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SOMEWHAT CONTINUITY IN A UNIFIED FRAMEWORK FOR CONTINUITIES OF RELATIONS

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ABSTRACT. By using a unifying scheme for continuities of relations, we show that our paratopological mild, upper and lower semicontinuities are natural generalizations of the feeble continuity of Zdeněk Frolík which was later studied by several authors under the name somewhat continuity.

Introduction

A nonvoid family $\mathcal{R}$ of relations on one nonvoid set X to another Y is called a *relator* on X to Y, and the ordered pair $(X,Y)(\mathcal{R}) = \big((X,Y),\mathcal{R}\big)$ is called a *relator space*. Note that relator spaces of the simpler type $X(\mathcal{R}) = (X,X)(\mathcal{R})$ are already substantial generalizations of ordered sets and uniform spaces.

If $\mathcal{R}$ is a relator on X to Y, then the members of the family

$$\mathcal{D}_{\mathcal{R}} = \big\{A \subset Y\colon \quad \forall x \in X \quad \forall R \in \mathcal{R}\colon R(x) \cap A \neq \emptyset\big\}$$

are called the dense subsets of $(X,Y)(\mathcal{R})$. Thus, $\mathcal{D}_{\mathcal{R}}$ is a proper stack (ascending system) in Y. And the dense subsets are frequently more important tools than the closed ones.

Moreover, if $\mathcal{R}$ is a relator on X to Y, then the relator

$$\mathcal{R}^{\triangle} = \big\{S \subset X \times Y\colon \quad \forall x \in X \quad \exists u \in X \quad \exists R \in \mathcal{R}\colon R(u) \subset S(x)\big\}$$

is called the *paratopological refinement* of $\mathcal{R}$. Thus, $\triangle$ is a closure operation on the family of all relators on X to Y such that, for every relator $\mathcal{R}$ on X to Y, $\mathcal{R}^{\triangle}$ is the largest relator on X to Y such that $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\triangle}}$. It is curious that, in general, there is no such operation for the closed sets.

If F is a relation on X to Y and G is a relation on Z to W, then we say that (F,G) is a pair of relations on (X,Z) to (Y,W). Moreover, if $\mathcal{R}$ is a relator

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on X to Y, then we write $\mathcal{R}^{-1}=\{R^{-1}: R\in\mathcal{R}\}$. And, if $\mathcal{S}$ is a relator on Y to Z, then we also write $\mathcal{S}\circ\mathcal{R}=\{S\circ R: R\in\mathcal{R},\ S\in\mathcal{S}\}$.

Now, a pair (F,G) of relations on one relator space $(X,Y)(\mathcal{R})$ to another $(Z,W)(\mathcal{S})$ is called *paratopologically mildly, upper* and *lower semicontinuous* if

$$G^{-1}\circ\mathcal{S}^{\Delta}\circ F\subset\mathcal{R}^{\Delta},\quad \mathcal{S}^{\Delta}\circ F\subset(G\circ\mathcal{R}^{\Delta})^{\Delta}\quad\text{and}\quad G^{-1}\circ\mathcal{S}^{\Delta}\subset(\mathcal{R}^{\Delta}\circ F^{-1})^{\Delta},$$

respectively. Moreover, the relation G is called *denseness preserving* if $A\in\mathcal{D}_{\mathcal{R}}$ implies $G(A)\in\mathcal{D}_{\mathcal{S}}$.

We show that, in some particular cases, the denseness preservingness of the relation G is equivalent to some of the paratopological semicontinuities of the pair (F,G). Therefore, the *feeble continuity* introduced first by Frolík [5] can also be included in our unifying scheme for continuities. Note that feeble continuity was later called somewhat continuity by Gentry and Hoyle [6], and it has been intensively investigated by several authors.

1. A few basic facts on relations and relators

A subset F of a product set $X\times Y$ is called a *relation* on X to Y. In particular, the relations $\Delta_X=\{(x,x): x\in X\}$ and $X^2=X\times X$ are called the identity and the universal relations on X, respectively.

Namely, if in particular $X=Y$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X\cup Y$. Therefore, it is frequently not a severe restriction to assume that $X=Y$.

If F is a relation on X to Y, and moreover $x\in X$ and $A\subset X$, then the sets $F(x)=\{y\in Y: (x,y)\in F\}$ and $F[A]=\bigcup_{x\in A}F(x)$ are called the *images* of x and A under F, respectively. Whenever $A\in X$ is unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X to Y, then the sets $D_F=\{x\in X: F(x)\neq\emptyset\}$ and $R_F=F(D_F)$ are called the *domain* and the *range* of F, respectively. Whenever, $X=D_F$ (and $Y=R_F$), we say that F is a relation of X *into* (*onto*) Y.

A relation F on X to Y is said to be a *function* if for each $x\in D_F$ there exists a unique $y\in Y$ such that $y\in F(x)$. In this case, by identifying singletons with their elements, we usually write $y=F(x)$ in place of $\{y\}=F(x)$.

If F is a relation on X to Y, then the values $F(x)$, where $x\in X$, uniquely determine F since we have $F=\bigcup_{x\in X}\{x\}\times F(x)$. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(y)=\{x\in X: y\in F(x)\}$ for all $y\in Y$. Note that thus we have $D_F=F^{-1}(Y)$.

Moreover, if F is a relation on X to Y and G is a relation on Y to Z, then the composition $G \circ F$ of G and F can be defined such that $(G \circ F)(x) = G\big(F(x)\big)$ for all $x \in X$. Note that thus we have $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

A relation R on X is called *reflexive*, *symmetric* and *transitive* if $\Delta_X \subset R$, $R^{-1} \subset R$, and $R \circ R \subset R$, respectively. Moreover, a reflexive relation is called a *preorder* (*tolerance*) relation if it is transitive (symmetric).

