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LOCALLY BEST AND UNIFORMLY BEST ESTIMATORS IN LINEAR MODEL WITH NUISANCE PARAMETERS

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Dedicated to Lubomír Kubáček on the occasion of his 70th birthday

ABSTRACT. The regular linear model in which the vector of the first order parameters is divided into two parts—to the vector of the useful parameters and to the vector of the nuisance parameters—is considered. We study the estimators of the useful parameters.

1. Introduction, notations

The properties of the regular linear model are well known (see [1], Chapter 1). In this paper the situation where the vector of the first order parameters is divided into two parts (to the vector of the useful parameters and to the vector of the nuisance parameters) is considered.

Let us describe one example (author L. Kubáček). When laying the foundations for a large building it is necessary to determine the moment at which the subsoil (after large landscaping has been done) stabilizes to the point that it is possible to continue construction without risk of following damage.

There are k points chosen at the building site and their heights are repeatedly measured at the moments $t_1, \dots, t_m$. It is necessary to create a model describing the subsidence of the subsoil at the chosen points and to estimate the unknown parameters of this model on the basis of the results of the repeated measurements.

The result of the measurement at the ith point in the jth epoch could be described as follows:

$$\eta_i(t_j) = \kappa_i - \beta_1^i\big(1 - e^{-\beta_2^i t_j}\big) + \varepsilon_{ij}\,, \qquad i = 1, \dots, k\,, \quad j = 1, \dots, m\,, \tag{1}$$

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where κ_i is the height of the ith point at time t_0, the function $\beta_1^i\big(1-e^{-\beta_2^i t}\big)$ describes the movement of the earth-strata at the ith point. The parameters $\beta_1^i > 0$, $\beta_2^i > 0$ are different at the different points, i.e., we suppose that the geological composition of the subsoil is not homogeneous. The aim is to estimate the unknown parameters $\kappa_1,\dots,\kappa_m$ and β_1^i, β_2^i, $i=1,\dots,k$.

The civil engineer needs to know when it is possible to continue the construction, i.e., when the subsidence of the subsoil at all the points is insignificant. It means that it is necessary to determine such τ_i, $i=1,\dots,k$ that

$$\beta_1^i\big(1-e^{-\beta_2^i\tau_i}\big) = C\beta_1^i\,, \qquad i=1,\dots,k\,,$$

where $0 < C < 1$ is a suitable constant which is sufficiently close to 1. It is possible to continue the construction at the time $\tau = \max\{\tau_1,\dots,\tau_k\}$.

Remark 1. The condition mentioned above is equivalent to more simple one: to determine such τ_i that

$$e^{-\beta_2^i\tau_i} = \gamma\,, \qquad i=1,\dots,k\,,$$

where $0 < \gamma < 1$ is sufficiently close to 0. The time when it is possible to continue the construction is again $\tau = \max\{\tau_1,\dots,\tau_k\}$.

The model (1) is not linear in parameters; we linearize it by using the first two members of the Taylor expansion of the function $\beta_1^i\big(1-e^{-\beta_2^i t}\big)$ at the suitable point $(\beta_{1,0}^i, \beta_{2,0}^i)$, $\beta_{1,0}^i > 0$, $\beta_{2,0}^i > 0$.

We get the model

$$\begin{aligned}\eta_i(t_j) = \kappa_i - \big[\beta_{1,0}^i\big(1-e^{-\beta_{2,0}^i t_j}\big) + \big(1-e^{-\beta_{2,0}^i t_j}\big)\big(\beta_1^i-\beta_{1,0}^i\big) + \beta_{1,0}^i t_j e^{-\beta_{2,0}^i t_j}\big(\beta_2^i-\beta_{2,0}^i\big)\big]&\\ + \varepsilon_{ij}\,, \qquad i=1,\dots,k\,, \quad j=1,\dots,m\,.&\end{aligned}$$

