Tatra Mt. Math. Publ. **21** (2001), 51–59

Tatra Mountains
Mathematical Publications

LINEAR SYSTEMS WITH FUZZY POINTS

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ABSTRACT. The paper describes fuzzy point solutions of linear systems with fuzzy point coefficients. Fuzzy points are considered as a singular case of fuzzy numbers.

1. Introduction

A fuzzy interpretation of linear equations was considered in many papers (cf. references in [3]). Recently fuzzy linear systems were considered by B. I. Eke [2]. We are going to discuss in details this point of view.

The concept of fuzzy singletons was introduced by S. Gottwald [4] and L. A. Zadeh [8]. It was a notion of a fuzzy point as a special case of a fuzzy set. Another interpretation of fuzzy points was presented by C. K. Wong [7] as a fuzzy element of fuzzy sets. Contradictory consequences of such interpretation were explained by S. Gottwald [5]. Therefore fuzzy points were not popular during the last 20 years.

We consider fuzzy points as a special case of fuzzy numbers, i.e., suitable fuzzy subsets of the real line. Our considerations are motivated by paper [2]. However, we do not follow the interpretation of fuzzy points as elements of a fuzzy field.

After a short presentation of algebraic properties of fuzzy numbers we describe linear systems with fuzzy point coefficients. Next, we discuss the existence of fuzzy point solutions. Finally, we describe the family of all solutions of such linear systems.

2. Fuzzy points

Let $F(X)$ denote the family of all fuzzy subsets $A\colon X \to [0,1]$ of given universe X. A fuzzy point is a special case of fuzzy subset.

2000 Mathematics Subject Classification: 15A06, 15A33, 03E72.
Keywords: fuzzy point, extension principle, linear system, fuzzy solution.

Definition 1. By the *fuzzy point* with a support $c \in X$ and a grade of membership $t \in [0,1]$ we call the fuzzy set $\langle c,t\rangle \colon X \to [0,1]$,

$$\langle c,t\rangle(x) = \begin{cases} t & \text{if } x = c\,, \\ 0 & \text{if } x \neq c\,. \end{cases} \tag{1}$$

The family of all fuzzy points will be denoted by FP.

We apply Zadeh's [9] extension principle to binary operations on a universe X.

Definition 2. By the *extension* of a binary operation $H \colon X \times X \to X$ we call the operation $\widetilde{H} \colon F(X) \times F(X) \to F(X)$,

$$\widetilde{H}(A,B)(z) = \bigvee_{H(x,y)=z} A(x) \wedge B(y)\,, \qquad A,B \in F(X)\,, \quad z \in X. \tag{2}$$

D. Dubois and H. Prade [1] observed that

Theorem 1. *If a binary operation H is associative and commutative, then its extension $\widetilde{H}$ is also associative and commutative.*

We shall show that the extension of a binary operation transforms fuzzy points onto fuzzy points.

Theorem 2. *Let $\langle a,s\rangle, \langle b,t\rangle \in$ FP. If H is a binary operation, then*

$$\widetilde{H}\big(\langle a,s\rangle, \langle b,t\rangle\big) = \big\langle H(a,b), s \wedge t\big\rangle\,. \tag{3}$$

Proof. Using formulas (1) and (2) we get

$$\widetilde{H}\big(\langle a,s\rangle, \langle b,t\rangle\big)(z) = \bigvee_{H(x,y)=z} \langle a,s\rangle(x) \wedge \langle b,t\rangle(y)$$
$$= \left\{ \begin{matrix} s \wedge t & \text{if } x = a,\ y = b,\ z = H(a,b) \\ 0 & \text{if } z \neq H(a,b) \end{matrix} \right\} = \big\langle H(a,b)\,,\ s \wedge t\big\rangle\,.$$

□

Further we shall omit the sign $\sim$ (an application of the extension may be recognized by arguments). Putting $X = \mathbb{R}$ we consider arithmetic operations on fuzzy points. As a consequence of (3) we get