If R is a relation on X, then we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_X$. Moreover, we define $R^\infty = \bigcup\limits_{n=0}^{\infty} R^n$. Note that thus R^∞ is the smallest preorder relation on X such that $R \subset R^\infty$.

A nonvoid family $\mathcal{R}$ of relations on one nonvoid set X to another Y is called a *relator* on X to Y, and the ordered pair $(X,Y)(\mathcal{R}) = \big((X,Y),\mathcal{R}\big)$ is called a *relator space*. Whenever $X = Y$, by identifying singletons with their elements, we may write $X(\mathcal{R})$ in place of $(X,X)(\mathcal{R})$ since $(X,X) = \big\{\{X\}\big\}$.

Relator spaces of the latter, more simple type are already substantial generalizations of ordered sets and uniform spaces [20]. They deserve to be widely investigated because of the following two facts.

If $\mathcal{D}$ is a nonvoid family of certain distance functions on X, then the relator $\mathcal{R}_{\mathcal{D}}$ consisting of all surroundings $B_r^d = \big\{(x,y) \in X^2 : d(x,y) < r\big\}$, where $d \in \mathcal{D}$ and $r > 0$, is a more convenient mean of defining the basic notions of analysis in the space $X(\mathcal{D})$ than the family of all open subsets of $X(\mathcal{D})$, or even the family $\mathcal{D}$ itself.

Moreover, all reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be easily derived from relators (according to the results of [22] and [19]), and thus they need not be studied separately.

For instance, if $\mathcal{A}$ is a certain generalized topology or a nonvoid stack (ascending system) in X, then $\mathcal{A}$ can be easily derived (according to the forthcoming definitions of the families $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$) from the Davis–Pervin relator $\mathcal{R}_{\mathcal{A}}$ consisting of all preorders $R_A = A^2 \cup (X \setminus A) \times X$, where $A \in \mathcal{A}$.

Note that, in contrast to these preorders R_A, the surroundings B_r^d are usually tolerances. Therefore, besides preorder relators, tolerance relators are also important particular cases of reflexive relators. Unfortunately, the class of all reflexive relators proved insufficient for several important purposes.

2. Some important set-valued functions for relators

A function $\mathfrak{F}$ of the family of all relators on X to Y into a family of sets is called a *set-valued function* for relators on X to Y. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X to Y.

The set-valued function $\mathfrak{F}$ is called *increasing* (*decreasing*) if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y with $\mathcal{S} \subset \mathcal{R}$ we have $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$). Moreover, such a set-valued function $\mathfrak{F}$ is called *normal* if for every relator $\mathcal{R}$ on X to Y we have $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_R$ ($\mathfrak{F}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathfrak{F}_R$), where $\mathfrak{F}_R = \mathfrak{F}_{\{R\}}$.

If $\mathcal{R}$ is a relator on X to Y, then for any $x \in X$ and $A \subset Y$ we write:

(1) $x \in \operatorname{int}_{\mathcal{R}}(A)$ if $R(x) \subset A$ for some $R \in \mathcal{R}$;

(2) $x \in \operatorname{cl}_{\mathcal{R}}(A)$ if $R(x) \cap A \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $A \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset$;

(4) $A \in D_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) = X$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we write:

(5) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \operatorname{int}_{\mathcal{R}}(A)$;

(6) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subset A$.

The relation $\operatorname{int}_{\mathcal{R}}$ ($\operatorname{cl}_{\mathcal{R}}$) on $\mathcal{P}(Y)$ to X is called the interior (closure) induced by $\mathcal{R}$ on (X, Y). And the members of the families $\mathcal{E}_{\mathcal{R}}$ ($\mathcal{D}_{\mathcal{R}}$) and $\mathcal{T}_{\mathcal{R}}$ ($\mathcal{F}_{\mathcal{R}}$) are called the fat (dense) and the open (closed) subsets of $(X, Y)(\mathcal{R})$ and $X(\mathcal{R})$, respectively.

The interior and the fat sets are frequently more important tools than the open sets. For instance, if $\prec$ is a preorder on X, then $\mathcal{E}_{\prec}$ and $\mathcal{T}_{\prec}$ are precisely the families of all residual and ascending subsets of the preordered set $X(\prec)$, respectively.

Therefore, a function of a relator space may also be naturally called a net. And the theory of the convergence and adherence of the usual nets can naturally be extended. However, preordered nets proved sufficient for our purposes.

Concerning the above basic tools, we shall only quote here the following theorems which have been mostly established in [22] and [14].

Theorem 2.1. *If $\mathcal{R}$ is a relator on X to Y and $A \subset Y$, then*

$$\operatorname{cl}_{\mathcal{R}}(A) = X \setminus \operatorname{int}_{\mathcal{R}}(Y \setminus A).$$

Theorem 2.2. *If $\mathcal{R}$ is a relator on X to Y, then*

$$\mathcal{D}_{\mathcal{R}} = \left\{A \subset Y : Y \setminus A \notin \mathcal{E}_{\mathcal{R}}\right\} = \left\{A \subset Y : \forall B \in \mathcal{E}_{\mathcal{R}} : A \cap B \neq \emptyset\right\}.$$

Theorem 2.3. *If $\mathcal{R}$ is a relator on X to Y and $A \subset Y$, then*

$$\operatorname{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}(A) \qquad \text{and} \qquad \mathcal{D}_{\mathcal{R}} = \left\{A \subset Y : \forall R \in \mathcal{R} : R^{-1}(A) = X\right\}.$$

Theorem 2.4.

(1) int *and* $\mathcal{E}$ *are normal increasing set-valued functions for relators on* X *to* Y*;*

(2) cl *and* $\mathcal{D}$ *are normal decreasing set-valued functions for relators on* X *to* Y*.*

Remark 2.5. Unfortunately, the increasing set-valued functions $\mathcal{T}$ and $\mathcal{F}$ are not normal in general.