Denote

$$\begin{aligned} Y_i^{(j)} &= \eta_i(t_j) + \beta_{1,0}^i\big(1-e^{-\beta_{2,0}^i t_j}\big)\,,\\ \varphi_1^i(t) &= -\big(1-e^{-\beta_{2,0}^i t}\big)\,, \quad \varphi_2^i(t) = -\beta_{1,0}^i t e^{-\beta_{2,0}^i t}\,,\end{aligned}$$

$$\delta\beta_1^i = \beta_1^i - \beta_{1,0}^i\,, \qquad \delta\beta_2^i = \beta_2^i - \beta_{2,0}^i\,, \quad i=1,\dots,k\,, \quad j=1,\dots,m\,.$$

Thus

$$Y_i^{(j)} = \kappa_i + \varphi_1^i(t_j)\delta\beta_1^i + \varphi_2^i(t_j)\delta\beta_2^i + \varepsilon_{ij}\,, \qquad i=1,\dots,k\,, \quad j=1,\dots,m\,.$$

Let us consider the observation vector $\mathbf{Y} = \big(\mathbf{Y}_1^{(1)},\dots,\mathbf{Y}_1^{(m)},\dots,\mathbf{Y}_k^{(1)},\dots,\mathbf{Y}_k^{(m)}\big)'$.

The model described above could be rewritten in the form

$$\mathbf{Y} = (\mathbf{W},\mathbf{Z})\begin{pmatrix}\beta\\ \kappa\end{pmatrix} + \varepsilon\,,$$

where

$$\mathbf{W} = \begin{pmatrix} \varphi_1^1(t_1) & \varphi_2^1(t_1) & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varphi_1^1(t_m) & \varphi_2^1(t_m) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \varphi_1^2(t_1) & \varphi_2^2(t_1) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \varphi_1^2(t_m) & \varphi_2^2(t_m) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \varphi_1^k(t_1) & \varphi_2^k(t_1) \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \varphi_1^k(t_m) & \varphi_2^k(t_m) \end{pmatrix},$$

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \dots & 0 \\ \dots & \dots & & \dots \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \delta\beta_1^1 \\ \delta\beta_2^1 \\ \delta\beta_1^2 \\ \delta\beta_2^2 \\ \vdots \\ \delta\beta_1^k \\ \delta\beta_2^k \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_k \end{pmatrix}.$$

The matrices $\mathbf{W}$, $\mathbf{Z}$ are known, the vector β of the type $2k \times 1$ is a vector of the useful parameters, κ is a vector of nuisance parameters.

The following notation will be used throughout the paper:

$\mathbb{R}^n$	the space of all n-dimensional real vectors;
$\mathbf{u}_p$, $\mathbf{A}_{m,n}$	the real column p-dimensional vector, the real $m \times n$ matrix;
$\mathbf{A}'$, $r(\mathbf{A})$	the transpose, the rank of the matrix $\mathbf{A}$;
$\mathscr{R}(\mathbf{A})$, $\mathrm{Ker}(\mathbf{A})$	the range, the null space of the matrix $\mathbf{A}$;
$\mathbf{A} \otimes \mathbf{B}$	the Kronecker (tensor) product of the matrices $\mathbf{A}, \mathbf{B}$;
$\mathbf{A}^-$	a generalized inverse of a matrix $\mathbf{A}$(satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$);
$\mathbf{A}^+$	the Moore-Penrose generalized inverse of a matrix $\mathbf{A}$(satisfying $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $(\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+$, $(\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$);
$\mathbf{P}_A$	the orthogonal projector onto $\mathscr{R}(\mathbf{A})$;
$\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$	the orthogonal projector onto $\mathscr{R}^\perp(\mathbf{A}) = \mathrm{Ker}(\mathbf{A}')$;

$\mathbf{I}_k$	the $k \times k$ identity matrix;
$0_{m,n}$	the $m \times n$ null matrix;
1_k	$= (1, \dots, 1)' \in \mathbb{R}^k$
$\chi^2_r(0)$	χ^2-distribution with r degrees of freedom;
$\chi^2_r(0, 1-\alpha)$	$(1-\alpha)$-quantile of this distribution.

