^*},$$

which expresses the change of $k_{\mathbf{G}}$ *caused by the change* $\boldsymbol{\Delta}$ *of the parameter* $\boldsymbol{\vartheta}^*$. *Then*

$$\begin{aligned}\zeta_{k_G} = & -2\big(\hat{\boldsymbol\beta}(\boldsymbol\vartheta^*) - \boldsymbol\beta^*\big)'\mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}\\ & \times \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\boldsymbol\Sigma(\boldsymbol\Delta)\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\boldsymbol{v}(\boldsymbol\vartheta^*)\\ & - \big(\hat{\boldsymbol\beta}(\boldsymbol\vartheta^*) - \boldsymbol\beta^*\big)'\mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}\\ & \times \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\boldsymbol\Sigma(\boldsymbol\Delta)\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\\ & \times \mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}\big(\hat{\boldsymbol\beta}(\boldsymbol\vartheta^*) - \boldsymbol\beta^*\big)\,.\end{aligned}$$

Further, its mean value is

$$E[\zeta_{k_G}] = -\boldsymbol\Delta'\big[\mathrm{Tr}(\mathbf{U}_{\mathbf{G}}\mathbf{V}_1),\dots,\mathrm{Tr}(\mathbf{U}_{\mathbf{G}}\mathbf{V}_p)\big]',$$

where $\mathrm{Tr}(\mathbf{U}_{\mathbf{G}})$ *means the trace of the matrix* $\mathbf{U}_{\mathbf{G}}$,

$$\begin{aligned}\mathbf{U}_{\mathbf{G}} = & \ \boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}\\ & \times \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\end{aligned}$$

and its variance is

$$\mathrm{var}_{\vartheta^*}[\zeta_{k_G}] = \boldsymbol\Delta'\mathbf{A}\boldsymbol\Delta\,,$$

where

$$\mathbf{A} = 2\mathbf{S}_{\mathbf{U_G}} + 4\mathbf{C}_{\mathbf{U_G},[\mathbf{M_X}\boldsymbol\Sigma(\boldsymbol\vartheta^*)\mathbf{M_X}]^+ + \boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{P}^{\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)}_{\mathbf{XC}^{-1}(\boldsymbol\vartheta^*)\mathbf{B}'}}\,,$$

$$\{\mathbf{S}_{\mathbf{U_G}}\}_{i,j} = \mathrm{Tr}(\mathbf{U}_{\mathbf{G}}\mathbf{V}_i\mathbf{U}_{\mathbf{G}}\mathbf{V}_j)\,,\quad i,j=1,\dots,p\,, \tag{4.2}$$

and

$$\begin{aligned}&\Big\{\mathbf{C}_{\mathbf{U_G},[\mathbf{M_X}\boldsymbol\Sigma(\boldsymbol\vartheta^*)\mathbf{M_X}]^+ + \boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{P}^{\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)}_{\mathbf{XC}^{-1}(\boldsymbol\vartheta^*)\mathbf{B}'}}\Big\}_{i,j}\\ &\qquad = \mathrm{Tr}\Big[\mathbf{U}_{\mathbf{G}}\mathbf{V}_i\big(\big[\mathbf{M_X}\boldsymbol\Sigma(\boldsymbol\vartheta^*)\mathbf{M_X}\big]^+ + \boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{P}^{\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)}_{\mathbf{XC}^{-1}(\boldsymbol\vartheta^*)\mathbf{B}'}\big)\mathbf{V}_j\Big]\,.\end{aligned}$$

P r o o f . The first part follows from expressions:

$$\boldsymbol{v}(\boldsymbol\vartheta^*) = \mathbf{Y} - \mathbf{X}\hat{\boldsymbol\beta}(\boldsymbol\vartheta^*)\,,$$

$$\frac{\partial\hat{\boldsymbol\beta}(\boldsymbol\vartheta^*)}{\partial\vartheta_i} = -\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{V}_i\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\boldsymbol{v}(\boldsymbol\vartheta^*)\,,$$

$$\begin{aligned}&\frac{\partial(\mathbf{G}[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}]^+\mathbf{G}')^{-1}}{\partial\vartheta_i}\\ &\quad = -\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{V}_i\\ &\qquad \times \boldsymbol\Sigma^{-1}(\boldsymbol\vartheta^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol\vartheta^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\end{aligned}$$

for $i = 1,\dots,p$.