Theorem 3. *The extensions of arithmetic operations on fuzzy points produce fuzzy points, and for $\langle a,s\rangle, \langle b,t\rangle \in$ FP we have*

$$\langle a,s\rangle + \langle b,t\rangle = \big\langle (a+b), s \wedge t\big\rangle\,, \qquad \langle a,s\rangle - \langle b,t\rangle = \langle a-b, s \wedge t\rangle\,,$$
$$\langle a,s\rangle \cdot \langle b,t\rangle = \langle a \cdot b, s \wedge t\rangle\,, \qquad b \neq 0 \ \Rightarrow\ \frac{\langle a,s\rangle}{\langle b,t\rangle} = \left\langle \left(\frac{a}{b}\right), s \wedge t\right\rangle.$$

By the induction rule we obtain

Corollary 1. *For arbitrary $n \in \mathbb{N}$, $\langle a_k, s_k \rangle \in \mathrm{FP}$, $k = 1, \dots, n$ we get*

$$\sum_{k=1}^{n} \langle a_k, s_k \rangle = \left\langle \sum_{k=1}^{n} a_k, \bigwedge_{k=1}^{n} s_k \right\rangle, \qquad \prod_{k=1}^{n} \langle a_k, s_k \rangle = \left\langle \prod_{k=1}^{n} a_k, \bigwedge_{k=1}^{n} s_k \right\rangle. \tag{4}$$

Lemma 1. *The multiplication of fuzzy points is distributive with respect to the addition.*

P r o o f . From Theorem 3 for $\langle a, s \rangle, \langle b, t \rangle, \langle c, u \rangle \in \mathrm{FP}$ we have

$$\langle a, s \rangle \langle b+c, t \wedge u \rangle = \langle ab+ac, s \wedge t \wedge u \rangle = \langle ab+ac, s \wedge t \wedge s \wedge u \rangle = \langle ab, s \wedge t \rangle + \langle ac, s \wedge u \rangle .$$

□

The above result leads to the description of an algebraic structure in FP. From Theorem 1, Theorem 3 and Lemma 1 we see that

Theorem 4. $\big(\mathrm{FP}, +, \cdot, \langle 0, 1 \rangle, \langle 1, 1 \rangle\big)$ *is a commutative semiring with unit* $\langle 1, 1 \rangle$ *and zero* $\langle 0, 1 \rangle$.

Due to Theorem 3 we can use the subtraction and the division of fuzzy points, but they are not strict inverses of the addition and the multiplication.

Example 1. Let $\langle a, s \rangle \in \mathrm{FP}$. If $s < 1$ then

$$\langle a, s \rangle + \langle -a, s \rangle = \langle 0 \, , s \wedge s \rangle \neq \langle 0, 1 \rangle , \quad \langle a, s \rangle \cdot \left\langle \frac{1}{a}, s \right\rangle = \langle 1, s \wedge s \rangle \neq \langle 1, 1 \rangle.$$

Therefore semigroups $(\mathrm{FP}, +)$ and $(\mathrm{FP}, \cdot)$ do not create a field or a ring of fuzzy points.

3. Linear systems in FP

Now we pay attention to a fuzzification of linear systems (cf. Definition 2).

Definition 3. A linear system with fuzzy points coefficients has the form

$$\sum_{k=1}^{n} \langle a_{ik}, r_{ik} \rangle \langle x_k, t_k \rangle = \langle b_i, s_i \rangle , \qquad i = 1, \dots, m , \tag{5}$$

where $\langle a_{ik}, r_{ik} \rangle, \langle b_i, s_i \rangle \in \mathrm{FP}$ are given and $\langle x_k, t_k \rangle \in \mathrm{FP}$ are unknown for $i = 1, \dots, m$, $k = 1, \dots, n$.