If $\mathscr{R}(\mathbf{A}) \subset \mathscr{R}(\mathbf{S})$, S p.s.d., then the symbol $\mathbf{P}_A^{S^-}$ denotes the projector projecting vectors in $\mathscr{R}(\mathbf{S})$ onto $\mathscr{R}(\mathbf{A})$ along $\mathscr{R}(\mathbf{SA}^\perp)$. A general representation of all such projectors $\mathbf{P}_A^{S^-}$ is given by $\mathbf{A}(\mathbf{A}'\mathbf{S}^-\mathbf{A})^-\mathbf{A}'\mathbf{S}^- + \mathbf{B}(\mathbf{I} - \mathbf{SS}^-)$, where $\mathbf{B}$ is arbitrary, (see [4], (2.14)). $\mathbf{M}_A^{S^-} = \mathbf{I} - \mathbf{P}_A^{S^-}$.

2. Locally best linear unbiased estimators

Let us consider the following linear model

$$\mathbf{Y} = (\mathbf{W}, \mathbf{Z}) \begin{pmatrix} \beta \\ \kappa \end{pmatrix} + \varepsilon \,, \tag{2}$$

where

$\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n)'$ is a random observation vector,

$\beta \in \mathbb{R}^r$ is a vector of the useful parameters,

$\kappa \in \mathbb{R}^s$ is a vector of the nuisance parameters,

$\mathbf{W}_{n,r}$ is a design matrix belonging to the vector β,

$\mathbf{Z}_{n,s}$ is a design matrix belonging to the vector κ.

We suppose that

1. $E(\mathbf{Y}) = \mathbf{W}\beta + \mathbf{Z}\kappa$, $\forall \beta \in \mathbb{R}^r$, $\forall \kappa \in \mathbb{R}^s$,
2. $\mathrm{var}(\mathbf{Y}) = \Sigma_\vartheta = \sum_{i=1}^{p} \vartheta_i V_i$, $\forall \vartheta = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta} \subset \mathbb{R}^p$, $V_1, \dots, V_p$ given symmetric matrices,
3. $\underline{\vartheta} \subset \mathbb{R}^p$ contains an open sphere in $\mathbb{R}^p$,
4. if $\vartheta \in \underline{\vartheta}$, the matrix Σ_ϑ is positive semidefinite,
5. the matrix Σ_ϑ is not a function of the vector $(\beta', \kappa')'$.

If the matrix Σ_ϑ is positive definite for any $\vartheta \in \underline{\vartheta}$ and $r(\mathbf{W}, \mathbf{Z}) = r + s < n$, the model is said to be *regular*, (see [1], p.13).

Notation. For the sake of simplicity we use the symbol Σ_0 instead of Σ_{ϑ_0}.

THEOREM 1. *In the regular model* (2) *the* ϑ_0-LBLUE *of the parameter* $(\beta', \kappa')'$ *is given as*

$$\begin{pmatrix}\hat\beta\\ \hat\kappa\end{pmatrix} = \begin{pmatrix}(\mathbf{W}'\Sigma_0^{-1}\mathbf{M}_Z^{\Sigma_0^{-1}}\mathbf{W})^{-1}\mathbf{W}'\Sigma_0^{-1}\mathbf{M}_Z^{\Sigma_0^{-1}}\\ (\mathbf{Z}'\Sigma_0^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_0^{-1}\mathbf{M}_W^{\Sigma_0^{-1}M_Z^{\Sigma_0^{-1}}}\end{pmatrix}\mathbf{Y}. \tag{3}$$

P r o o f . According to the Theorem 1.1.1 in [1] using the following Rohde's formula for inverse of partitioned p.d. matrix (see [2], Lemma 13)

$$\begin{pmatrix}\mathbf{A}, \mathbf{B}\\ \mathbf{B}', \mathbf{C}\end{pmatrix}^{-1}$$
$$= \begin{pmatrix}\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1}, & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\\ -(\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1}, & (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\end{pmatrix} \tag{4}$$
$$= \begin{pmatrix}(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}, & -(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\\ -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}, & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\end{pmatrix}, \tag{5}$$

