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ω-PRIMITIVES ON σ-DISCRETE METRIC SPACES

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Dedicated to Professor Mišík

ABSTRACT. This paper continues the investigation started in [Z. Duszyński, Z. Grande, S. P. Ponomarev: *On the ω-primitive*, Math. Slovaca, **51** (2001), 469–476], [J. Ewert, S. P. Ponomarev: *Oscillation and ω-primitives*, Real Anal. Exchange, **26** (2000–2001), 687–702], [P. Kostyrko: *Some properties of oscillation*, Math. Slovaca, **30** (1980), 157–162], in the case of σ-discrete metric spaces. It is shown that given an upper semicontinuous function $f\colon X \to [0,\infty]$, where X is a σ-discrete dense in itself metric space, there exists a function $F\colon X \to \mathbb{R}$ (called an ω-primitive for f) whose oscillation equals f.

1. Preliminaries

Let (X,d) be a metric space. Given a function $F\colon X \to \mathbb{R}$, it will be convenient to define its oscillation at each $x \in X$ via the upper and lower Baire functions

$$M(F,x) := \lim_{r\to 0}\bigl(\sup\{F(y)\colon y \in B(x,r)\}\bigr)\,, \tag{1}$$

$$m(F,x) := \lim_{r\to 0}\bigl(\inf\{F(y)\colon y \in B(x,r)\}\bigr)\,, \tag{2}$$

where $B(x,r) := \bigl\{y \in X\colon d(x,y) < r\bigr\}$.

Then the oscillation of F at each x is defined as

$$\omega(F,x) = M(F,x) - m(F,x)\,, \tag{3}$$

along with the convention

$$\infty - \infty = -\infty - (-\infty) = 0\,, \quad \infty - (-\infty) = \infty\,.$$

It is well known that $\omega(F,\cdot)\colon X \to [0,\infty]$ is an upper semicontinuous (we shall abbreviate: USC) function.

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By the upper limit of F at x we mean

$$\limsup_{t\to x} F(t) := \lim_{r\to 0}\bigl(\sup\{F(y)\colon y\in B(x,r)\setminus\{x\}\}\bigr)\,. \tag{4}$$

The lower limit is defined analogously. By E^d we denote the derived set of $E\subset X$, and we let $\operatorname{dist}(x,E)=\inf\{d(x,y)\colon y\in E\}$. In the sequel (cf. Theorem 1) we also deal with lower semicontinuous functions, abbreviated: LSC. The closure of E is denoted by $\operatorname{cl} E$.

The general question we study is this. Given a USC function $f\colon X\to[0,\infty]$ on a metric space X, does there exist $F\colon X\to\mathbb{R}$ such that $\omega(F,x)$ equals $f(x)$ at each $x\in X$? If such an F exists we call it an ω-primitive for f (cf. [1]). The positive answer to this problem was given in [3] in the case of Baire metric spaces and, quite recently in [1], in the case of the so-called massive metric spaces.

In the present paper we restrict our study of ω-primitives exclusively to spaces dense in themselves. Note that an ω-primitive for f does not exist if f is positive at an isolated point of X.

We recall the basic definitions from [1].

A metric space (X,d) is said to be

- *σ-discrete at a point* $x\in X$ if there is a neighborhood W of x such that $W=\bigcup\limits_{n=1}^{\infty} A_n$ where each A_n is a discrete subset of X, an empty set is considered as discrete;
- *massive* if it is not σ-discrete at any of its points.

We note that each massive space is obviously dense in itself.

Now we will give a motivation and the purpose of the present paper.

It was shown in [1] (Theorems 1, 1', 2) that given any USC function $f\colon X\to[0,\infty]$, where X is a massive metric space, there exists an ω-primitive $F\colon X\to[0,\infty)$ for f in the form

$$F=f\varphi\,, \tag{5}$$

where φ is the characteristic function of a set $E\subset X$ which in particular can be chosen of type $\mathcal{F}_\sigma$ or $\mathcal{G}_\delta$, so that an ω-primitive F can always be found in at most Baire class two. Similar result was obtained in the case when X is a massive topological space satisfying some additional conditions [3].

But the situation changes if X is σ-discrete. Indeed, in that case an ω-primitive in the form of the product (5) might not exist as it is shown by the following simple example.

Example. Let $X=\mathbb{Q}$ be the set of all rational numbers with the usual metric. Define the USC function $f\colon X\to[0,\infty)$ letting $f(0)=2$ and $f\bigl(\frac{p}{q}\bigr)=1+\frac{1}{q}$ if the fraction $x=\frac{p}{q}\neq 0$ is irreducible.

It is clear that $X = \mathbb{Q}$ is a σ-discrete space and that at each $x \in X$ we have $\limsup_{t\to x} f(t) < f(x)$. This inequality implies, by a very elementary argument, that no function of the form (5), where φ is the characteristic function of a set, could be an ω-primitive for f.

Therefore in the case of σ-discrete spaces one had to look for new methods. We find it necessary to note that the idea of "proceeding inductively" in construction of ω-primitives in the case $X = \mathbb{Q}$ was first suggested by Jan Kolář in a private communication to the second of the authors.