Moreover, if $\mathcal{R}$ is a relator on X, then in contrast to Theorem 2.2 we can only state that $\mathcal{F}_{\mathcal{R}} = \{A \subset X : X \setminus A \in \mathcal{T}_{\mathcal{R}}\}$.

3. Some important unary operations for relators

A function $\square$ of the family of all relators on X to Y into itself is called a *unary operation* for relators on X to Y. And we write $\mathcal{R}^{\square} = \square(\mathcal{R})$ for every relator $\mathcal{R}$ on X to Y.

The operation $\square$ is called *expansive* and *idempotent* if for every relator $\mathcal{R}$ on X to Y we have $\mathcal{R} \subset \mathcal{R}^{\square}$ and $\mathcal{R}^{\square} = \mathcal{R}^{\square\square}$, respectively. And the operation $\square$ is called *stable* if $\mathcal{R} = \mathcal{R}^{\square}$ whenever $\mathcal{R} = \{X \times Y\}$.

Moreover, an increasing and expansive (idempotent) operation is called an *extension* (*modification*) *operation.* And an idempotent extension operation is called a *refinement operation.*

A set-valued function $\mathfrak{F}$ for relators on X to Y is called $\square$*-increasing* ($\square$*-decreasing*) for some unary operation $\square$ for relators on X to Y if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$).

If $\mathfrak{F}$ is an increasing (decreasing) set-valued function for relators on X to Y, then the operation $\square_{\mathfrak{F}}$ defined by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \{S \subset X \times Y : \mathfrak{F}_S \subset \mathfrak{F}_{\mathcal{R}}\} \qquad \big(\mathcal{R}^{\square_{\mathfrak{F}}} = \{S \subset X \times Y : \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_S\}\big)$$

for every relator $\mathcal{R}$ on X to Y, is called the *operation induced by the set-valued function* $\mathfrak{F}$.

Moreover, a monotonic set-valued function $\mathfrak{F}$ for relators on X to Y is called *regular* if for every relator $\mathcal{R}$ on X to Y we have $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$. Note that thus a normal set-valued function is, in particular, regular.

The appropriateness of the above definitions is apparent from the following theorems which have been mostly proved in [14].

Theorem 3.1. *If* $\square$ *is a unary operation for relators on* X *to* Y*, then* $\square$ *is a refinement if and only if it is* $\square$*-increasing.*

THEOREM 3.2. *If $\square$ is an unary operation and $\mathfrak{F}$ is a $\square$-increasing ($\square$-decreasing) set-valued function for relators on X to Y, then*

(1) *$\square$ is a refinement and $\square = \square_{\mathfrak{F}}$;*

(2) *$\mathfrak{F}$ is increasing (decreasing) and regular;*

(3) *for every relator $\mathcal{R}$ on X to Y, $\mathcal{R}^{\square}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.*

THEOREM 3.3. *If $\mathfrak{F}$ is a regular increasing (decreasing) set-valued function for relators on X to Y, then $\mathfrak{F}$ is $\square_{\mathfrak{F}}$-increasing ($\square_{\mathfrak{F}}$-decreasing).*

Remark 3.4. If the increasing (decreasing) set-valued function $\mathfrak{F}$ fails to be regular, then we can only state that $\square_{\mathfrak{F}}$ is an extension operation for relators on X to Y such that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ implies $\mathcal{S} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$ ($\mathcal{R} \subset \mathcal{S}^{\square_{\mathfrak{F}}}$).

If $\mathcal{R}$ is a relator on X to Y, then the relators

$$\mathcal{R}^{\wedge} = \big\{ S \subset X \times Y : \forall x \in X : x \in \operatorname{int}_{\mathcal{R}}\big(S(x)\big) \big\}$$

and

$$\mathcal{R}^{\vartriangle} = \big\{ S \subset X \times Y : \forall x \in X : S(x) \in \mathcal{E}_{\mathcal{R}} \big\}$$

are called the *topological* and the *paratopological refinements* of $\mathcal{R}$, respectively.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^{\infty} = \{ R^{\infty} : R \in \mathcal{R} \} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \{ S \subset X^2 : S^{\infty} \in \mathcal{R} \}$$

are called the *direct* and the *inverse preorder modifications* of $\mathcal{R}$, respectively. And the relators $\mathcal{R}^{\wedge\infty} = (\mathcal{R}^{\wedge})^{\infty}$ and $\mathcal{R}^{\curlywedge} = (\mathcal{R}^{\wedge})^{\partial}$ are called the *quasi-topological modification* and *extension* of $\mathcal{R}$, respectively.

The appropriateness of the above definitions is apparent from the following theorem which has been mostly proved in [14].

THEOREM 3.5. *If $\mathcal{R}$ is a relator on X to Y, then*

$$\mathcal{R}^{\wedge} = \mathcal{R}^{\square_{\operatorname{int}}} = \mathcal{R}^{\square_{\operatorname{cl}}}, \qquad \mathcal{R}^{\vartriangle} = \mathcal{R}^{\square_{\mathcal{E}}} = \mathcal{R}^{\square_{\mathcal{D}}}, \qquad \mathcal{R}^{\curlywedge} = \mathcal{R}^{\square_{\mathcal{T}}} = \mathcal{R}^{\square_{\mathcal{F}}}.$$

Hence, by Theorems 2.4, 3.3 and 3.2, it is clear that in particular we have

COROLLARY 3.6. *If $\mathcal{R}$ is a relator on X to Y, then $\mathcal{R}^{\wedge}$ and $\mathcal{R}^{\vartriangle}$ are the largest relators on X to Y such that $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\wedge}}$ ($\operatorname{cl}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{R}^{\wedge}}$) and $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\vartriangle}}$ ($\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\vartriangle}}$), respectively.*

Remark 3.7. Unfortunately, the extension operation $\curlywedge$ is not, in general, idempotent and stable. Moreover, if $\mathcal{R}$ is a relator on X, then in general there does not exist the largest relator $\mathcal{R}^{\square}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$ ($\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\square}}$).

