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A CONVERGENCE PRESERVING UPPER AND LOWER SEMICONTINUITY OF MULTIFUNCTIONS

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ABSTRACT. By using open covers, a uniform type convergence of nets of multifunctions from one topological space to another is introduced. Under some general conditions, the limit multifunctions of this convergence are proved to be upper and lower semicontinuous.

1. Prerequisites

In this paper a new type of convergence for multifunctions is discussed. It resembles the strong convergence, which was defined for functions (see [4]). We show that this new type of convergence preserves some continuity properties of multifunctions.

First, we shall establish some terminology. Let X and Y be two sets; let $P(Y)$ denote the set of all nonempty subsets of Y. A *multifunction* from X to Y is a function F from X to $P(Y)$. We write $F\colon X \to Y$. If X and Y are topological spaces, and $a \in X$ then a multifunction F from X to Y is called *lower semicontinuous* (*upper semicontinuous*) at the point a if for each open subset G of Y such that $F(a) \cap G \neq \emptyset$ ($F(a) \subset G$) there exists an open neighborhood $V \subset X$ of a such that $V \subset \{x\colon F(x) \cap G \neq \emptyset\}$ ($V \subset \{x\colon F(x) \subset G\}$) holds.

In what follows a topological space X is called regular (normal) if for every closed subset B of X and every x from X such that $x \notin B$ (every closed set $A \subset X$ such that $A \cap B = \emptyset$) there exist two disjoint open subsets W, V of X such that $x \in W$ and $B \subset V$ ($A \subset W$ and $B \subset V$). So regular (normal) spaces are not automatically supposed to be T_1.

The following definition appeared in [4]:

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Definition 1. Let X be an arbitrary set, let (Y, T) be a topological space. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a net of functions from X to Y. We say that $\{f_\gamma\}_{\gamma\in\Gamma}$ *converges to a function f from X to Y strongly* if for every open cover r of Y there exists an index α from Γ such that for every $x \in X$ and for every $\gamma \geq \alpha$ there exists an open set U from r such that $\{f(x), f_\gamma(x)\} \subseteq U$.

In [4] the following result was proved: *Let X and Y be topological spaces and Y be regular. Let $= \{f_\gamma\}_{\gamma\in\Gamma}$ be a net of continuous functions from X to Y that converges strongly to a function $f\colon X \to Y$. Then f is continuous.* Some results concerning the strong convergence were published in [1], [3], [6] and [7].

Before defining our new notion of convergence in Definition 3 we need one more definition:

Definition 2. Let X be a topological space, let r be an open cover of X. Let A be a subset of X. The set $r^*(A) = \bigcup\{V \in r\colon A \cap V \neq \emptyset\}$ is called the *r-star* of A.

Definition 3. Let X and Y be two topological spaces. Let $n = \{F_\gamma\}_{\gamma\in\Gamma}$ be a net of multifunctions from X to Y. Let F be a multifunction from X to Y. We say that n *converges strongly to* F if for every open cover r of Y there exists an index γ_0 such that for all $\gamma \geq \gamma_0$ and for all $x \in X$, $F(x) \subseteq r^*\big(F_\gamma(x)\big)$ and $F_\gamma(x) \subseteq r^*\big(F(x)\big)$.

Definition 4. Let X and Y be two topological spaces. Let $n = \{F_\gamma\}_{\gamma\in\Gamma}$ be a net of multifunctions from X to Y. It is called *frequently upper (lower) semicontinuous at a point $a \in X$* if for each $\gamma_0 \in \Gamma$ there exists a $\gamma \geq \gamma_0$ such that the multifunction F_γ is upper (lower) semicontinuous at a.

2. Strong convergence and continuity of multifunctions

The following two theorems show that under quite general conditions, the strong convergence preserves upper-semicontinuity and lower-semicontinuity of multifunctions.

Theorem 1. *Let X be a topological space, let Y be a normal topological space. Let $n = \{F_\gamma\}_{\gamma\in\Gamma}$ be a net of multifunctions from X to Y. Let F be a multifunction from X to Y. Let a be an element of X, let the set $F(a)$ be closed. If the net n converges strongly to F and if it is frequently upper semicontinuous at a, then F is upper semicontinuous at a.*

Proof. Let V be an open subset of Y such that $F(a) \subset V$. Since Y is normal and $F(a)$ closed, there exist two open subsets O, W of Y such that

$$F(a) \subset O \subseteq \overline{O} \subseteq W \subseteq \overline{W} \subseteq V\,.$$

It is clear that $r = \{V, Y - \overline{W}\}$, $c = \{W, Y - \overline{O}\}$ are open covers of Y.

Since the net n converges to F strongly, and the set Γ is directed, there exists an index γ_0 such that

$$F(x) \subseteq r^*\big(F_\gamma(x)\big) \qquad \text{and} \qquad F_\gamma(x) \subseteq c^*\big(F(x)\big)\,. \tag{$*$}$$

for all $\gamma \geq \gamma_0$ and $x \in X$.

Moreover, since the net n is frequently upper semicontinuous at the point a, there exists a $\gamma \in \Gamma$, with $\gamma \geq \gamma_0$, such that the multifunction F_γ is upper semicontinuous at the point a.

Since $F_\gamma(a) \subseteq c^*\big(F(a)\big) = W$, and W is open, there exists an open neighborhood U of a such that $\forall x \in U$, $F_\gamma(x) \subseteq W$. Because of $(*)$ we obtain $\forall x \in U$ $F(x) \subseteq r^*\big(F_\gamma(x)\big) = V$.