In the present paper we give a construction of ω-primitives, in the case of an arbitrary σ-discrete metric space dense in itself. Our construction of an ω-primitive consists in a consecutive modification of a given USC function $f\colon X \to [0,\infty)$ at infinitely many points of X. The cardinality of X is of no importance.

2. Main results

Let (X,d) be an arbitrary metric space. For each nonempty set $E \subset X$ we let (cf. [1])

$$\Delta E = \begin{cases} +\infty & \text{if } \operatorname{card} E = 1\,, \\ \inf\{d(x,y)\colon\, x,y \in E\,, x \neq y\} & \text{if } \operatorname{card} E > 1\,. \end{cases}$$

A set $E \subset X$ will be called *dispersed* if $\Delta E > 0$. Notice that each dispersed set is closed. Moreover, if X is dense in itself, then each dispersed set $E \subset X$ is nowhere dense in X.

Lemma 1. *Each σ-discrete dense in itself metric space (X,d) can be represented in the form*

$$X = \bigcup_{n=1}^{\infty} A_n\,, \tag{6}$$

where A_n are nonempty, pairwise disjoint and dispersed.

Proof. This is an immediate consequence of Lemma 2 of [1] which says that each σ-discrete subset A of a metric space X can be represented in the form

$$A = \bigcup_{i\in I\subset\mathbb{N}} C_i\,, \tag{7}$$

where C_i are nonempty, pairwise disjoint and $\Delta C_i > 0$. Indeed, since by assumption X is dense in itself, it is easy to see that the set I of indexes in (7), written for $A = X$, should be infinite (cf. also Lemma 3), whence clearly that representation can be rewritten in the form (6), as required. $\square$

Corollary 1. *If X is a σ-discrete dense in itself metric space, then each function $F\colon X \to \mathbb{R}$ is at most Baire 1.*

Indeed, since each A_n in (6) is dispersed, we have that each subset of A_n is dispersed too, hence closed. Therefore all subsets of X, in particular sets of the form $\{x \in X\colon F(x) < a\}$, $\{x \in X\colon F(x) > a\}$, are $\mathcal{F}_\sigma$, whence clearly our claim follows.

It is maybe worth noting that the Corollary 1 remains valid if we drop the assumption that X is dense in itself. The unique reason for this assumption is that only such spaces are considered throughout the paper.

Lemma 2. *Let (X, d) be a dense in itself metric space, and $f\colon X \to [0, \infty)$ a* USC *function. Then the function $f^*\colon X \to [0, \infty)$ defined by*

$$f^*(x) = f(x) - \limsup_{t \to x} f(t)\,, \qquad x \in X\,, \tag{8}$$

has the following property: for each open set $W \subset X$

$$\inf\{f^*(x)\colon x \in W\} = 0\,. \tag{9}$$

P r o o f. Suppose the contrary. Then there is an open set $W_0 \subset X$ and $\varepsilon > 0$ such that $f^*(x) > \varepsilon$ for each $x \in W_0$. Let $x_0 \in W_0$ be fixed. Then

$$f(x_0) - \varepsilon > \limsup_{t \to x_0} f(t)\,.$$

It follows by (4) that there is $r > 0$ such that $B(x_0, r) \subset W_0$ and

$$\sup\{f(t)\colon t \in B(x_0, r) \setminus \{x_0\}\} < f(x_0) - \varepsilon\,.$$

Therefore there exists $x_1 \in B(x_0, r)$, $x_1 \neq x_0$, for which $f(x_1) < f(x_0) - \varepsilon$. Taking the point x_1 instead of x_0 and repeating the above arguments, we obtain a point $x_2 \in B(x_0, r)$ such that $f(x_2) < f(x_1) - \varepsilon$. Proceeding in that way, we construct inductively a sequence $\{x_n\colon n = 0, 1, 2, \dots\}$, $x_n \in W_0$, such that

$$f(x_n) - f(x_{n+1}) > \varepsilon$$

for $n = 0, 1, 2, \dots$ Since

$$f(x_0) - f(x_{n+1}) = \sum_{k=0}^{n} \bigl(f(x_k) - f(x_{k+1})\bigr) > (n + 1)\varepsilon\,,$$

we get $f(x_{n+1}) < f(x_0) - (n + 1)\varepsilon$. It follows that $f(x_n) < 0$ for all sufficiently large n, which is impossible since $f \geq 0$, and we are done. $\square$

Lemma 3. *Let $A_1, A_2, \dots, A_n$ be pairwise disjoint and dispersed subsets of an arbitrary metric space (X,d). Then there exists a family of pairwise disjoint balls*

$$\mathcal{B} = \{B(x, r(x)) : x \in A_1 \cup \dots \cup A_n\}. \tag{10}$$

P r o o f. For each $x \in A_1 \cup \dots \cup A_n$, it suffices to take $r(x) = \frac{1}{2} \min\{\operatorname{dist}(x, A_j \setminus \{x\}) : j = 1, \dots, n\}$. □