the ϑ_0-LBLUE of the vector parameter $(\beta', \kappa')'$ is given by

$$\begin{aligned}\begin{pmatrix}\hat\beta\\ \hat\kappa\end{pmatrix} &= \left[\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}\Sigma_0^{-1}(\mathbf{W}, \mathbf{Z})\right]^{-1}\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}\Sigma_0^{-1}\mathbf{Y}\\ &= \begin{bmatrix}\mathbf{W}'\Sigma_0^{-1}\mathbf{W}, & \mathbf{W}'\Sigma_0^{-1}\mathbf{Z}\\ \mathbf{Z}'\Sigma_0^{-1}\mathbf{W}, & \mathbf{Z}'\Sigma_0^{-1}\mathbf{Z}\end{bmatrix}^{-1}\begin{pmatrix}\mathbf{W}'\Sigma_0^{-1}\\ \mathbf{Z}'\Sigma_0^{-1}\end{pmatrix}\mathbf{Y}\\ &= \begin{bmatrix}\mathbf{U}, & -\mathbf{U}\mathbf{W}'\Sigma_0^{-1}\mathbf{Z}\mathbf{V}\\ -\mathbf{V}\mathbf{Z}'\Sigma_0^{-1}\mathbf{W}\mathbf{U}, & \mathbf{V} + \mathbf{V}\mathbf{Z}'\Sigma_0^{-1}\mathbf{W}\mathbf{U}\mathbf{W}'\Sigma_0^{-1}\mathbf{Z}\mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{W}'\Sigma_0^{-1}\\ \mathbf{Z}'\Sigma_0^{-1}\end{bmatrix}\mathbf{Y},\end{aligned}$$

where

$$\mathbf{V} = (\mathbf{Z}'\Sigma_0^{-1}\mathbf{Z})^{-1}, \qquad \mathbf{U} = \left[\mathbf{W}'\boldsymbol{\Sigma}_0^{-1}\mathbf{W} - \mathbf{W}'\Sigma_0^{-1}\mathbf{Z}\mathbf{V}\mathbf{Z}'\Sigma_0^{-1}\mathbf{W}\right]^{-1}.$$

Utilizing that

$$\mathbf{P}_Z^{\Sigma_0^{-1}} = \mathbf{Z}(\mathbf{Z}'\Sigma_0^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_0^{-1}, \qquad \mathbf{M}_Z^{\Sigma_0^{-1}} = \mathbf{I} - \mathbf{P}_Z^{\Sigma_0^{-1}},$$

we get the statement. □

Remark 2. As Σ_0 is supposed to be positive definite, we can write (see [2], Lemma 16)

$$\Sigma_0^{-1}\mathbf{M}_Z^{\Sigma_0^{-1}} = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{Z}(\mathbf{Z}'\Sigma_0^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_0^{-1} = (\mathbf{M}_Z\Sigma_0\mathbf{M}_Z)^+.$$

Thus the statement of the Theorem 1 has the equivalent form

$$\begin{pmatrix}\hat\beta\\ \hat\kappa\end{pmatrix} = \begin{pmatrix}\left[\mathbf{W}'(\mathbf{M}_Z\Sigma_0\mathbf{M}_Z)^+\mathbf{W}\right]^{-1}\mathbf{W}'(\mathbf{M}_Z\Sigma_0\mathbf{M}_Z)^+\\ (\mathbf{Z}'\Sigma_0^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\Sigma_0^{-1}\mathbf{M}_W^{\Sigma_0^{-1}M_Z^{\Sigma_0^{-1}}}\end{pmatrix}\mathbf{Y}.$$

Theorem 2. *Let in the regular model* (2) $\mathbf{Y} \sim N_n\left[(\mathbf{W},\mathbf{Z})\begin{pmatrix}\beta\\ \kappa\end{pmatrix}, \Sigma_0\right]$, $\beta = (\beta_1, \dots \beta_r)'$. *Then*

(i) *the* ϑ_0-BLUE *of the useful parameter* β *given by* (3) *is a* r*-dimensional normally distributed vector*

$$\hat\beta \sim N_r\big(\beta, \left[\mathbf{W}'(\mathbf{M}_Z\Sigma_0\mathbf{M}_Z)^{+}\mathbf{W}\right]^{-1}\big)\,,$$

