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KATZ PARTIAL SUMMATIONS FAMILY

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. A certain generalization of partial summations is presented. It is proved that there is a correspondence between the Katz family of discrete probability distributions and a wide class of summation processes. The correspondence is demonstrated on some examples. Relations between probability generating functions and moments of generating and generated distributions are investigated.

I. An invariance of distributions from the Katz family

A discrete probability distribution $\{P_j^*\}_{j=0}^{\infty}$ belongs to the *Katz family* of distributions if

$$P_{j+1}^* = \frac{\alpha+\beta j}{1+j} P_j^*, \qquad j = 0,1,2,\dots,$$

where $\alpha \ge 0$, $\beta < 1$ and for such a number $n \in \{1,2,\dots\}$ that $\alpha + \beta n < 0$ it holds $P_{n+j}^* = 0$, $j = 1,2,\dots$ (see, e.g., in [3]). Let us consider the summation process

$$P_x = \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0,1,2,\dots \tag{1}$$

THEOREM I.1. *Let $\{P_j^*\}_{j=0}^{\infty}$ be a distribution from the Katz family. $\{P_j^*\}_{j=0}^{\infty}$ is invariant in regard to the summation process* (1) *(i.e., $P_x^* = \sum\limits_{j=x}^{\infty} g(j)P_j^*$, $x = 0,1,2,\dots$) if and only if $g(j) = 1 - \frac{\alpha+\beta j}{1+j}$.*

Proof.

Let

$$P_x^* = \sum_{j=x}^{\infty} g(j) P_j^*, \qquad x = 0,1,2,\dots$$

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Let $n \in \{0, 1, 2, \dots\}$ be such a number that $P_n^* \neq 0$ (if $P_n^* = 0$ then $P_{n+1}^* = 0$, too). Consequently

$$P_n^* - P_{n+1}^* = g(n) P_n^* \implies g(n) = 1 - \frac{P_{n+1}^*}{P_n^*} = 1 - \frac{\alpha + \beta n}{1 + n}\,.$$

Let $g(j) = 1 - \frac{\alpha + \beta j}{1+j}$, $j = 0, 1, 2, \dots$ For $x = 0, 1, 2, \dots$ it holds

$$\sum_{j=x}^{\infty} \left(1 - \frac{\alpha + \beta j}{1 + j}\right) P_j^* = \sum_{j=x}^{\infty} P_j^* - \sum_{j=x}^{\infty} \frac{\alpha + \beta j}{1 + j} P_j^* = \sum_{j=x}^{\infty} P_j^* - \sum_{j=x+1}^{\infty} P_j^* = P_x^*\,.$$

□

Examples.

a) Let $\{P_j^*\}_{j=0}^{\infty}$ have the geometric distribution with the parameter p (i.e., $\alpha = \beta = 1 - p$, $0 < p \leq 1$). The geometric distribution is invariant in regard to the summation process

$$P_x = p \sum_{j=x}^{\infty} P_j^*\,.$$

The invariance of the geometric distribution in regard to the summation $P_x = c \sum\limits_{j=x+1}^{\infty} P_j^*$, where c is a proper constant, is proved in [5]. This type of the partial summation can be found also in [1] and [4] and it is generalized in [2].

b) Let $\{P_j^*\}_{j=0}^{\infty}$ have the Poisson distribution with the parameter λ (i.e., $\alpha = \lambda$, $\beta = 0$, $\lambda \geq 0$). The Poisson distribution is invariant in regard to the summation process

$$P_x = \sum_{j=x}^{\infty} \left(1 - \frac{\lambda}{j + 1}\right) P_j^*\,.$$

c) Let $\{P_j^*\}_{j=0}^{\infty}$ have the binomial distribution with the parameters n, p (i.e., $\alpha = \frac{np}{1-p}$, $\beta = -\frac{p}{1-p}$, $0 \leq p < 1$, $n \in \{1, 2, \dots\}$). The binomial distribution is invariant in regard to the summation process

$$P_x = \sum_{j=x}^{n} \left(1 - \frac{p(n - j)}{(1 - p)(1 + j)}\right) P_j^*\,.$$