Lemma 4. *Let (X,d) be a dense in itself metric space, $G \subset X$, $\Delta G > 0$, and a family $\{B(x, r(x)) : x \in G\}$ of pairwise disjoint open balls be given such that $r(x) < \Delta G$, $x \in G$. Assume that to each $x \in G$ there corresponds a point $\xi = \xi_x$ satisfying the inequality*

$$0 < d(\xi, x) < r(x)/3. \tag{11}$$

Then the set $A = \{\xi_x : x \in G\}$ of all such ξ is dispersed, more precisely,

$$\Delta A \geq \frac{1}{3} \Delta G. \tag{12}$$

P r o o f. Take any two points $\xi, \xi' \in A$. By our assumption and (11) they belong to disjoint balls, say $\xi \in B(x, r(x)/3)$ and $\xi' \in B(x', r(x')/3)$. It follows readily

$$\begin{aligned} 0 < \Delta G \leq d(x, x') &\leq d(x, \xi) + d(\xi, \xi') + d(\xi', x') \\ &< \frac{1}{3}\bigl(r(x) + r(x')\bigr) + d(\xi, \xi') < \frac{2}{3} \Delta G + d(\xi, \xi') \end{aligned}$$

which shows that

$$\forall\, \xi,\ \xi' \in A : d(\xi, \xi') > \frac{1}{3} \Delta G$$

whence (12) follows. □

Now we may pass to our first main result.

Theorem 1. *Let (X,d) be a σ-discrete dense in itself metric space and $f : X \to [0, \infty)$ a USC function. Then, given any lower semicontinuous function $g : X \to (0, \infty)$, there exist a dense set $S \subset X$ and a function $F : X \to \mathbb{R}$, at most Baire 1, such that*

$$\omega(F, \cdot) = f, \qquad -g < F \leq f, \qquad F|S \leq 0. \tag{13}$$

P r o o f. By Lemma 1, the space X can be written in the form

$$X = \bigcup_{n=1}^{\infty} A_n, \tag{14}$$

where $\Delta A_n > 0$ for all n.

The construction of an ω-primitive F will be made inductively, each induction step consisting, in a very general outline, of the three operations:

- adding of new points to the set already defined on a previous step;
- applying Lemma 3 to the obtained set, so to have all its points separated by disjoint balls;
- determining the values of F (yet defined on the previous set) at new points.

First induction step. Since $\Delta A_1 > 0$ there is a number $r_1 > 0$ such that $\mathcal{B}_1 = \{B(x, r_1) \colon x \in A_1\}$ is the family of pairwise disjoint balls. We may always assume that

$$0 < r_1 < \frac{1}{10} \min\{\Delta A_1,\ 1\}. \tag{15}$$

Set $X_1 := A_1$. Since $g > 0$ is LSC then for each $x \in X_1$ there is a positive number $r_x < r_1$ such that

$$\forall x' \in B(x, r_x) \colon g(x') > g(x)/2. \tag{16}$$

Using Lemma 2 we can fix $\xi \in B(x, r_x)$ such that

$$0 < d(x, \xi) < \min\{r_1, g(x)/2\} \tag{17}$$

and

$$f^*(\xi) < \frac{1}{10} \min\{1, g(x)/2\},$$

whence by (16) we get

$$0 < d(x, \xi) < \min\{r_1, g(\xi)\}, \tag{18}$$

$$f^*(\xi) < \frac{1}{10} \min\{1, g(\xi)\}. \tag{19}$$

Denote by $X_{1,1}$ the set of all ξ thus chosen; i.e., satisfying (18), (19). By Lemma 4, it follows from (15) and (17) that the set $X_{1,1}$ is dispersed.

Denote

$$\mathcal{R}_1 := \{X_1, X_{1,1}\}, \quad Y_1 := X_1 \cup X_{1,1}, \tag{20}$$

and define the function $F_1 \colon Y_1 \to \mathbb{R}$ letting

$$F_1(x) = \begin{cases} f(x) & \text{if } x \in X_1, \\ -f^*(x) & \text{if } x \in X_{1,1}. \end{cases}$$

But the "typical" induction step is the second one.

Second induction step. Let $X_2 = A_2 \setminus Y_1$. It may happen that $X_2 = \emptyset$. In that case we consider $X_2 := A_j \setminus Y_1$ where $j = \min\{k \colon A_k \setminus Y_1 \neq \emptyset\}$. Notice that such j exists, for otherwise we would get $X = X_1 \cup X_{1,1}$ which is impossible since X is dense in itself (cf., e.g., Lemma 3). So without loss of generality (and to simplify notations) we may assume that $X_2 = A_2 \setminus Y_1 \neq \emptyset$. Thus we have that the sets

$$X_1,\ X_{1,1}\,,\ X_2$$

are disjoint and dispersed.

Therefore by Lemma 3 there is a family

$$\mathcal{B}_2 = \{B(x, r_2(x)) \colon x \in Y_1 \cup X_2\} \tag{21}$$

consisting of disjoint open balls. It is clear that we may also assume that

$$\forall x \in Y_1 \cup X_2 \colon 0 < r_2(x) < 10^{-2} \min\{r_1,\ \Delta X_1,\ \Delta X_{1,1},\ \Delta X_2\}\,. \tag{22}$$

Now using the lower semicontinuity of g, Lemma 2 and the family (21) we apply the arguments of the first step to X_1 and X_2. Finally, for each $x \in X_1$ we fix a point $\xi \in B(x, r_2(x))$ satisfying

$$0 < d(x, \xi) < \min\{r_2(x)\,,\ g(\xi)\}\,, \tag{23}$$

$$f^*(\xi) < 10^{-2} \min\{1, g(\xi)\}\,. \tag{24}$$

Denote the set of all such ξ by $X_{1,2}$.