(ii) *if the* $p \times r$ *matrix* $\mathbf{G}$ *is such that* $\mathbf{G}\left[\operatorname{var}(\hat\beta)\right]\mathbf{G}'$ *is positive definite, the ellipsoid*

$$\mathscr{E}_{1-\alpha}(\mathbf{G}\beta) = \Big\{u \in \mathbb{R}^p\colon (\mathbf{u}-\mathbf{G}\hat\beta)'\left[\mathbf{G}\big(\mathbf{W}'(\mathbf{M}_Z\Sigma_0\mathbf{M}_Z)^{+}\mathbf{W}\big)^{-1}\mathbf{G}'\right]^{-1}(\mathbf{u}-\mathbf{G}\hat\beta) \le \chi^2_p(0, 1-\alpha)\Big\}\,,$$

is the $(1-\alpha)$*-confidence ellipsoid for the vector* $\mathbf{G}\beta$.

Proof. (i) The assertion is a consequence of the fact that $\hat\beta$ is a linear transform of the normally distributed random vector $\mathbf{Y}$.

(ii) According to Pearson lemma ($\mathbf{X} \sim N_n(0,\mathbf{V}) \;\Rightarrow\; \mathbf{X}'\mathbf{V}^{-}\mathbf{X} \sim \chi^2_{r(V)}$)

$$P\Big[(\mathbf{G}\hat\beta-\mathbf{G}\beta)'\left[\mathbf{G}\operatorname{var}(\hat\beta)\mathbf{G}'\right]^{-1}(\mathbf{G}\hat\beta-\mathbf{G}\beta) \le \chi^2_p(0,1-\alpha)\Big] = 1-\alpha\,. \tag{6}$$

If we denote

$$\mathscr{E}_{1-\alpha}(\mathbf{G}\beta) = \Big\{\mathbf{u} \in \mathbb{R}^p\colon (\mathbf{u}-\mathbf{G}\hat\beta)'\left[\mathbf{G}(\mathbf{W}'(\mathbf{M}_Z\Sigma_0\mathbf{M}_Z)^{+}\mathbf{W})^{-1}\mathbf{G}'\right]^{-1}(\mathbf{u}-\mathbf{G}\hat\beta) \le \chi^2_p(0,1-\alpha)\Big\}\,,$$

then

$$P\left[\mathbf{G}\beta \in \mathscr{E}_{1-\alpha}(\mathbf{G}\beta)\right] = 1-\alpha\,.$$

□

Remark 3. If the vector β is divided into k "natural" subvectors $\beta_{(1)}, \dots, \beta_{(k)}$ of the same dimension $l < r$ (for example the "natural" subvectors could be $\beta_{(i)} = \left(\delta\beta_1^i, \delta\beta_2^i\right)'$, $i = 1, \dots, k$, in the situation described in the introduction), the joint confidence ellipsoids for these subvectors can be derived.

Denote $A_i = (0_{l,l}, \dots, \mathbf{I}_l, \dots, 0_{l,l})$, $i = 1, \dots, k$, the $(1 \times k)$ block matrix, where the ith block is the matrix $\mathbf{I}_l$, the other blocks are zero matrices. There holds

Theorem. (Generalized Scheffé theorem). *Let* $\eta \sim N_s(\mu, \mathbf{V})$, *where* $\mathbf{V}$ *is a positive definite matrix. Let* $\mathscr{A}$ *be a class of* $t \times s$ *matrices* $\mathbf{A}$ *of the rank* $r(\mathbf{A}) = t < s$. *Then*

$$P\left[(\eta-\mu)'\mathbf{V}^{-1}(\eta-\mu) \le \chi^2_s(0, 1-\alpha)\right] = 1-\alpha$$

$$\iff \quad P\Big[\forall\{\mathbf{A}\in\mathscr{A}\}\colon\ \big[\mathbf{A}(\eta-\mu)\big]'(\mathbf{AVA}')^{-1}\mathbf{A}(\eta-\mu)\le\chi_s^2(0,1-\alpha)\Big]=1-\alpha\,. \tag{7}$$