However, concerning the operation $\wedge\infty$, we fortunately have the following theorem which has been mostly proved in [10].

Theorem 3.8. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(1) $\mathcal{S}^{\wedge\infty} \subset \mathcal{R}^{\wedge}$;

(2) $\mathcal{S}^{\wedge\infty\wedge} \subset \mathcal{R}^{\wedge\infty\wedge}$;

(3) $\mathcal{S}^{\wedge\infty} \subset \mathcal{R}^{\wedge\infty}$;

(4) $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$ $(\mathcal{F}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{R}})$.

Corollary 3.9. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\wedge\infty}}$ $(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\wedge\infty}})$.*

4. A unifying scheme for continuities of relations

Definition 4.1. If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$, and $\square$ is a unary operation for relators, then we say that:

(1) (F, G) is mildly $\square$-continuous if $(G^{-1} \circ \mathcal{S}^{\square} \circ F)^{\square} \subset \mathcal{R}^{\square\square}$;

(2) (F, G) is upper $\square$-semicontinuous if $(\mathcal{S}^{\square} \circ F)^{\square} \subset (G \circ \mathcal{R}^{\square})^{\square}$;

(3) (F, G) is lower $\square$-semicontinuous if $(G^{-1} \circ \mathcal{S}^{\square})^{\square} \subset (\mathcal{R}^{\square} \circ F^{-1})^{\square}$;

(4) (F, G) is $\square$-continuous if it is both upper and lower $\square$-semicontinuous.

Remark 4.2. Now, the pair (F, G) may, for instance, be called properly continuous if it is $\square$-continuos with $\square$ being the identity operation on relators.

Moreover, the pair (F, G) may, for instance, be called topologically and paratopologically continuous if it is $\square$-continuous with $\square = \wedge$ and $\vartriangle$, respectively.

If in particular $\square$ is a refinement operation, then by Theorem 3.1 we can at once state the following

Theorem 4.3. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$, and $\square$ is a refinement operation, then*

(1) *(F, G) is mildly $\square$-continuous* $\iff$ $G^{-1} \circ \mathcal{S}^{\square} \circ F \subset \mathcal{R}^{\square}$;

(2) *(F, G) is upper $\square$-semicontinuous* $\iff$ $\mathcal{S}^{\square} \circ F \subset (G \circ \mathcal{R}^{\square})^{\square}$;

(3) *(F, G) is lower $\square$-semicontinuous* $\iff$ $G^{-1} \circ \mathcal{S}^{\square} \subset (\mathcal{R}^{\square} \circ F^{-1})^{\square}$.

Moreover, if in particular the operation $\square$ is inversion compatible in the sense that $(\mathcal{R}^{\square})^{-1} = (\mathcal{R}^{-1})^{\square}$ for every relator $\mathcal{R}$, then we can also easily establish the following

Theorem 4.4. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$, and $\square$ is an inversion compatible operation, then*

(1) *(F, G) is mildly $\square$-continuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if and only if (G, F) is mildly $\square$-continuous with respect to the relators $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$;*

(2) *(F, G) is lower $\square$-semicontinuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if and only if (G, F) is upper $\square$-semicontinuous with respect to the relators $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$.*

Remark 4.5. Unfortunately, the operations considered in Section 3 are not, in general, inversion compatible.

Therefore, the above theorem can have only a limited range of applicability. However, it helps us to keep in mind the definition of lower $\square$-semicontinuity.

Because of [18, Theorem 9.4], we may also naturally have the following

Definition 4.6. If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$, and $\square$ is a unary operation for relators, then we say that:

(1) (F, G) is *lower selectionally mildly $\square$-semicontinuous* if (f, G) is mildly $\square$-continuous for every selection function f of F;

(2) (F, G) is *upper selectionally mildly $\square$-semicontinuous* if (F, g) is mildly $\square$-continuous for every selection function g of G.

Remark 4.7. Note that by using the various operations on relators some mixed type continuity properties of (F, G) may also be introduced.

However, in the sequel, we shall only be interested in the paratopological continuities of (F, G) which turn out to be closely related to the denseness preservingness of G.

5. Fatness and denseness preserving relations

Definition 5.1. If $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces and G is a relation on Y to W, then we say that:

(1) G is *fatness preserving* if $A \in \mathcal{E}_{\mathcal{R}}$ implies $G(A) \in \mathcal{E}_{\mathcal{S}}$;

(2) G is *fatness reversing* if $A \in \mathcal{E}_{\mathcal{R}}$ implies $G(A) \in \mathcal{D}_{\mathcal{S}}$.

Remark 5.2. The denseness preserving and reversing relations are to be defined analogously.

Theorem 5.3. *If $(X,Y)(\mathcal{R})$ and $(Z,W)(\mathcal{S})$ are relator spaces and G is a relation on Y to W, then the following assertions are equivalent:*

(1) *G is denseness preserving;*

(2) *G^{-1} is fatness preserving.*

Proof. Suppose that $A \in \mathcal{E}_{\mathcal{S}}$ and $B \in \mathcal{D}_{\mathcal{R}}$. Then, if the assertion (1) holds, we have $G(B) \in \mathcal{D}_{\mathcal{S}}$. Hence, by Theorem 2.2, it follows that $A \cap G(B) \neq \emptyset$. This implies that $G^{-1}(A) \cap B \neq \emptyset$. Therefore, by Theorem 2.2, we also have $G^{-1}(A) \in \mathcal{E}_{\mathcal{R}}$. And thus, the assertion (2) also holds.