Exactly in the same way, applying the arguments of the first step to X_2, we obtain the corresponding set $X_{2,2}$ whose points ξ satisfy the same relations (23), (24). Using the same arguments as in the first induction step, we obtain from the above construction and Lemma 4 that the disjoint sets $X_{1,2}, X_{2,2}$ are also dispersed.

Yet we need to apply one more construction in the second step, which will also become typical in all further steps.

Namely, since X is dense in itself, then for each $\xi \in X_{1,1}$ there is a sequence $\{t_m \colon m \in \mathbb{N}\}$, $t_m \to \xi$, such that

$$\lim_{m\to\infty} f(t_m) = \limsup_{t\to\xi} f(t)\,.$$

Therefore, for each $\xi \in X_{1,1}$ a point $\eta = \eta_\xi$ can be fixed, so that the following relations hold

$$0 < d(\eta, \xi) < r_2(\xi)\,, \qquad \Big| f(\eta) - \limsup_{t\to\xi} f(t) \Big| < 10^{-2}. \tag{25}$$

Denote by $X_{1,1,2}$ the set of all $\eta = \eta_\xi$ defined in that way for all $\xi \in X_{1,1}$. As previously, it follows from our constructions by Lemma 4 that the set $X_{1,1,2}$ is dispersed.

Thus on the second induction step we get the family

$$\mathcal{R}_2 = \{X_1, X_2, X_{1,1}, X_{1,2}, X_{1,1,2}, X_{2,2}\} = \mathcal{R}_1 \cup \{X_{1,2}, X_{1,1,2}, X_2, X_{2,2}\}$$

composed of disjoint dispersed sets.

Denote by Y_2 the union of all sets belonging to the family $\mathcal{R}_2$.

Now define the function $F_2 \colon Y_2 \to \mathbb{R}$ letting

$$F_2(x) = \begin{cases} f(x) & \text{if } x \in X_1 \cup X_2 \cup X_{1,1,2}\,, \\ -f^*(x) & \text{if } x \in X_{1,1} \cup X_{1,2} \cup X_{2,2}\,. \end{cases}$$

Evidently $F_2|Y_1 = F_1$. In order to give the reader more feeling about the idea of the whole construction, we will describe the

Third induction step. Let $X_3 := A_3 \setminus Y_2$. To simplify notations, we will assume just as on step two, that $X_3 \neq \emptyset$. It is clear that the family $\mathcal{R}_2 \cup \{X_3\}$ consists of pairwise disjoint dispersed sets. Therefore by Lemma 3 there exists a family

$$\mathcal{B}_3 = \bigl\{B\bigl(x, r_3(x)\bigr) \colon x \in Y_2 \cup X_3\bigr\}$$

of pairwise disjoint open balls such that

$$\forall x \in Y_2 \cup X_3 \colon 0 < r_3(x) < 10^{-3} \min\bigl\{r_1,\, \Delta E \colon E \in \mathcal{R}_2 \cup \{X_3\}\bigr\}\,. \tag{26}$$

Using Lemma 2 we can fix, for each $x \in X_1 \cup X_2 \cup X_3 \cup X_{1,1,2}$ a point $\xi = \xi_x$ such that

$$0 < d(x, \xi) < \min\bigl\{r_3(x)\,,\, g(\xi)\bigr\}\,, \quad f^*(\xi) < 10^{-3} \min\bigl\{1, g(\xi)\bigr\}\,. \tag{27}$$

Denote by $X_{1,3}, X_{2,3}, X_{3,3}, X_{1,1,2,3}$ the sets of thus obtained points ξ satisfying (27) and corresponding to the sets $X_1, X_2, X_3, X_{1,1,2}$ respectively.

Since the space X is dense in itself, for each $\xi \in X_{1,1} \cup X_{1,2} \cup X_{2,2}$ there is a sequence $\{t_m \colon m \in \mathbb{N}\}$ such that $\xi = \lim\limits_{m\to\infty} t_m$ and $\lim\limits_{m\to\infty} f(t_m) = \limsup\limits_{t\to\xi} f(t)$. Therefore for each $\xi \in X_{1,1} \cup X_{1,2} \cup X_{2,2}$ we can choose a point $\eta = \eta_\xi$ such that

$$0 < d(\eta, \xi) < r_3(\xi)\,, \quad \bigl|f(\eta) - \limsup_{t\to\xi} f(t)\bigr| < 10^{-3}. \tag{28}$$

Quite analogously to the preceding item, we denote by $X_{1,1,3}, X_{1,2,3}, X_{2,2,3}$ the sets of the points η satisfying relations (28) and corresponding to the sets $X_{1,1}, X_{1,2}, X_{2,2}$ respectively.