P r o o f . [3], Theorem 2.2. ☐

Using the preceding theorem and the form (6) we get

$$P\Big[\forall i=1,\dots,k\colon\ \big[\mathbf{A}_i(\hat\beta-\beta)\big]'\big[\mathbf{A}_i\big(\mathrm{var}(\hat\beta)\big)\mathbf{A}_i'\big]^{-1}\big[\mathbf{A}_i(\hat\beta-\beta)\big]\le\chi_r^2(0,1-\alpha)\Big]\ge 1-\alpha\,,$$

i.e.,

$$P\Big[\forall i=1,\dots,k\colon\ \big(\hat\beta_{(i)}-\beta_{(i)}\big)'\big[\mathrm{var}\big(\hat\beta_{(i)}\big)\big]^{-1}\big(\hat\beta_{(i)}-\beta_{(i)}\big)\le\chi_r^2(0,1-\alpha)\Big]\ge 1-\alpha\,.$$

Thus

$$\mathscr{E}^{\,s}_{1-\alpha}\big(\beta_{(i)}\big)=\Big\{u\in\mathbb{R}^l\colon\ \big(u-\hat\beta_{(i)}\big)'\big[\mathrm{var}\big(\hat\beta_{(i)}\big)\big]^{-1}\big(u-\hat\beta_{(i)}\big)\le\chi_r^2(0,1-\alpha)\Big\}\,,$$

are *the joint confidence ellipsoids for the subvectors* $\beta_{(1)},\dots,\beta_{(k)}$ *of the vector* β.

3. Uniformly best linear unbiased estimators

Lemma 1. *Let the matrix* $(\mathbf{W},\mathbf{Z})$ *is of full rank in columns. Then for the projection matrix* $\mathbf{M}_{(W,Z)}$ *it holds*

$$\mathbf{M}_{(W,Z)}=\mathbf{P}_W+\mathbf{M}_W\mathbf{P}_Z^{M_W}=\mathbf{P}_Z+\mathbf{M}_Z\mathbf{P}_W^{M_Z}=\mathbf{M}_W\mathbf{M}_Z^{M_W}=\mathbf{M}_Z\mathbf{M}_W^{M_Z}\,.$$

P r o o f . a) Owing to the form (4) of the Rohde's formula

$$\begin{aligned}\mathbf{P}_{(W,Z)}&=(\mathbf{W},\mathbf{Z})\left[\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}(\mathbf{W},\mathbf{Z})\right]^{-1}\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}\\ &=(\mathbf{W},\mathbf{Z})\begin{bmatrix}\mathbf{W}'\mathbf{W}, & \mathbf{W}'\mathbf{Z}\\ \mathbf{Z}'\mathbf{W}, & \mathbf{Z}'\mathbf{Z}\end{bmatrix}^{-1}\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}\\ &=(\mathbf{W},\mathbf{Z})\begin{bmatrix}\mathbf{A}_{11}, & \mathbf{A}_{12}\\ \mathbf{A}_{21}, & \mathbf{A}_{22}\end{bmatrix}\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}&\mathbf{A}_{11}=(\mathbf{W}'\mathbf{W}-\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W})^{-1}=(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1},\\ &\mathbf{A}_{12}=-(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1},\quad \mathbf{A}_{21}=(\mathbf{A}_{12})',\\ &\mathbf{A}_{22}=(\mathbf{Z}'\mathbf{Z})^{-1}+(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1},\end{aligned}$$

we get

$$\begin{aligned}\mathbf{P}_{(W,Z)} &= \mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}' - \mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{P}_Z \\ &\quad - \mathbf{P}_Z\mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}' + \mathbf{P}_Z + \mathbf{P}_Z\mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{P}_Z \\ &= \mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{M}_Z - \mathbf{P}_Z\mathbf{W}(\mathbf{W}'\mathbf{M}_Z\mathbf{W})^{-1}\mathbf{W}'\mathbf{M}_Z + P_Z \\ &= \mathbf{P}_W^{M_Z} - \mathbf{P}_Z\mathbf{P}_W^{M_Z} + \mathbf{P}_Z = \mathbf{P}_Z + \mathbf{M}_Z\mathbf{P}_W^{M_Z}\,.\end{aligned}$$