Suppose now that $A \in \mathcal{D}_{\mathcal{R}}$ and $B \in \mathcal{E}_{\mathcal{S}}$. Then, if the assertion (2) holds, we have $G^{-1}(B) \in \mathcal{E}_{\mathcal{R}}$. Hence, by Theorem 2.2, it follows that $A \cap G^{-1}(B) \neq \emptyset$. This implies that $G(A) \cap B \neq \emptyset$. Therefore, by Theorem 2.2, we also have $G(A) \in \mathcal{D}_{\mathcal{S}}$. And thus the assertion (1) also holds. $\square$

Similar arguments show that the following theorem is also true.

Theorem 5.4. *If $(X,Y)(\mathcal{R})$ and $(Z,W)(\mathcal{S})$ are relator spaces and G is a relation on Y to W, then the following assertions are equivalent:*

(1) *G is denseness reversing;*

(2) *G^{-1} is denseness reversing.*

Remark 5.5. The corresponding theorem for fatness reversing relations can also be proved with the help of following theorem.

Theorem 5.6. *If $(X,Y)(\mathcal{R})$ and $(Z,W)(\mathcal{S})$ are relator spaces and G is a relation on Y to W, then the following assertions are equivalent:*

(1) *G is fatness reversing;*

(2) *$\mathcal{S}^{-1} \circ G \circ \mathcal{R} = \{X \times Z\}$;*

(3) *for all $x \in X$ and $R \in \mathcal{R}$ and $S \in \mathcal{S}$ we have $Z = S^{-1}\big(G\big(R(x)\big)\big)$.*

Proof. If $x \in X$ and $R \in \mathcal{R}$, then by the definition of $\mathcal{E}_{\mathcal{R}}$ we have $R(x) \in \mathcal{E}_{\mathcal{R}}$. Hence, if the assertion (1) holds, we can infer that $G\big(R(x)\big) \in \mathcal{D}_{\mathcal{S}}$. Therefore, by Theorem 2.3, we also have $S^{-1}\big(G\big(R(x)\big)\big) = Z$ for all $S \in \mathcal{S}$. And thus the assertion (3) also holds.

While, if $A \in \mathcal{E}_{\mathcal{R}}$, then by the definition of $\mathcal{E}_{\mathcal{R}}$ there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) \subset A$. This implies that $S^{-1}\big(G\big(R(x)\big)\big) \subset S^{-1}\big(G(A)\big)$ for all $S \in \mathcal{S}$. Hence, if the assertion (3) holds, we can infer that $Z = S^{-1}\big(G(A)\big)$ for all $S \in \mathcal{S}$. Therefore, by Theorem 2.3, we also have $G(A) \in \mathcal{D}_{\mathcal{S}}$. And thus the assertion (1) also holds.

The equivalence of the assertions (2) and (3) is immediate from the corresponding definitions. $\square$

6. Paratopologically mildly continuous relations

Theorem 6.1. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$, then the following assertions are equivalent:*

(1) *(F, G) is mildly Δ-continuous;*

(2) *$G^{-1} \circ \mathcal{S}^{\Delta} \circ F \subset \mathcal{R}^{\Delta}$;*

(3) *for all $x \in X$ and $V \in \mathcal{S}^{\Delta}$ there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset G^{-1}\big(V\big(F(x)\big)\big)$.*

Proof. By Theorem 4.3, the assertions (1) and (2) are equivalent. Moreover, if the assertion (2) holds, then for each $V \in \mathcal{S}^{\Delta}$ we have $G^{-1} \circ V \circ F \in \mathcal{R}^{\Delta}$. Hence, by using the definition of $\mathcal{R}^{\Delta}$, we can infer that for each $x \in X$ there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset (G^{-1} \circ V \circ F)(x)$. Therefore, the assertion (3) also holds. Moreover, a reverse argument shows that the converse implication $(3) \Rightarrow (2)$ is also true. □

Remark 6.2. The inclusion $R(u) \subset G^{-1}\big(V\big(F(x)\big)\big)$, in a more detailed form, means only that for all $y \in R(u)$ we have $G(y) \cap V\big(F(x)\big) \neq \emptyset$. That is, there exists $z \in F(x)$ such that $G(y) \cap V(z) \neq \emptyset$.

Now, by using Theorem 6.1, we can also easily prove the following

Theorem 6.3. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ such that F is a function onto Z, then the following assertions are equivalent:*

(1) *(F, G) is mildly Δ-continuous;*

(2) *$G^{-1} \circ \mathcal{S} \circ F \subset \mathcal{R}^{\Delta}$;*

(3) *for all $x \in X$ and $S \in \mathcal{S}$ there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset G^{-1}\big(S\big(F(x)\big)\big)$.*

Proof. Since $\mathcal{S} \subset \mathcal{S}^{\Delta}$, the implication $(1) \Rightarrow (3)$ is immediate from Theorem 6.1. Moreover, from the proof of Theorem 6.1, it is clear that the assertions (2) and (3) are equivalent.

Therefore, we must only prove the implication $(3) \Rightarrow (1)$. For this, note that if $x \in F^{-1}(Z)$ and $V \in \mathcal{S}^{\Delta}$, then by the definition of $\mathcal{S}^{\Delta}$ there exist $z \in Z$ and $S \in \mathcal{S}$ such that $S(z) \subset V\big(F(x)\big)$. Moreover, since $Z = F(X)$, there exists $v \in X$ such that $z = F(v)$. Therefore, we also have $S\big(F(v)\big) \subset V\big(F(x)\big)$, and hence $G^{-1}\big(S\big(F(v)\big)\big) \subset G^{-1}\big(V\big(F(x)\big)\big)$. On the other hand, if the assertion (3) holds, then there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset G^{-1}\big(S\big(F(v)\big)\big)$. Therefore, we also have $R(u) \subset G^{-1}\big(V\big(F(x)\big)\big)$. Note that the same conclusion can also be obtained whenever $x \in X \setminus F^{-1}(Z)$, and hence $F(x) = \emptyset$. Namely, in this case, by the assertion (3) there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) = \emptyset$. Therefore, by Theorem 6.1, the assertion (1) also holds. □

Moreover, by using Theorem 6.1, we can also easily prove the following two theorems.