As a result, on the third step we obtain the family

$$\mathcal{R}_3 := \mathcal{R}_2 \cup \{X_{1,3}, X_{1,1,3}, X_{1,2,3}, X_{1,1,2,3}, X_{2,3}, X_{2,2,3}, X_3, X_{3,3}\} \tag{29}$$

of pairwise disjoint dispersed sets. Let Y_3 be the union of all sets belonging to $\mathcal{R}_3$. Define the function $F_3 \colon Y_3 \to \mathbb{R}$ as follows

$$F_3(x) = \begin{cases} f(x) & \text{if } x \in X_1 \cup X_2 \cup X_3 \cup X_{1,1,2} \cup X_{1,1,3} \cup X_{1,2,3} \cup X_{2,2,3}\,, \\ -f^*(x) & \text{if } x \in X_{1,1} \cup X_{1,2} \cup X_{1,3} \cup X_{1,1,2,3} \cup X_{2,2} \cup X_{2,3} \cup X_{3,3}\,. \end{cases}$$

We have obviously $F_3|Y_2 = F_2$ and this completes the third induction step.

Now we will pass to the general situation.

Induction assumptions. Suppose that for each natural k, $k \le n$, we have constructed the following objects.

(I). The family

$$\mathcal{R}_k = \{X_1, \dots, X_k, X_{k,k}\} \cup \{X_{p,i_1\dots i_m} : p = 1, \dots, k,\ p \le i_1 < \cdots < i_m \le k\} \tag{30}$$

of pairwise disjoint dispersed subsets of X.

(II). The set Y_k defined as the union of all sets belonging to the family $\mathcal{R}_k$.

(III). The family

$$\mathcal{B}_k = \{B(x, r_k(x)) : x \in Y_{k-1} \cup X_k\} \tag{31}$$

of pairwise disjoint open balls, where $X_k := A_k \setminus Y_{k-1}$, such that

$$0 < r_k(x) < 10^{-k} \min\{r_1,\ \Delta E \colon E \in \mathcal{R}_{k-1} \cup \{X_k\}\} \tag{32}$$

holds for $k = 2, \dots, n$ and $x \in Y_{k-1} \cup X_k$.

(IV). For each $z \in Y_k \setminus Y_{k-1}$ there exist exactly one $B(x, r_k(x)) \in \mathcal{B}_k$ and exactly one set $X_{p,i_1\dots i_m,k} \in \mathcal{R}_k$ such that the following conditions are satisfied

(a) $x \in X_{p,i_1\dots i_m}$ and $z \in B(x, r_k(x))$;

(b) $0 < d(z, x) < \min\{r_k(x), g(z)\}$ and $f^*(z) < 10^{-k} \min\{1, g(z)\}$ if m is even;

(c) $0 < d(z, x) < r_k(x)$ and $\left|f(z) - \limsup\limits_{t \to x} f(t)\right| < 10^{-k}$ if m is odd.

(V). The function $F_k \colon Y_k \to \mathbb{R}$, such that $F_k|Y_{k-1} = F_{k-1}$, defined by $F_k(x) = f(x)$ if $x \in \bigcup\limits_{p=1}^{k} X_p \cup \bigcup\limits_{p=1}^{k} \bigcup\{X_{p,i_1\dots i_m} : p \le i_1 < \cdots < i_m \le k\,, m \text{ is even}\}$, and $F_k(x) = -f^*(x)$ if $x \in \bigcup\limits_{p=1}^{k} \bigcup\{X_{p,i_1\dots i_m} : p \le i_1 < \cdots < i_m \le k\,,\ m \text{ is odd}\}$.

It follows from (I), (II) that

$$Y_k \setminus Y_{k-1} = \{X_k \cup X_{k,k}\} \cup \{X_{p,i_1\dots i_m,k} \colon 1 \le p \le k-1, p \le i_1 < \cdots < i_m \le k-1\} \tag{33}$$

for $1 \le k \le n$. We let $Y_0 := \emptyset$.

Next we will define $\mathcal{R}_{n+1}$ and F_{n+1}. Setting $X_{n+1} = A_{n+1} \setminus Y_n$ we obtain the family $\mathcal{R}_n \cup \{X_{n+1}\}$ of pairwise disjoint dispersed sets. So by Lemma 3 there exists a family $\mathcal{B}_{n+1} = \{B(x, r_{n+1}(x)) \colon x \in Y_n \cup X_{n+1}\}$ of pairwise disjoint balls such that

$$\forall x \in Y_n \cup X_{n+1} \colon 0 < r_{n+1}(x) < 10^{-n-1} \min\{r_1,\ \Delta E \colon E \in \mathcal{R}_n \cup \{X_{n+1}\}\}.$$

Then, applying Lemma 2, for each point $x \in X_p$, $p = 1, 2, \ldots, n+1$, and for each $x \in X_{p,i_1\ldots i_m}$, $p = 1, \ldots, n$, $p \le i_1 < \cdots < i_m \le n$, if m is even, we fix a point $\xi = \xi_x$ such that

$$0 < d(x, \xi) < \min\{r_{n+1}(x),\, g(\xi)\}, \quad f^*(\xi) < 10^{-n-1} \min\{1,\, g(\xi)\}.$$

Denote by $X_{p,n+1}$ and $X_{p,i_1\ldots i_m,n+1}$ respectively the sets of points ξ thus chosen. Furthermore, for each $\xi \in X_{p,i_1\ldots i_m}$, with m odd, we choose, in the same way as it has been made on steps two and three (cf. (25), (28)), a point $\eta = \eta_\xi$ satisfying

$$0 < d(\eta, \xi) < r_{n+1}(\xi), \quad \bigl|f(\eta) - \limsup_{t\to\xi} f(t)\bigr| < 10^{-n-1}, \tag{34}$$

and we will denote by $X_{p,i_1\ldots i_m,n+1}$ the set of all such η.