Since the minimum operation $\wedge$ is associative and commutative, then

$$\bigwedge_{k=1}^{n} (r_{ik} \wedge t_k) = \left(\bigwedge_{k=1}^{n} r_{ik} \right) \wedge \left(\bigwedge_{k=1}^{n} t_k \right), \qquad i = 1, \dots, m .$$

Now, from Theorem 3 and Corollary 1 we see that (cf. [2], Proposition 2.2)

Lemma 2. *If the system (5) has a solution, then*

$$\sum_{k=1}^{n} a_{ik} x_k = b_i\,, \qquad i = 1, \ldots, m\,, \tag{6}$$

$$\left(\bigwedge_{k=1}^{n} r_{ik}\right) \wedge \left(\bigwedge_{k=1}^{n} t_k\right) = s_i\,, \qquad i = 1, \ldots, m\,. \tag{7}$$

As a consequence of the above consideration we get

Theorem 5. *Let $\langle a_{ik}, r_{ik}\rangle$, $\langle b_i, s_i\rangle \in \mathrm{FP}$ for $i = 1, \ldots, m$, $k = 1, \ldots, n$. If the system (5) has a solution $\langle x_k, t_k\rangle \in \mathrm{FP}$ for $k = 1, \ldots, n$, then the system (6) has a solution $x_1, \ldots, x_n$ and*

$$r_i \wedge t = s_i\,, \qquad i = 1, \ldots, m\,, \tag{8}$$

$$r_i \geq s_i\,, \qquad i = 1, \ldots, m\,, \tag{9}$$

$$t \geq s\,, \tag{10}$$

where

$$r_i = \bigwedge_{k=1}^{n} r_{ik}\,, \qquad i = 1, \ldots, m\,, \qquad t = \bigwedge_{k=1}^{n} t_k\,, \qquad s = \bigvee_{i=1}^{m} s_i\,. \tag{11}$$

The above theorem brings necessary conditions for the existence of fuzzy point solutions of system (5). In particular condition (9) is necessary. However, it is not sufficient for the existence of membership grade t fulfilling (8) and (10).

Example 2. Let us consider the following fuzzy point system

$$\begin{cases} \langle 2, \frac{1}{2}\rangle\langle x, u\rangle + \langle 3, \frac{1}{3}\rangle\langle y, t\rangle = \langle 5, s_1\rangle\,, \\ \langle 3, \frac{1}{3}\rangle\langle x, u\rangle + \langle 4, \frac{1}{4}\rangle\langle y, t\rangle = \langle 7, s_2\rangle\,. \end{cases}$$

The conventional system has the unique solution $x = 1$, $y = 1$, but for $s_1 = \frac{1}{5}$, $s_2 = \frac{1}{7}$ there does not exist $t \in [0, 1]$ fulfilling (8).

4. Families of solutions

Here we assume that system (6) has a solution $x_1, \ldots, x_n$. Usually grades of membership in (5) are not unique and we have a rich family of fuzzy points which fulfil (5). Conditions (9) can be strengthened in order to obtain (8). We can do it in two ways. The first case was indicated by E k e [2], Theorem 3.16. It is evident that

Lemma 3. *If grades of membership in* (5) *fulfil condition*

$$r_i = s_i \qquad \text{for} \quad i = 1, \ldots, m\,, \tag{12}$$

then every $t \in [s, 1]$ *fulfils* (8).

The second case needs an additional notation. Assuming (9) we put

$$I = \{i \colon r_i > s_i\} \subset \{1, 2, \ldots, m\}\,. \tag{13}$$

Lemma 4. *If grades of membership in* (5) *fulfil conditions* (9) *and*

$$\forall_{i \in I}\, (s_i = s)\,, \tag{14}$$

then $t = s$ *fulfils* (8).

P r o o f . Let us assume (9). If $i \notin I$, then from (13) we get $r_i = s_i \leq s$ and $r_i \wedge s = s_i$. If $i \in I$, then $r_i \wedge s = s = s_i$ which gives (8) for $t = s$. □

Now we present the full description of the solution family.