F is proved to be upper semicontinuous at a. □

THEOREM 2. *Let X be a topological space, let Y be a regular one. Let $n = \{F_\gamma\}_{\gamma\in\Gamma}$ be a net of multifunctions from X to Y. Let F be a multifunction from X to Y. Let a be an element of X. If the net n converges strongly to F and if it is frequently lower semicontinuous at a, then F is lower semicontinuous at a.*

P r o o f . Let $U \subset Y$ be an open set such that $F(a) \cap U \neq \emptyset$. Let us consider a point $b \in F(a) \cap U$. Since Y is regular, there exist two open subsets V, W of U such that

$$b \in V \subseteq \overline{V} \subseteq W \subseteq \overline{W} \subseteq U\,.$$

Denote $r = \{W, Y - \overline{V}\}$, $c = \{U, Y - \overline{W}\}$. Since the net n converges to F strongly, there exist an index α, such that

$$\forall x \in X\,, \qquad F(x) \subseteq r^*\big(F_\alpha(x)\big) \quad \text{and} \quad F_\alpha(x) \subseteq c^*\big(F(x)\big)\,. \tag{$**$}$$

Moreover, we can consider an index α such that F_α is lower semicontinuous at a.

First we prove $F_\alpha(a) \cap W \neq \emptyset$. Let us suppose, contrary to what we wish to prove, that $F_\alpha(a) \cap W = \emptyset$.

Since by $(**)$, $F(a) \subseteq r^*\big(F_\alpha(a)\big) = Y - \overline{V}$, this is a contradiction with the fact that $F(a) \cap V \supset \{b\} \neq \emptyset$.

The multifunction F_α is lower semicontinuous at a, so there exists an open neighborhood O of a such that $\forall t \in O$, $F_\alpha(t) \cap W \neq \emptyset$.

Hence, because of $(**)$ we obtain $\forall t \in O$, $F_\alpha(t) \subseteq c^*\big(F(t)\big)$. Therefore $\forall t \in O$, $F(t) \cap U \neq \emptyset$, because if for a t' from O the set $F(t') \cap U$ were an empty set, then $F_\alpha(t') \subseteq c^*\big(F(t')\big) = Y - \overline{W}$ would lead to a contradiction.

F is proved to be lower semicontinuous at a. □

The following example of [4] shows that the regularity conditions of Theorem 1 and Theorem 2 cannot be omitted.

EXAMPLE 1. (Example 2 in [4]). Let $Y = \{1,2\}$ with the topology $T = \{\emptyset, \{1\}, \{1,2\}\}$. Let us define functions $f\colon \mathbb{R} \to Y$ and $g\colon \mathbb{R} \to Y$ as follows: $\forall x \in \mathbb{R} \setminus \{0\}\colon f(x) = 2$, $f(0) = 1$; $\forall x \in \mathbb{R}\colon g(x) = 2$. Put $\forall n \in \mathbb{N}\colon f_n = g$. Then the sequence $(f_n : n \in \mathbb{N})$ converges strongly to f. But it does not converge pointwise to f. Of course, f is not continuous.

The following theorem shows that under certain conditions the strong convergence is weaker than the uniform one.

Let X be a set, let (Y, U) be a uniform space. We say that a net n of multifunctions converges uniformly to a multifunction $F\colon X \to Y$ if for each $V \in U$ there exists an index γ_0 such that for every $\gamma \geq \gamma_0$ and for every $x \in X$

$$F(x) \subseteq V\big(F_\gamma(x)\big) \quad \text{and} \quad F_\gamma(x) \subseteq V\big(F(x)\big) \qquad \text{hold}.$$

THEOREM 3. *Let X be a set, let (Y, U) be a compact uniform space. Let n be a net of multifunctions which converges uniformly to a multifunction $F\colon X \to Y$.*

Then n converges to F strongly.

Proof. Let r be an open cover of Y. Then, by the Lebesgue Covering Theorem [2, p. 199] there exists a $W \in U$ such that for each $y \in Y$ there exists an $O_y \in r$ such that $W(y) \subseteq O_y$. Moreover, by the uniform convergence of the net $n = \{F_\gamma\}_{\gamma \in \Gamma}$ to F, there exists a $\gamma \in \Gamma$ such that, for all $\alpha \geq \gamma$ and $x \in X$, we have $F(x) \subseteq W\big(F_\alpha(x)\big)$ and $F_\alpha(x) \subseteq W\big(F(x)\big)$. So $F(x) \subseteq W\big(F_\alpha(x)\big) \subseteq r^*\big(F_\alpha(x)\big)$ and $F_\alpha(x) \subseteq W\big(F(x)\big) \subseteq r^*\big(F(x)\big)$ hold. $\square$

REFERENCES

[1] GODET-THOBIE, CH.—KUPKA, I.: *Strong convergence and fixed points*, preprint, 1993.

[2] KELLEY, J. L.: *General Topology*, D. Van Nostrand, Princeton, 1957.

[3] KUPKA, I.: *A generalized uniform convergence and Dini's theorem*, New Zealand J. Math. **26** (1997), 67–72.

[4] KUPKA, I.—TOMA, V.: *A uniform convergence for non-uniform spaces*, Publ. Math. Debrecen **47/3–4** (1995), 299–309.

[5] KURATOWSKI, K.: *Topologie I*, PWN, Warszawa, 1952.

[6] PÉRAIRE, Y.: *Convergence forte et relation forte*, preprint, 1997.

[7] TOMA, V.: *Strong convergence and Dini theorems for non-uniform spaces*, Ann. Math. Blaise Pascal **4** (1997), 97–102.

[8] TUKEY, J. W.: *Convergence and Uniformity in Topology*, Princeton University Press, Princeton, 1940.

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