Theorem 6.4. *If (F, G) is a mildly Δ-continuous pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ such that F is a function onto Z, then G is denseness preserving.*

P r o o f. By Theorem 5.3, it suffices to show that G^{-1} is fatness preserving. For this, note that if $A \in \mathcal{E}_{\mathcal{S}}$, then by the definition of fat sets there exist $z \in Z$ and $S \in \mathcal{S}$ such that $S(z) \subset A$. Moreover, since F is a function and $Z = F(X)$, there exists $x \in X$ such that $z = F(x)$. Therefore, we also have $S\big(F(x)\big) \subset A$, and hence $G^{-1}\big(S\big(F(x)\big)\big) \subset G^{-1}(A)$. Moreover, by Theorem 6.1, there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset G^{-1}\big(S\big(F(x)\big)\big)$. Therefore, $R(u) \subset G^{-1}(A)$, and thus the inclusion $G^{-1}(A) \in \mathcal{E}_{\mathcal{R}}$ is also true. □

Theorem 6.5. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ such that F^{-1} is onto X and G is denseness preserving, then (F, G) is mildly Δ-continuous.*

P r o o f. If $x \in X$ and $V \in \mathcal{S}^{\Delta}$, then since $X = F^{-1}(Z)$, and thus in particular $F(x) \neq \emptyset$, it is clear that $V\big(F(x)\big) \in \mathcal{E}_{\mathcal{S}}$. Hence, by using that G^{-1} is fatness preserving, we can infer that $G^{-1}\big(V\big(F(x)\big)\big) \in \mathcal{E}_{\mathcal{R}}$. Therefore, there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset G^{-1}\big(V\big(F(x)\big)\big)$. And thus, by Theorem 6.1, the pair (F, G) is mildly Δ-continuous. □

Now, as an immediate consequence of Theorems 6.4 and 6.5, we also can state

Corollary 6.6. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ such that F is a function of X onto Z, then the following assertions are equivalent:*

(1) *G is denseness preserving;*

(2) *(F, G) is mildly Δ-continuous.*

7. Paratopologically upper semicontinuous relations

Theorem 7.1. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$, then the following assertions are equivalent:*

(1) *(F, G) is upper Δ-semicontinuous;*

(2) $\mathcal{S}^{\Delta} \circ F \subset (G \circ \mathcal{R})^{\Delta}$;

(3) $\mathcal{S}^{\Delta} \circ F \subset \big(G \circ \mathcal{R}^{\Delta}\big)^{\Delta}$;

(4) *for all* $x \in X$ *and* $V \in \mathcal{S}^{\Delta}$ *there exist* $u \in X$ *and* $R \in \mathcal{R}$ *such that* $G\big(R(u)\big) \subset V\big(F(x)\big)$.

P r o o f . By Theorem 5.3, the assertions (1) and (3) are equivalent. Moreover, since $\mathcal{R} \subset \mathcal{R}^{\Delta}$, and hence $(G \circ \mathcal{R})^{\Delta} \subset (G \circ \mathcal{R}^{\Delta})^{\Delta}$, the implication (2)$\Rightarrow$(3) is also true.

On the other hand, if the assertion (3) holds, then for each $V \in \mathcal{S}^{\Delta}$ we have $V \circ F \in (G \circ \mathcal{R}^{\Delta})^{\Delta}$. Hence, by using the definition of the paratopological refinement, we can infer that for each $x \in X$ there exist $v \in X$ and $U \in \mathcal{R}^{\Delta}$ such that $(G \circ U)(v) \subset (V \circ F)(x)$. Moreover, since $v \in X$ and $U \in \mathcal{R}^{\Delta}$, there exist $u \in X$ and $R \in \mathcal{R}$ such that $R(u) \subset U(v)$. Therefore, we have

$$G\big(R(u)\big) \subset G\big(U(v)\big) = (G \circ U)(v) \subset (V \circ F)(x) = V\big(F(x)\big).$$

And thus, the assertion (4) also holds. Moreover, a reverse argument shows that the implication (4)$\Rightarrow$(2) is also true. $\square$

Remark 7.2. The inclusion $G\big(R(u)\big) \subset V\big(F(x)\big)$, in a more detailed form, means only that for all $y \in R(u)$ we have $G(y) \subset V\big(F(x)\big)$. That is, for each $w \in G(y)$ there exists $z \in F(x)$ such that $w \in V(z)$.

Now, analogously to Theorems 6.3, 6.4 and 6.5, one can also easily prove the following three theorems.

Theorem 7.3. *If* (F, G) *is a pair of relations on one relator space* $(X, Y)(\mathcal{R})$ *to another* $(Z, W)(\mathcal{S})$ *such that* F *is a function onto* Z, *then the following assertions are equivalent:*

(1) (F, G) *is upper* Δ*-semicontinuous;*

(2) $\mathcal{S} \circ F \subset \big(G \circ \mathcal{R}\big)^{\Delta}$;

(3) *for all* $x \in X$ *and* $S \in \mathcal{S}$ *there exist* $u \in X$ *and* $R \in \mathcal{R}$ *such that* $G\big(R(u)\big) \subset S\big(F(x)\big)$.