Theorem 6. *Let system* (6) *have a solution* $x_1, \ldots, x_n$.

(a) *If we assume* (12), *then for every* $t_k \in [s, 1]$, *the sequence* $\langle x_k, t_k \rangle$, $k = 1, \ldots, n$, *is a solution of* (5).

(b) *If we assume* (9), (14), *and* $t_{k_0} = s$ *for fixed* $k_0 \in \{1\, dots, n\}$, *then for every* $t_k \in [s, 1]$, $k \neq k_0$, *the sequence* $\langle x_k, t_k \rangle$, $k = 1, \ldots, n$, *is a solution of* (5).

(c) *If there exists* $i_0 \in \{1, \ldots, m\}$ *such that*

$$r_{i_0} < s_{i_0} \qquad \text{or} \qquad s_{i_0} < r_{i_0} \wedge s\,, \tag{15}$$

then system (5) *does not have a solution with supports* $x_1, \ldots, x_n$.

P r o o f . Case (a) is a direct consequence of Lemma 3. Similarly by Lemma 4 we get case (b). The solution family is empty if we contradict (9) or (14) which gives (15) and we obtain case (c). □

Let us observe that condition (12) is a special case of (9) with (14) for $I = \emptyset$. Thus we have

Corollary 2. *Let system* (6) *have a solution. System* (5) *has a solution*

(a) *iff grades of membership fulfil* (9) *and* (14);

(b) *iff there exists a constant* $t \in [0, 1]$ *which fulfils* (8).

5. Classical methods with fuzzy points

Now we intend to discuss classical solution methods with the arithmetic of fuzzy points. Let us assume that $m = n$ with Cramer's system (6). Since the determinant of a matrix essentially depends on all its elements, then from Corollary 1 we get

LEMMA 5. *For a square matrix of fuzzy points we have*

$$\det[\langle a_{ik}, r_{ik}\rangle] = \left\langle \det[a_{ik}]\,,\ \bigwedge_{i,k=1}^{n} r_{ik} \right\rangle .$$

THEOREM 7. *Let* (6) *be a Cramer's system. The Cramer's solution with fuzzy points has the smallest grade of membership in the set of all coefficients cf.* (11)

$$t_k = \bigwedge_{i=1}^{n} (r_i \wedge s_i)\,, \qquad k = 1, \dots, n\,. \tag{16}$$

P r o o f. In virtue of Lemma 5, fuzzy point determinants in Cramer's rule have the form $\langle \det A, u\rangle$, $\langle \det A_k, u_k\rangle$, $k = 1, \dots, n$, where

$$u = \bigwedge_{i=1}^{n} \bigwedge_{k=1}^{n} r_{ik}, \qquad u_k = \left(\bigwedge_{i=1}^{n} \bigwedge_{\substack{j=1 \\ j\neq k}}^{n} r_{ij} \right) \wedge \left(\bigwedge_{i=1}^{n} s_i \right). \tag{17}$$

Therefore from Theorem 3 we get

$$\langle x_k, t_k\rangle = \left\langle \frac{\det A_k}{\det A}\,,\ u \wedge u_k \right\rangle, \qquad k = 1, \dots, n\,, \tag{18}$$

which proves (16). □

COROLLARY 3. *If* $s_i \neq \mathrm{const}.$ *in the system* (5), *then the Cramer's solution* (18) *is not a fuzzy point solution.*

P r o o f. From (16) we get

$$t = \bigwedge_{k=1}^{n} t_k = \bigwedge_{i=1}^{n} (r_i \wedge s_i) \leq s_j\,, \qquad j = 1, \dots, n\,.$$

If there exists $j_0 \in \{1, \dots, n\}$ such that $s_{j_0} < s$, then $t < s$, contradictory to (10). Thus fuzzy points (18) do not form a solution of (5) (cf. Theorem 5). □

EXAMPLE 3. Let us consider Cramer's system from Example 2 with $s_1 = \frac{1}{3}$, $s_2 = \frac{1}{4}$. Since formula (16) gives $t_1 = t_2 = \frac{1}{4} < s = \frac{1}{3}$, then fuzzy points $\langle x, u\rangle = \langle 1, t_1\rangle$, $\langle y, t\rangle = \langle 1, t_2\rangle$ determined by (18) are not a solution.