Therefore, using Lemma 4, we get the family

$$\mathcal{R}_{n+1} := \mathcal{R}_n \cup \{X_{n+1}, X_{n+1,n+1}\} \cup \{X_{p,i_1\ldots i_m,n+1} \colon X_{p,i_1\ldots i_m} \in \mathcal{R}_n\} \tag{35}$$

consisting of pairwise disjoint dispersed sets. Denote by Y_{n+1} the union of all sets from $\mathcal{R}_{n+1}$.

Next we define the function $F_{n+1} \colon Y_{n+1} \to \mathbb{R}$ as follows:
$F_{n+1}(x) = F_n(x)$, if $x \in Y_n$;
$F_{n+1}(x) = f(x)$,
 if $x \in X_{n+1} \cup \bigcup\{X_{p,i_1\ldots i_m,n+1} \colon X_{p,i_1\ldots i_m} \in \mathcal{R}_n, m \text{ is odd}\}$;
$F_{n+1}(x) = -f^*(x)$,
 if $x \in X_{n+1,n+1} \cup \bigcup\{X_{p,i_1\ldots i_m,n+1} \colon X_{p,i_1\ldots i_m} \in \mathcal{R}_n, m \text{ is even}\}$.

It is not difficult to check that we have thus constructed all the objects satisfying conditions (I)–(V) for $n+1$. Therefore, by induction principle, we obtain the infinite sequences of such objects satisfying the above conditions for all n. Since $A_n \subset Y_n$, the equality

$$X = \bigcup_{n=1}^{\infty} Y_n = Y_1 \cup \bigcup_{n=2}^{\infty} (Y_n \setminus Y_{n-1})$$

holds, all the terms on the right being disjoint (cf. also (33)).

Now we define the function $F \colon X \to \mathbb{R}$ letting $F|Y_n := F_n$ for each $n \in \mathbb{N}$. It follows at once from our construction that $-g < F \le f$. We will show soon that F is indeed an ω-primitive for f. But before proving it, we are going to establish some auxiliary results.

Denote by S the union of all the sets $X_{p,i_1\ldots i_m}$ with m odd. First we will show that

$$S \text{ is dense in } X. \tag{36}$$

To this end, let $z \in X$ and $B(z,r)$ be fixed. There exist a unique natural $k \geq 1$ and a unique set in $\mathcal{R}_k$ containing z. This set can be either of the form X_p for some $p \in \{1,\dots,k\}$ or of the form $X_{p,i_1\dots i_m}$, where $1 \leq p \leq k$, $p \leq i_1 < \dots < i_m \leq k$.

If $z \in X_{p,i_1\dots i_m}$ with m odd, then $z \in S$, so $B(z,r) \cap S \neq \emptyset$. Suppose $z \in X_{p,i_1\dots i_m}$ with m even or $z \in X_p$. Since $\lim\limits_{j\to\infty} r_j(z) = 0$, there is $j_0 > k$ such that $B\big(z, r_j(z)\big) \subset B(z,r)$ for $j > j_0$. Fix any $j > j_0$. Since $Y_k \subset Y_j$, we have, by the property of the family $\mathcal{B}_{j+1}$, that $B\big(z, r_{j+1}(z)\big) \cap (Y_j \cup X_{j+1}) = \{z\}$. There exists a point $\xi \in B\big(z, r_{j+1}(z)\big) \cap Y_{j+1}$ such that $0 < d(z,\xi) < r_{j+1}(z)$. Then $\xi \in X_{p,i_1\dots i_m,j+1} \subset S$ or $\xi \in X_{p,j+1} \subset S$, so we have $S \cap B(z,r) \neq \emptyset$ which yields (36).

Next we will prove that

$$\forall z \in X\colon \liminf_{x\to z} F(x) = 0\,. \tag{37}$$

Let $z \in X$. Since the function $r \mapsto \inf\big\{F(x)\colon x \in B(z,r) \setminus \{z\}\big\}$ is monotone and $\lim\limits_{j\to\infty} r_j(z) = 0$, it is sufficient to prove that

$$\lim_{j\to\infty} \Big(\inf\big\{F(x)\colon x \in B\big(z, r_j(z)\big) \setminus \{z\}\big\}\Big) = 0\,. \tag{38}$$

From the properties of the families $\mathcal{B}_j$ we get

$$\forall j \in \mathbb{N}\colon \big(B\big(z, r_j(z)\big) \setminus \{z\}\big) \cap (Y_{j-1} \cup X_j) = \emptyset\,.$$

This implies

$$B\big(z, r_j(z)\big) \setminus \{z\} = \bigcup_{k=j}^{\infty} \big(B\big(z, r_j(z)\big) \setminus \{z\}\big) \cap (Y_k \setminus Y_{k-1})\,.$$