In the next step we use the notation

$$\mathbf{F} = \mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \\ \times \boldsymbol{\Sigma}(\boldsymbol{\Delta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\,,$$

$$\mathbf{D} = \mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \\ \times \boldsymbol{\Sigma}(\boldsymbol{\Delta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1}\mathbf{G}$$

and $\boldsymbol{\xi}(\boldsymbol{\vartheta}^*) = \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^*) - \boldsymbol{\beta}^*$. Then

$$\boldsymbol{\xi}(\boldsymbol{\vartheta}^*) \sim N_k\Big(\mathbf{0}, \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\Big)\,,$$
$$\boldsymbol{v}(\boldsymbol{\vartheta}^*) \sim N_n\Big(\mathbf{0}, \boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*) - \mathbf{X}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}'\Big)$$

and $\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)$ and $\boldsymbol{v}(\boldsymbol{\vartheta}^*)$ are stochastically independent. The random variable ζ_{k_G} can be rewritten in the form

$$\zeta_{k_G} = -2\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\mathbf{F}\boldsymbol{v}(\boldsymbol{\vartheta}^*) - \boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\mathbf{D}\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)$$

and the proof is finished by using the following relations

$$E\big[\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\mathbf{D}\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\big] = \operatorname{Tr}\big(\mathbf{D}\operatorname{var}_{\vartheta^*}\big[\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\big]\big)\,,$$
$$E\big[\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\mathbf{F}\boldsymbol{v}(\boldsymbol{\vartheta}^*)\big] = 0\,,$$
$$\operatorname{var}_{\vartheta^*}\big[\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\mathbf{D}\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\big] = 2\operatorname{Tr}\big(\mathbf{D}\operatorname{var}_{\vartheta^*}\big[\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\big]\mathbf{D}\operatorname{var}_{\vartheta^*}\big[\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\big]\big)\,,$$
$$E\big[\boldsymbol{v}(\boldsymbol{\vartheta}^*)'\mathbf{F}'\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\mathbf{F}\boldsymbol{v}(\boldsymbol{\vartheta}^*)\big] = \operatorname{Tr}\big(\mathbf{F}'\operatorname{var}_{\vartheta^*}\big[\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\big]\mathbf{F}\operatorname{var}_{\vartheta^*}\big[\boldsymbol{v}(\boldsymbol{\vartheta}^*)\big]\big)\,,$$
$$E\big[\boldsymbol{v}'(\boldsymbol{\vartheta}^*)\mathbf{F}'\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\mathbf{D}\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)\big] = 0\,.$$

□

Let the random variable $k_{\mathbf{G}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta})$ be linearly approximated by

$$k_{\mathbf{G}}(\mathbf{Y}, \boldsymbol{\vartheta}^* + \boldsymbol{\Delta}) \approx k_{\mathbf{G}}(\mathbf{Y}, \boldsymbol{\vartheta}^*) + \zeta_{k_G}\,.$$

Obviously, “the dangerous movement” of the random variable $k_{\mathbf{G}}$, which makes a decrease of the confidence level is to the right, i.e., $\zeta_{k_G}(\mathbf{y}) \to +\infty$. The movement to the left is not interesting since the confidence level increases.

With respect to the Chebyshev inequality we have

$$P\Big\{\big|\zeta_{k_G} - E[\zeta_{k_G}]\big| \geq t\sqrt{\operatorname{var}_{\vartheta^*}\big[\zeta_{k_G}\big]}\Big\} \leq \frac{1}{t^2}\,, \qquad t > 0\,. \tag{4.3}$$

If t is sufficiently large, then (4.3) is sufficiently near to 0. This implies the inequality

$$\zeta_{k_G} \leq E[\zeta_{k_G}] + t\sqrt{\operatorname{var}_{\vartheta^*}\big[\zeta_{k_G}\big]}\,,$$

which is valid with sufficiently high probability. Thus for ζ_{k_G} limited in this way we get

$$P\{\chi_s^2(0)+\zeta_{k_G} \geq \chi_s^2(0,1-\alpha)\} \leq P\Big\{\chi_s^2(0)+E[\zeta_{k_G}]+t\sqrt{\mathrm{var}_{\vartheta^*}[\zeta_{k_G}]} \geq \chi_s^2(0,1-\alpha)\Big\}.$$

If we admit that the confidence level $1-\alpha$ decreases with a given probability ε, there holds

$$P\{k_{\mathbf{G}}(\mathbf{Y},\boldsymbol{\vartheta}^*) + \zeta_{k_G} \geq \chi_s^2(0,1-\alpha)\} \leq \alpha + \varepsilon\,.$$