$$\mathbf{M}_{(W,Z)} = \mathbf{I} - \mathbf{P}_{(W,Z)} = \mathbf{M}_Z\mathbf{M}_W^{M_Z}\,.$$

b) According to the form (5) of the Rohde's formula

$$\mathbf{P}_{(W,Z)} = (\mathbf{W},\mathbf{Z})\begin{bmatrix}\mathbf{B}_{11}, & \mathbf{B}_{12} \\ \mathbf{B}_{21}, & \mathbf{B}_{22}\end{bmatrix}\begin{pmatrix}\mathbf{W}' \\ \mathbf{Z}'\end{pmatrix},$$

where

$$\begin{aligned}\mathbf{B}_{11} &= (\mathbf{W}'\mathbf{W})^{-1} + (\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{Z}\big[\mathbf{Z}'\mathbf{Z} - \mathbf{Z}'\mathbf{W}(\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{Z}\big]^{-1}\mathbf{Z}'\mathbf{W}(\mathbf{W}'\mathbf{W})^{-1} \\ &= (\mathbf{W}'\mathbf{W})^{-1} + (\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}(\mathbf{W}'\mathbf{W})^{-1}\,,\end{aligned}$$

$$\mathbf{B}_{12} = -(\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}, \quad \mathbf{B}_{21} = (\mathbf{B}_{12})', \quad \mathbf{B}_{22} = (\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}.$$

Thus

$$\begin{aligned}\mathbf{P}_{(W,Z)} &= \mathbf{P}_W + \mathbf{P}_W\mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{P}_W - \mathbf{P}_W\mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}' \\ &\quad - \mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{P}_W + \mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}' \\ &= \mathbf{P}_W - \mathbf{P}_W\mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{M}_W + \mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{M}_W \\ &= \mathbf{P}_W + \mathbf{M}_W\mathbf{Z}(\mathbf{Z}'\mathbf{M}_W\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{M}_W = \mathbf{P}_W + \mathbf{M}_W\mathbf{P}_Z^{M_W}\,,\end{aligned}$$

i.e.,

$$\mathbf{M}_{(W,Z)} = \mathbf{I} - \mathbf{P}_{(W,Z)} = \mathbf{M}_W\mathbf{M}_Z^{M_W}\,.$$

□

THEOREM 3. *In the regular model (2) a statistics $\mathbf{g}'\mathbf{Y}$ is the* UBLUE *of its expected value $\mathbf{g}'(\mathbf{W}\beta + \mathbf{Z}\kappa)$ if and only if*

$$\begin{aligned}\mathbf{g} \in \operatorname{Ker}\left[\sum_{i=1}^{p} V_i M_{(W,Z)} V_i\right] &= \operatorname{Ker}\left[\sum_{i=1}^{p} V_i M_Z M_W^{M_Z} V_i\right] \\ &= \operatorname{Ker}\left[\sum_{i=1}^{p} V_i M_W M_Z^{M_W} V_i\right].\end{aligned} \tag{8}$$

P r o o f . The statement is a consequence of [1], Theorem 1.2.1. and Lemma 1.

□

Theorem 4. *In the regular model* (2) *the* UBLUE *of a linear function*

$$\mathbf{f}'\begin{pmatrix}\beta\\ \kappa\end{pmatrix} = (\mathbf{f}'_1, \mathbf{f}'_2)\begin{pmatrix}\beta\\ \kappa\end{pmatrix} = \mathbf{f}'_1\beta + \mathbf{f}'_2\kappa$$

exists if and only if

$$\mathbf{f} \in \mathscr{R}\left(\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix} B\right),$$

where

$$\mathscr{R}(B) = \mathrm{Ker}\left[\sum_{i=1}^{p} V_i M_W M_Z^{M_W} V_i\right].$$