Since we may write (cf. (33))

$$Y_k \setminus Y_{k-1} = X_k \cup X_{k,k} \cup \bigcup \{X_{p,i_1\dots i_m,k}\colon X_{p,i_1\dots i_m} \in \mathcal{R}_{k-1}\}\,,$$

we get

$$S \cap (Y_k \setminus Y_{k-1}) = X_{k,k} \cup \bigcup \{X_{p,i_1\dots i_m,k}\colon X_{p,i_1\dots i_m} \in \mathcal{R}_{k-1},\ m \text{ is even}\}\,.$$

Hence $0 \geq F(\xi) > -10^{-k}$ for all $\xi \in S \cap (Y_k \setminus Y_{k-1})$.

So we have shown that

$$\forall j \in \mathbb{N}\colon 0 \geq \inf\big\{F(x)\colon x \in B\big(z, r_j(z)\big) \setminus \{z\}\big\} \geq -10^{-j}.$$

This obviously yields (38), whence (37) follows.

Finally we shall show now that F is an ω-primitive for f. Let $x \in X$. There is a unique set $X_{p,i_1\dots i_m} \in \mathcal{R}_{i_m}$ containing x. Since the sequence $\{\mathcal{R}_n : n \in \mathbb{N}\}$ is increasing, we have $x \in X_{p,i_1\dots i_m} \in \mathcal{R}_j$ for each $j > i_m$.

Suppose first that m is odd. Then the following holds

$$-10^{-i_m} < F(x) = -f^*(x) = -\big(f(x) - \limsup_{t\to x} f(t)\big) \le 0\,.$$

Consider the sequence of balls $\big\{B\big(x, r_j(x)\big) : j = i_m, i_m+1, \dots\big\}$. Since m is odd, it follows from our construction (cf., e.g., (34)) that there is a sequence $\{\eta_j : j = i_m, i_m+1, \dots\}$, $\eta_j \in X_{p,i_1\dots i_m,j}$, such that

$$0 < d(x, \eta_j) < r_j(x)\,, \qquad \big|f(\eta_j) - \limsup_{t\to x} f(t)\big| < 10^{-j}.$$

Then we have $F(\eta_j) = f(\eta_j)$. Since $r_j(x) \to 0$ as $j \to \infty$, we conclude that $\lim\limits_{j\to\infty} \eta_j = x$, $\eta_j \ne x$, and

$$\lim_{j\to\infty} F(\eta_j) = \limsup_{t\to x} f(t)\,.$$

Let $\varepsilon > 0$, $\delta > 0$ be given. There exists j_0 such that

$$\forall j \ge j_0 \colon d(x, \eta_j) < \delta \quad \text{and} \quad \big|F(\eta_j) - \limsup_{t\to x} f(t)\big| < \varepsilon\,.$$

This implies that

$$\begin{aligned} \forall j \ge j_0 \colon \limsup_{t\to x} f(t) - \varepsilon \le F(\eta_j) &\le \sup\big\{F(t) \colon t \in B(x,\delta)\big\} \\ &\le \sup\big\{f(t) \colon t \in B(x,\delta) \setminus \{x\}\big\}\,. \end{aligned}$$

Letting $\delta \to 0$ we obtain (cf. (1), (2), (3))

$$\limsup_{t\to x} f(t) - \varepsilon \le M(F,x) \le \limsup_{t\to x} f(t)\,,$$

and since $\varepsilon > 0$ is arbitrary, we thus get

$$M(F,x) = \limsup_{t\to x} f(t)\,.$$

In the considered case we have $F(x) < 0$, therefore from the last equality and (37) we obtain

$$m(F,x) = F(x) = -f(x) + \limsup_{t\to x} f(t) = -f(x) + M(F,x)\,.$$

Thus we get

$$\omega(F,x) = M(F,x) - m(F,x) = f(x)\,.$$

Finally, suppose that $x \in X_{p,i_1\dots i_m}$ with m even, or $x \in X_p$. In that case

$F(x) = f(x)$. Therefore, since $F \le f$ and f is USC, we get

$$f(x) = F(x) \le M(F,x) \le M(f,x) = f(x), \quad \text{hence} \quad M(F,x) = f(x). \tag{39}$$

Since $F(x) \ge 0$ and $F|S \le 0$, we may write, in view of (36), that

$$\inf\{F(t) \colon t \in B(x,\delta)\} = \inf\{F(t) \colon t \in B(x,\delta) \setminus \{x\}\}.$$

It follows by (37), letting $\delta \to 0$, that $m(F,x) = 0$, which yields by (39) that $\omega(F,x) = f(x)$.

We have thus shown that

$$\forall x \in X \colon \omega(F,x) = f(x),$$

i.e., F is an ω-primitive for f. It remains to prove that F is Baire 1. But this is immediate by Corollary 1 to Lemma 1. □

Now we are in a position to prove the analogous result in the case when f may take on the value $+\infty$. In the next proof we use Teichmüller–Tukey's lemma. Though equivalent to Zorn's lemma, technically it turned out to be useful in proofs of basic results in [1], [3].