From the last two inequalities it follows for a given probability ε that

$$E[\zeta_{k_G}] + t\sqrt{\mathrm{var}_{\vartheta^*}[\zeta_{k_G}]} \leq \delta_\varepsilon\,,$$

where

$$\delta_\varepsilon = \chi_s^2(0,1-\alpha) - \chi_s^2(0,1-\alpha-\varepsilon)$$

is a sufficient condition that ζ_{k_G} caused by the change $\boldsymbol{\Delta}$ of the parameter $\boldsymbol{\vartheta}^*$ is acceptable. Thus this inequality determines those changes $\boldsymbol{\Delta}$ of $\boldsymbol{\vartheta}^*$ which make only tolerable change of a confidence level.

The mean value $E[\zeta_{k_G}]$ depends on $\boldsymbol{\Delta}$ linearly and the term $t\sqrt{\mathrm{var}_{\vartheta^*}[\zeta_{k_G}]}$ depends linearly on the norm $\|\boldsymbol{\Delta}\| = \sqrt{(\boldsymbol{\Delta})'(\boldsymbol{\Delta})}$. Let us denote

$$\mathbf{a} = \big[\mathrm{Tr}(\mathbf{U}_{\mathbf{G}}\mathbf{V}_1), \cdots, \mathrm{Tr}(\mathbf{U}_{\mathbf{G}}\mathbf{V}_p)\big]'. \tag{4.4}$$

Let the function $\Phi(\boldsymbol{\Delta})$, $\boldsymbol{\Delta} \in \mathbb{R}^p$, be defined as follows

$$\Phi(\boldsymbol{\Delta}) = -\boldsymbol{\Delta}'\mathbf{a} + t\sqrt{\boldsymbol{\Delta}'\mathbf{A}\boldsymbol{\Delta}}\,.$$

Definition 4.4. Let

$$\mathcal{K}_\varepsilon = \{\boldsymbol{\Delta}\colon \boldsymbol{\Delta} \in \mathbb{R}^p,\ \Phi(\boldsymbol{\Delta}) \leq \delta_\varepsilon\}\,,$$

where δ_ε is given by the relationship

$$P\{\chi_s^2(0) \geq \chi_s^2(0,1-\alpha) - \delta_\varepsilon\} = \alpha + \varepsilon\,.$$

The set $\mathcal{K}_\varepsilon$ is called the *insensitivity region of the confidence ellipsoid.*

Lemma 4.5. *Matrices* $\mathbf{S}_{\mathbf{U}_{\mathbf{G}}}$ *and* $\mathbf{C}_{\mathbf{U}_{\mathbf{G}},[\mathbf{M}_{\mathbf{X}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{X}}]^+ + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}}$ *are at least p.s.d.*

Proof. Matrices $\mathbf{U}_{\mathbf{G}}$, $[\mathbf{M}_{\mathbf{X}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{X}}]^+$ and $\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}$ are p.s.d. since by the assumption the matrix $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)$ is p.d. Thus there exist matrices $\mathbf{J}$ and $\mathbf{K}$ such that

$$\mathbf{U}_{\mathbf{G}} = \mathbf{J}\mathbf{J}', \quad \big[\mathbf{M}_{\mathbf{X}}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M}_{\mathbf{X}}\big]^+ + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'} = \mathbf{K}\mathbf{K}'.$$

The matrix $\mathbf{S}_{\mathbf{U_G}}$ is the Gram matrix of the p-tuple $\left\{\mathbf{J}'\mathbf{V}_1\mathbf{J}, \dots, \mathbf{J}'\mathbf{V}_p\mathbf{J}\right\}$ in the vector space $\mathcal{S}$ of symmetric matrices with the inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Tr}(\mathbf{AB}), \qquad \forall \mathbf{A}, \mathbf{B} \in \mathcal{S},$$

and according to Lemma 2.2 the matrix $\mathbf{S}_{\mathbf{U_G}}$ is at least p.s.d.