P r o o f. According to the Theorem 3 the UBLUE of the linear function

$$\mathbf{f}'\begin{pmatrix}\beta\\ \kappa\end{pmatrix} = \mathbf{f}'_1\beta + \mathbf{f}'_2\kappa$$

exists iff there exists a vector $\mathbf{g}$ for which

$$\mathbf{g} \in \mathrm{Ker}\left[\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i\right] = \bigcap_{i=1}^{p} \mathrm{Ker}\big(\mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i\big) = \mathscr{R}(\mathbf{B}),$$

such that

$$E(\mathbf{g}'\mathbf{Y}) = \mathbf{g}'(\mathbf{W}\beta + \mathbf{Z}\kappa) = \mathbf{f}'_1\beta + \mathbf{f}'_2\kappa, \qquad \forall\beta,\ \forall\kappa,$$

i.e., such that

$$\mathbf{f}_1 = \mathbf{W}'\mathbf{g} \quad \text{and} \quad \mathbf{f}_2 = \mathbf{Z}'\mathbf{g} \qquad \Longleftrightarrow \qquad \mathbf{f} = \begin{pmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{pmatrix} = \begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}\mathbf{g}.$$

Thus the UBLUE of $\mathbf{f}'\beta + \mathbf{f}'_2\kappa$ exists iff

$$\mathbf{f} = (\mathbf{f}'_1, \mathbf{f}'_2)' \in \mathscr{R}\left(\begin{pmatrix}\mathbf{W}'\\ \mathbf{Z}'\end{pmatrix}\mathbf{B}\right).$$

□

Theorem 5. *In the regular linear model* (2) *the* UBLUE *of a linear function $f'_1\beta$ of the useful parameters exists if and only if*

$$\mathbf{f}_1 \in \mathscr{R}(\mathbf{W}'\mathbf{M}_Z\mathbf{C}), \qquad \textit{where} \quad \mathscr{R}(\mathbf{C}) = \mathrm{Ker}\left[\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i \mathbf{M}_Z\right].$$

P r o o f. $\mathbf{g}'\mathbf{Y}$ is an unbiased estimator of the linear function $\mathbf{f}'_1\beta$, of the useful parameters, if

$$\mathbf{g}'(\mathbf{W}\beta + \mathbf{Z}\kappa) = \mathbf{f}'_1\beta, \qquad \forall\beta,\ \forall\kappa,$$

i.e., if

$$\mathbf{g}'\mathbf{W} = \mathbf{f}_1' \quad \text{and} \quad \mathbf{g}'\mathbf{Z} = 0\,,$$

thus if

$$\mathbf{f}_1 = \mathbf{W}'\mathbf{g} \quad \text{and} \quad \mathbf{g} = \mathbf{M}_Z\mathbf{v} \iff \mathbf{f}_1 = \mathbf{W}'\mathbf{M}_Z\mathbf{v}\,, \quad \mathbf{v} \in \mathbb{R}^n.$$

According to the Theorem 3 $\mathbf{g}'\mathbf{Y}$ is the UBLUE od its mean value $\mathbf{g}'(\mathbf{W}\beta + \mathbf{Z}\kappa)$ iff

$$\mathbf{g} \in \operatorname{Ker}\left[\sum_{i=1}^{p} \mathbf{V}_i \mathbf{M}_Z \mathbf{M}_Z^{M_W} \mathbf{V}_i\right] = \bigcap_{i=1}^{p} \operatorname{Ker}\big(\mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i\big)\,.$$

It means that $\forall\, i = 1, \dots, p$

$$\mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i \mathbf{g} = 0\,,$$
$$\mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i \mathbf{M}_Z \mathbf{v} = 0\,.$$

Thus

$$\mathbf{v} \in \bigcap_{i=1}^{p} \operatorname{Ker}(\mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i \mathbf{M}_Z) = \operatorname{Ker}\left[\sum_{i=1}^{p} (\mathbf{V}_i \mathbf{M}_Z \mathbf{M}_W^{M_Z} \mathbf{V}_i \mathbf{M}_Z)\right] = \mathscr{R}(\mathbf{C})\,.$$

It proves that the function $\mathbf{f}_1'\beta$ has its UBLUE iff

$$\mathbf{f}_1 \in \mathscr{R}(\mathbf{W}'\mathbf{M}_Z\mathbf{C})\,.$$

□

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