Lemma 5. (Teichmüller-Tukey, [2]). *Let P be a property of finite character pertained to subsets of a set X. Then each set $E \subset X$ is contained in a maximal, with respect to the inclusion relation, set E^* having the same property P.*

We will consider, in a metric space (X,d), the property P_n of finite character defined for each natural n as follows:

$$E \text{ has the property } P_n \iff \Delta E \ge 1/n.$$

Sets $E \subset X$ which are maximal with respect to P_n will be called P_n-maximal sets.

Lemma 6. *Given any metric space (Y,d) dense in itself, there exists a sequence $\{D_k \colon k \in \mathbb{N}\}$, $D_k \subset Y$, such that*

(i) *D_k are mutually disjoint and $\Delta D_k \ge 1/k$ for each k;*

(ii) *$\operatorname{dist}(y, D_k) \le 1/k$ for each k and each $y \in Y$;*

(iii) *the set $D = \bigcup_{k=1}^{\infty} D_{2k}$ is dense and boundary in Y.*

Proof. Using Lemma 5, we fix a P_1-maximal set $D_1 \subset Y$. The set D_1 being closed and nowhere dense in Y (cf. Remark after the definition of ΔE), the subspace $Y \setminus D_1$ is dense in itself. So we may apply Lemma 5 to $Y \setminus D_1$ and find a P_2-maximal set $D_2 \subset Y \setminus D_1$. We have again that $Y \setminus (D_1 \cup D_2)$ is dense in itself. Proceeding in that way inductively, we obtain a sequence $\{D_k \colon k \in \mathbb{N}\}$ of pairwise disjoint subsets of Y with the following properties.

(a) $\Delta D_1 \geq 1$ and D_1 is P_1-maximal in Y;
$\Delta D_k \geq 1/k$ and D_k is P_k-maximal in $Y \setminus (D_1 \cup \dots \cup D_{k-1})$ for each $k \geq 2$.
(b) For each k the set $Y \setminus (D_1 \cup \dots \cup D_k)$ is dense in Y and dense in itself.

Now the claim (i) is evident from the construction of sets D_k. Note that D_k are closed, since $\Delta D_k > 0$.

The claim (ii) follows from the P_k-maximality of D_k. Indeed, otherwise there would exist a natural k_0 and a $y_0 \in Y$ such that $\operatorname{dist}(y_0, D_{k_0}) > 1/k_0$. But then the set $D_{k_0} \cup \{y_0\} \supset D_{k_0}$ would have the property P_{k_0}, contrary to the fact that D_{k_0} is already P_{k_0}-maximal.

It is immediate from (ii) that $D = \bigcup_{k=1}^{\infty} D_{2k}$ is dense in Y. But since $D_{2k+1} \cap D = \emptyset$, $k \in \mathbb{N}$, we conclude, again by (ii), that D is boundary. $\square$

THEOREM 2. *Let (X, d) be a σ-discrete dense in itself metric space. Then for each* USC *function $f\colon X \to [0, \infty]$ there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$.*

P r o o f. It is natural to consider the general case where $E := \{x \in X : f(x) = \infty\} \neq \emptyset$. Without loss of generality we may also assume that $X \setminus E \neq \emptyset$ and $\operatorname{Int} E \neq \emptyset$. Then X is the union of disjoint subspaces, namely

$$X = (X \setminus E) \cup \operatorname{Int} E \cup (E \setminus \operatorname{Int} E).$$

Since E is a closed subset of X, the subspace $X \setminus E$ and the function

$$f\big|(X \setminus E)\colon X \setminus E \to [0, \infty)$$

satisfy assumptions of Theorem 1. Therefore, for $g = 1$, there exists a function

$$F_1\colon X \setminus E \to \mathbb{R}$$

such that

$$-1 < F_1 \leq f\big|(X \setminus E)$$

and

$$\omega(F_1, \cdot) = f\big|(X \setminus E).$$

Next we will construct an ω-primitive for $f \mid \operatorname{Int} E$. First observe that the set $\operatorname{Int} E$ is dense in itself. Let $D = \bigcup_{k=1}^{\infty} D_{2k}$ be the set constructed by Lemma 6 for the subspace $\operatorname{Int} E$.

Define $F_2\colon \operatorname{Int} E \to [0, \infty)$ letting

$$F_2(x) = \begin{cases} 0 & \text{if } x \in \operatorname{Int} E \setminus D, \\ k & \text{if } x \in D_{2k}, \ k \in \mathbb{N}. \end{cases}$$

It is straightforward, by (ii) and (iii) of Lemma 6, that F_2 is an ω-primitive for $f \,|\, \mathrm{Int}\, E$.

Define the function $T\colon X \setminus E \to (0, \infty)$ by $T(x) = \big(\mathrm{dist}(x, E)\big)^{-1}$, which is obviously continuous. Finally, define the function

$$F(x) = \begin{cases} F_1(x) + T(x) & \text{if } x \in X \setminus E\,, \\ 0 & \text{if } x \in E \setminus \mathrm{Int}\, E\,, \\ F_2(x) & \text{if } x \in \mathrm{Int}\, E\,. \end{cases}$$

The verification of the fact that the function F thus constructed, is an ω-primitive for f, is immediate. □

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