Theorem 7.4. *If* (F, G) *is an upper* Δ*-semicontinuous pair of relations on one relator space* $(X, Y)(\mathcal{R})$ *to another* $(Z, W)(\mathcal{S})$ *such that* F *is a function onto* Z *and* G^{-1} *is onto* Y, *then* G *is denseness preserving.*

Theorem 7.5. *If* (F, G) *is a pair of relations on one relator space* $(X, Y)(\mathcal{R})$ *to another* $(Z, W)(\mathcal{S})$ *such that* F^{-1} *is onto* X *and* G *is a denseness preserving function, then* (F, G) *is upper* Δ*-semicontinuous.*

Now, as an immediate consequence of Theorems 7.4 and 7.5, one can also state

Corollary 7.6. *If (f, g) is a pair of functions of one relator space $(X, Y)(\mathcal{R})$ into another $(Z, W)(\mathcal{S})$ such that f is onto Z, then the following assertions are equivalent:*

(1) *g is denseness preserving;*

(2) *(f, g) is upper Δ-semicontinuous.*

8. Paratopologically lower semicontinuous relations

Theorem 8.1. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$, then the following assertions are equivalent:*

(1) *(F, G) is lower Δ-semicontinuous;*

(2) $G^{-1} \circ \mathcal{S} \subset (\mathcal{R}^{\Delta} \circ F^{-1})^{\Delta}$;

(3) $G^{-1} \circ \mathcal{S}^{\Delta} \subset (\mathcal{R}^{\Delta} \circ F^{-1})^{\Delta}$;

(4) *for all $z \in Z$ and $S \in \mathcal{S}$ there exist $v \in Z$ and $U \in \mathcal{R}^{\Delta}$ such that $U\big(F^{-1}(v)\big) \subset G^{-1}\big(S(z)\big)$.*

P r o o f . By Theorem 4.3, the assertions (1) and (3) are equivalent. Moreover, since $\mathcal{S} \subset \mathcal{S}^{\Delta}$, and hence $G^{-1} \circ \mathcal{S} \subset G^{-1} \circ \mathcal{S}^{\Delta}$, the implication (3)$\Rightarrow$(2) is also true.

On the other hand, if the assertion (2) holds, then for each $S \in \mathcal{S}$ we have $G^{-1} \circ S \in (\mathcal{R}^{\Delta} \circ F^{-1})^{\Delta}$. Hence, by using the definition of the paratopological refinement, we can infer that for each $z \in Z$ there exist $v \in Z$ and $U \in \mathcal{R}^{\Delta}$ such that $(U \circ F^{-1})(v) \subset (G^{-1} \circ S)(z)$. Therefore, the assertion (4) also holds.

Now, it remains only to prove the implication (4)$\Rightarrow$(3). For this, note that if $z \in Z$ and $V \in \mathcal{S}^{\Delta}$, then by the definition of $\mathcal{S}^{\Delta}$, there exist $u \in Z$ and $S \in \mathcal{S}$ such that $S(u) \subset V(z)$, and hence $G^{-1}\big(S(u)\big) \subset G^{-1}\big(V(z)\big)$. Moreover, if the assertion (4) holds, then there exist $v \in Z$ and $U \in \mathcal{R}^{\Delta}$ such that $U\big(F^{-1}(v)\big) \subset G^{-1}\big(S(u)\big)$. Therefore, $(U \circ F^{-1})(v) \subset (G^{-1} \circ V)(z)$. Hence, it is clear that $G^{-1} \circ V \in (\mathcal{R}^{\Delta} \circ F^{-1})^{\Delta}$, and thus the assertion (3) also holds. □

Remark 8.2. The inclusion $U\big(F^{-1}(v)\big) \subset G^{-1}\big(S(z)\big)$, in a more detailed form, means only that for all $x \in F^{-1}(v)$ and $y \in U(x)$ we have $G(y) \cap S(z) \neq \emptyset$.

Now, by using Theorem 8.1, we can also easily prove the following

Theorem 8.3. *If (F, G) is a pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ such that F^{-1} is a function onto X, then the following assertions are equivalent:*

(1) *(F, G) is lower Δ-semicontinuous;*

(2) $G^{-1} \circ \mathcal{S} \subset (\mathcal{R} \circ F^{-1})^{\Delta}$;

(3) *for all* $z \in Z$ *and* $S \in \mathcal{S}$ *there exist* $v \in Z$ *and* $R \in \mathcal{R}$ *such that* $R\big(F^{-1}(v)\big) \subset G^{-1}\big(S(z)\big)$.

Proof. Since $\mathcal{R} \subset \mathcal{R}^{\Delta}$, the implication (3)$\Rightarrow$(1) is immediate from Theorem 8.1. Moreover, from the proof of Theorem 8.1, it is clear that the assertions (2) and (3) are equivalent.

Therefore, we must only prove that the implication (1)$\Rightarrow$(3) is also true. For this, note that if $z \in Z$ and $S \in \mathcal{S}$, and the assertion (1) holds, then by Theorem 8.1 there exist $u \in Z$ and $U \in \mathcal{R}$ such that $U\big(F^{-1}(u)\big) \subset G^{-1}\big(S(z)\big)$. Moreover, by the definition of $\mathcal{R}^{\Delta}$, there exist $v \in Z$ and $R \in \mathcal{R}$ such that $R\big(F^{-1}(v)\big) \subset U\big(F^{-1}(u)\big)$. Therefore, the assertion (3) also holds. $\square$

Moreover, by using Theorem 8.1, we can also easily prove the following two theorems.

Theorem 8.4. *If* (F, G) *is a lower* Δ*-semicontinuous pair of relations on one relator space* $(X, Y)(\mathcal{R})$ *to another* $(Z, W)(\mathcal{S})$ *such that* F *is onto* Z, *then* G *is denseness preserving.*

Proof. By Theorem 5.3, it suffices to show that G^{-1} is fatness preserving. For this, note that if $A \in \mathcal{E}_{\mathcal{S}}$, then by the definition of fat sets there exist $z \in Z$ and $S \in \mathcal{S}$ such that $S(z) \subset A$, and hence $G^{-1}\big(S(z)\big) \subset G^{-1}(A)$. Moreover, by Theorem 8.1, there exist $v \in Z$ and $U \in \mathcal{R}^{\Delta}$ such that $U\big(F^{-1}(v)\big) \subset G^{-1}\big(S(z)\big)$. Therefore, we also have $U\big(F^{-1}(v)\big) \subset G^{-1}(A)$. On the other hand, since $Z = F(X)$, there exists $x \in X$ such that $v \in F(x)$, i.e., $x \in F^{-1}(v)$. Hence, it is clear that $U(x) \subset U\big(F^{-1}(v)\big) \subset G^{-1}(A)$, and thus the inclusion $G^{-1}(A) \in \mathcal{E}_{\mathcal{R}}$ is also true. $\square$