d) Let $\{P_j^*\}_{j=0}^{\infty}$ have the negative binomial distribution with the parameters k, p (i.e., $\alpha = k(1 - p)$, $\beta = 1 - p$, $k \in \{1, 2, \dots\}$, $0 < p \leq 1$). The negative binomial distribution is invariant in regard to the summation process

$$P_x = \sum_{j=x}^{\infty} \left(1 - \frac{(1 - p)(k + j)}{1 + j}\right) P_j^*\,.$$

e) Let $\{P_j^*\}_{j=0}^{\infty}$ have the 2-shifted Flory distribution ($P_j^* = (j+1)\left(\frac{1}{2}\right)^{j+2}$, $j = 0, 1, 2, \ldots$; i.e., $\alpha = 1$, $\beta = \frac{1}{2}$). The 2-shifted Flory distribution is invariant in regard to the summation process

$$P_x = \frac{1}{2} \sum_{j=x}^{\infty} \frac{j}{1+j} P_j^* .$$

f) Let $\{P_j^*\}_{j=0}^{\infty}$ have the Katz-Powell distribution with the parameter n ($P_j^* = \binom{n(n-1)}{j} 2^{-n(n-1)}$, $j = 0, 1, \ldots, n(n-1)$; i.e., $\alpha = n(n-1)$, $\beta = -1$, $n \in \{1, 2, \ldots\}$). The Katz-Powell distribution is invariant in regard to the summation process

$$P_x = \sum_{j=x}^{n(n-1)} \left(1 - \frac{n^2 - n - j}{1+j}\right) P_j^* .$$

g) Let $\{P_j^*\}_{j=0}^{\infty}$ have the zero-one distribution with the parameter p ($P_j^* = p^j (1-p)^{1-j}$, $j \in \{0, 1\}$; i.e., $\alpha = -\beta = \frac{p}{1-p}$, $0 \le p < 1$). The zero-one distribution is invariant in regard to the summation process

$$P_x = \sum_{j=x}^{1} \left(1 - \frac{p(1-j)}{(1-p)(1+j)}\right) P_j^* .$$

II. Katz partial summations family

Definition II.1. The family of probability distributions with the probability mass function (pmf.)

$$P_x = c \sum_{j=x}^{\infty} \left(1 - \frac{\alpha + \beta j}{1+j}\right) P_j^* , \qquad x = 0, 1, 2, \ldots , \tag{2}$$

where $\alpha \ge 0$, $\beta < 1$, c is a proper constant and $\{P_j^*\}_{j=0}^{\infty}$ is a proper pmf. with the mean $\mu^* < \infty$ will be called the *Katz partial summations family* $(\alpha, \beta, \{P_j^*\}_{j=0}^{\infty})$.

Remark II.2. In general $\{P_j^*\}_{j=0}^{\infty}$ in the summation process (2) cannot be an arbitrary pmf. The resulting $\{P_j\}_{j=0}^{\infty}$ is a pmf. if and only if the sequence $\{S_x\}_{x=0}^{\infty} = \left\{\sum\limits_{j=x}^{\infty} (1 - \frac{\alpha+\beta j}{1+j}) P_j^*\right\}_{x=0}^{\infty}$ does not change the sign. If $\alpha \le 1$ then $\{P_j^*\}_{j=0}^{\infty}$ may be an arbitrary pmf. except for the case $\alpha = 1$, $P_0^* = 1$. For

$\alpha > 1$ the function $g(j) = 1 - \frac{\alpha+\beta j}{1+j}$ is increasing. Hence $\{S_x\}_{x=0}^{\infty}$ is non-negative if and only if $\sum\limits_{j=0}^{\infty}(1-\frac{\alpha+\beta j}{1+j})P_j^* \geq 0$. $\{S_x\}_{x=0}^{\infty}$ can be non-positive only under the condition that $P_{k+n}^* = 0$, $n = 1, 2, \dots$, k being such a number that $g(k) \leq 0$ and $g(k+1) > 0$.