Now, let us consider a vector space $\mathscr{H}$ of matrices with given dimension; the inner product is given by the relation $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Tr}(\mathbf{A}'\mathbf{B})$, $\mathbf{A}, \mathbf{B} \in \mathscr{H}$. Then the matrix $\mathbf{C}_{\mathbf{U_G}, [\mathbf{M_X}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\mathbf{M_X}]^+ + \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{P}^{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)}_{\mathbf{XC}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{B}'}}$ is the Gram matrix of the p-tuple $\{\mathbf{J}'\mathbf{V}_1\mathbf{K}, \dots, \mathbf{J}'\mathbf{V}_p\mathbf{K}\}$ in such a space. □

COROLLARY 4.6. *Let* $\mathbf{A}$ *and* $\mathbf{a}$ *be given by* (4.2) *and* (4.4), *respectively. Then* $\mathbf{a} \in \mathcal{M}(\mathbf{S}_{\mathbf{U_G}}) \subset \mathcal{M}(\mathbf{A})$ *and the equation* $(t^2\mathbf{A} - \mathbf{aa}')\mathbf{x}_0 = \mathbf{a}\delta_\varepsilon$ *is consistent with respect to* $\mathbf{x}_0$.

Proof. The first part follows from Lemma 2.1, Lemma 2.2 and Lemma 4.5. The consistency of the mentioned equation follows from the fact that $\mathbf{a} \in \mathcal{M}(\mathbf{A})$ and t can be a chosen number. □

THEOREM 4.7. *Let* $\mathbf{A}$ *and* $\mathbf{a}$ *be given by* (4.2) *and* (4.4). *Let* $\mathbf{G}$ *be a given matrix of the type* $s \times k$ *such that* $r(\mathbf{G}) = s \le k$ *and* $r(\mathbf{G}', \mathbf{B}') = s + q$. *The boundary of the insensitivity region of the confidence ellipsoid in the regular mixed linear model with constraints* (2.1) *is the set*

$$\overline{\mathcal{K}}_\varepsilon = \left\{ \boldsymbol{\Delta} \colon (\boldsymbol{\Delta} - \mathbf{u}_0)'(t^2\mathbf{A} - \mathbf{aa}')(\boldsymbol{\Delta} - \mathbf{u}_0) = \frac{\delta_\varepsilon^2 t^2}{t^2 - \mathbf{a}'\mathbf{A}^-\mathbf{a}} \right\},$$

where $\mathbf{u}_0 = \frac{\delta_\varepsilon}{t^2 - \mathbf{a}'\mathbf{A}^-\mathbf{a}}\mathbf{A}^-\mathbf{a}$ *and* $\delta_\varepsilon = \chi_s^2(0, 1-\alpha) - \chi_s^2(0, 1-\alpha-\varepsilon)$.

Proof. From Definition 4.4, the boundary of region $\mathcal{K}_\varepsilon$ is given by the relationship

$$\Phi(\mathbf{u}) = \delta_\varepsilon \qquad \Longleftrightarrow \qquad -\mathbf{u}'\mathbf{a} + t\sqrt{\mathbf{u}'\mathbf{A}\mathbf{u}} = \delta_\varepsilon,$$

where $\delta_\varepsilon = \chi_s^2(0, 1-\alpha) - \chi_s^2(0, 1-\alpha-\varepsilon)$. The proof can be finished by using Lemma 4.3 and Corollary (4.6). For detail cf. [4], where the problem is solved for the regular mixed model without constraints. □

THEOREM 4.8. *Let the regular mixed linear model with constraints* (2.1) *be under consideration. Let* $\mathbf{G}$ *be a given matrix of the type* $s \times k$ *such that* $r(\mathbf{G}) = s \le k$ *and* $r(\mathbf{G}', \mathbf{B}') = s + q$. *Then*

$$\boldsymbol{\Delta} \in \mathcal{K}_\varepsilon \quad \Longrightarrow$$

$$P\Big\{\mathbf{G}\boldsymbol{\beta}^* \in \big\{\mathbf{u} \colon \big[\mathbf{u} - \mathbf{G}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})\big]'\big(\mathbf{G}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{G}'\big)^{-1} \times \big[\mathbf{u} - \mathbf{G}\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}^* + \boldsymbol{\Delta})\big] \le \chi_s^2(0, 1-\alpha)\big\}\Big\} \ge 1 - \alpha - \varepsilon.$$

P r o o f . The proof follows from Definition 4.4 and Lemma 4.3. □

Remark 4.9. With respect to the Chebyshev inequality it seems that the proper value of the parameter t in the explicit expression of the insensitivity region $\mathcal{K}_\varepsilon$ lies in the interval $[3, 5]$. For determining the optimum value of the parameter t maximizing the size of the insensitivity region cf. [8].

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