Theorem 8.5. *If* (F, G) *is a pair of relations on one relator space* $(X, Y)(\mathcal{R})$ *to another* $(Z, W)(\mathcal{S})$ *such that* F^{-1} *is a function onto* X *and* G *is denseness preserving, then* (F, G) *is lower* Δ*-semicontinuous.*

Proof. If $z \in Z$ and $S \in \mathcal{S}$, then by the definition of fat sets $S(z) \in \mathcal{E}_{\mathcal{S}}$. Hence, by using that G^{-1} is fatness preserving, we can infer that $G^{-1}\big(S(z)\big) \in \mathcal{E}_{\mathcal{R}}$. Therefore, there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) \subset G^{-1}\big(S(z)\big)$. Moreover, since F^{-1} is a function and $X = F^{-1}(Z)$, there exists $v \in Z$ such that $x = F^{-1}(v)$. Hence, it follows that $R\big(F^{-1}(v)\big) \subset G^{-1}\big(S(z)\big)$. Therefore, by Theorem 8.1, the pair (F, G) is lower Δ-semicontinuous. $\square$

Now, as an immediate consequence of Theorems 8.4 and 8.5, we can also state

Corollary 8.6. *If* (F, G) *is a pair of relations on one relator space* $(X, Y)(\mathcal{R})$ *to another* $(Z, W)(\mathcal{S})$ *such that* F^{-1} *is a function of* Z *onto* X, *then the following assertions are equivalent:*

(1) G *is denseness preserving;*

(2) (F, G) *is lower* Δ*-semicontinuous.*

Remark 8.7. By using Theorems 6.4, 6.5, 7.4, 7.5, 8.3 and 8.4 we can easily establish some intimate connections among the various paratopological continuities. For instance, Theorems 8.4, 6.5 and 7.5 show that the following assertion is true.

If (F, G) is a lower Δ-semicontinuous pair of relations on one relator space $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ such that F is a relation of X onto Z (and G is a function), then (F, G) is mildly Δ-continuous (upper Δ-semicontinuous).

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REFERENCES

[1] CARLSON, S. C.—VOTAW, C.: *Para-uniformities, para-proximities, and H-closed extensions*, Rocky Mountain J. Math. **16** (1986), 805–835.

[2] CURTIS, D. W.—MATHEWS, J. C.: *Generalized uniformities for pairs of spaces*, Topology Conference, Arizona State Univ., Tempe, Arizona, 1968, pp. 212–246.

[3] DAVIS, A. S.: *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly **68** (1961), 886–893.

[4] DOBOŠ, J.: *A note on the invariance of Baire spaces under mappings*, Čas. Pěst. Mat. **108** (1983), 409–411.

[5] FROLÍK, Z.: *Remarks concerning the invariance of Baire spaces under mappings*, Czechoslovak Math. J. **11** (1961), 381–385.

[6] GENTRY, K. R.—HOYLE, H. B.: *Somewhat continuous functions*, Czechoslovak Math. J. **21** (1971), 5–12.

[7] HOLÁ, Ľ: *On determining sets for the class of somewhat continuous functions*, Math. Slovaca **39** (1989), 257–259.

[8] KONISHI, I.: *On uniform topologies in general spaces*, J. Math. Soc. Japan **4** (1952), 166–188.

[9] KRISHNAN, V. S.: *A note on semi-uniform spaces*, J. Madras Univ. **25** (1955), 123–124.

[10] MALA, J.—SZÁZ, Á.: *Modifications of relators*, Acta Math. Hungar. **77** (1997), 69–81.

[11] NAKANO, H.—NAKANO, K.: *Connector theory*, Pacific J. Math. **56** (1975), 195–213.

[12] NEUBRUNN, T.: *A generalized continuity and product spaces*, Math. Slovaca **26** (1976), 97–99.

[13] PAGE, W.: *Topological Uniform Structures*, John Wiley and Sons, New York, 1978.

[14] PATAKI, G.: *On the extensions, refinements and modifications of relators*, Math. Balkanica **15** (2001), 155–186.

[15] PERVIN, W. J.: *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.

[16] PRAKASH, A.—SRIVASTAVA, P.: *Somewhat continuity and some other generalizations of continuity*, Math. Slovaca **27** (1977), 243–248.

[17] Del PRETE, I.—Di IORIO, M.: *Semicontinuous set valued mappings and connected sets*, Matematiche Catania **46** (1991), 609–624.

[18] RAKACZKI, Cs.—SZÁZ, Á.: *Semicontinuity and closedness properties of relations in relator spaces*, Tech. Rep., Inst. Math. Inf., Univ. Debrecen **97/16**, 1–23.

[19] SZÁZ, Á.: *Coherences instead of convergences*, Proc. Conf., Convergence and Generalized Functions, Inst. Math. Polish Acad Sci., Warsaw, 1984, pp. 141–148.

[20] SZÁZ, Á.: *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.

[21] SZÁZ, Á.: *Inverse and symmetric relators*, Acta Math. Hungar. **60** (1992), 157–176.

[22] SZÁZ, Á.: *Structures derivable from relators*, Singularité **3** (1992), 14–30.

[23] SZÁZ, Á.: *Refinements of relators*, Tech. Rep., Inst. Math. Inf., Univ. Debrecen **93/76**, 1–19.

[24] SZÁZ, Á.: *An extension of Kelley's closed relation theorem to relator spaces*, Filomat (Nis) **14** (2000) (to appear).

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