Lemma II.3. *For c in (2) it holds*

$$c = \left[1 - \alpha + (1-\beta)\mu^*\right]^{-1}.$$

Proof.

$$\begin{aligned} 1 &= \sum_{j=0}^{\infty} P_j \\ &= c\left[(1-\alpha)P_0^* + 2\left(1 - \frac{\alpha+\beta}{2}\right)P_1^* + 3\left(1 - \frac{\alpha+2\beta}{3}\right)P_2^* + \dots\right] \\ &= c\left[(1-\alpha)\sum_{j=0}^{\infty} P_j^* + (1-\beta)\sum_{j=1}^{\infty} jP_j^*\right], \end{aligned}$$

$$\Longrightarrow \quad c = \left[1 - \alpha + (1-\beta)\mu^*\right]^{-1}.$$

□

Remark II.4. If $\{P_j^*\}_{j=0}^{\infty}$ is a distribution from the Katz family and is invariant in regard to the summation (2), then according to Theorem I.1 $c = 1$ and for such a distribution $\mu^* = \frac{\alpha}{1-\beta}$ (see [1], p. 78).

Let

$$G^*(t) = \sum_{j=0}^{\infty} P_j^* t^j,$$

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be the probability generating functions of $\{P_j^*\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$, respectively.

Theorem II.5.

$$G(t) = c\,\frac{(1-\beta)(1-tG^*(t)) - (\alpha-\beta)\int_t^1 G^*(z)\,dz}{1-t}.$$

Proof.

$$\begin{aligned}
G(t) &= \sum_{j=0}^{\infty} P_j t^j = c\sum_{j=0}^{\infty} t^j \sum_{x=j}^{\infty}\left(1-\frac{\alpha+\beta x}{1+x}\right)P_x^* \\
&= c\sum_{j=0}^{\infty}\left(1-\frac{\alpha+\beta j}{1+j}\right)P_j^* \sum_{x=0}^{j} t^x \\
&= \frac{c}{1-t}\sum_{j=0}^{\infty} P_j^*\left(1-\frac{\alpha+\beta j}{1+j}\right)(1-t^{j+1}) \\
&= c\,\frac{1-tG^*(t)-\int\limits_t^1 \sum\limits_{j=0}^{\infty} P_j^*(\alpha+\beta j)z^j\,dz}{1-t} \\
&= c\,\frac{(1-\beta)(1-tG^*(t))-(\alpha-\beta)\int_t^1 G^*(z)\,dz}{1-t}\,.
\end{aligned}$$

□

Let

$$J^*(t) = \sum_{j=0}^{\infty}\frac{{}^*\mu'_{[j]}t^j}{j!}\,,$$

$$J(t) = \sum_{j=0}^{\infty}\frac{\mu'_{[j]}t^j}{j!}$$

be the generating functions of the descending factorial moments of the distributions $\{P_j^*\}_{j=0}^{\infty}$ and $\{P_j\}_{j=0}^{\infty}$, respectively (see [1], p. 49). Let us suppose that these functions exist.

Theorem II.6. *For $r = 1, 2, \ldots$ we have*

$$\mu'_{[r]} = \frac{c}{r+1}\left[(1-\beta)^*\mu'_{[r+1]}(1-\alpha-\beta r + r^*\mu'_{[r]})\right].$$

Proof. The following equations hold (see, e.g., [1]):

$$J(t) = G(t+1)\,,$$

$$\mu'_{[r]} = J^{(r)}(t)\big|_{t=0}\,.$$

According to Theorem II.5 we have

$$tJ(t) = c\left[-(1-\beta)\big(1-(t+1)J^*(t)\big)-(\alpha-\beta)\int\limits_1^{t+1} G^*(z)\,dz\right].$$

Deriving $(r+1)$-times the last equation and substituting 0 for t we obtain

$$\begin{aligned} \big(tJ(t)\big)^{(r+1)}\Big|_{t=0} &= (r+1)\mu'_{[r]}\,, \\ -\big[1-(t+1)J^*(t)\big]^{(r+1)}\Big|_{t=0} &= {}^*\mu'_{[r+1]} + (r+1)\,{}^*\mu'_{[r]}\,, \\ \left(\int_0^{t+1} G^*(z)\,dz\right)^{(r+1)}\Bigg|_{t=0} &= {}^*\mu'_{[r]} \end{aligned}$$

and the proof can be easily completed. □

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