A similar situation appears in the Gaussian algorithm (cf. [6], Chapter 9):

$$a_{ik}^{(0)} = a_{ik}\,, \qquad a_{ik}^{(j)} = a_{ik}^{(j-1)} - a_{ij}^{(j-1)} \frac{a_{jk}^{(j-1)}}{a_{jj}^{(j-1)}}\,, \tag{19}$$

$$b_i^{(0)} = b_i\,, \qquad b_i^{(j)} = b_i^{(j-1)} - a_{ij}^{(j-1)} \frac{b_j^{(j-1)}}{a_{jj}^{(j-1)}} \tag{20}$$

for $j = 1, 2, \ldots, n-1$, $i, k = j+1, j+2, \ldots, n$,

$$x_j = \frac{\left(b_j^{(j-1)} - \sum\limits_{k=j+1}^{n} a_{jk}^{(j-1)} x_k\right)}{a_{jj}^{(j-1)}}\,, \qquad j = n, n-1, \ldots, 1\,. \tag{21}$$

LEMMA 6. *Grades of membership of fuzzy points in formulas* (20) *and* (21) *fulfil inequalities*

$$s_i^{(i-1)} \le \bigwedge_{k=1}^{i} s_k\,, \qquad i = 1, \ldots, n\,, \tag{22}$$

$$t_n \le \bigwedge_{i=1}^{n} s_i\,, \tag{23}$$

$$t_j \le \bigwedge_{k=j+1}^{n} t_k\,, \qquad j = 1, \ldots, n-1\,. \tag{24}$$

Proof. Using Theorem 3 in calculations (20) we observe that suitable grades of membership are bounded from above in the following way

$$\begin{aligned} s_i^{(0)} &= s_i\,, \\ s_i^{(j)} &\le s_i^{(j-1)} \wedge s_j^{(j-1)} \le s_j^{(j-2)} \wedge s_i^{(j-2)} \wedge s_{j-1}^{(j-2)} \\ &\le s_i^{(j-3)} \wedge s_j^{(j-3)} \wedge s_{j-1}^{(j-3)} \wedge s_{j-2}^{(j-3)} \le \cdots \le s_i \wedge \bigwedge_{k=1}^{j} s_k\,, \end{aligned}$$

which proves (22) ($j = i-1$). Thus (23) is implied by (21) for $j = n$ and (22) for $i = n$. Since x_j depends on $x_{j+1}, \ldots, x_n$ for $j = 1, 2, \ldots, n-1$, then from (21) we get (24). □

Immediately from (23) and (24) we get

THEOREM 8. *Let (6) be a Cramer's system. Gaussian algorithm with fuzzy points brings a solution with grades of membership fulfilling inequalities*

$$t_k \le \bigwedge_{i=1}^{n} s_i\,, \qquad k = 1, \dots, n\,. \tag{25}$$

In the same way as in Corollary 3 we get

COROLLARY 4. *If* $s_i \neq \mathrm{const}$ *in the system* (5), *then the solution from the Gaussian algorithm* (19)–(21) *is not a fuzzy point solution.*

6. Concluding remarks

From the above consideration we see that the problem of solution of linear systems with fuzzy point coefficients has rather marginal meaning. The main solvability assumption from Lemma 3 guarantees that system (5) has the solution with the maximal grades of membership $t_k = 1$, $k = 1, \dots, n$, which coincides with the conventional solution (support solution). A slight difference we have under assumptions from Lemma 4, where one grade of membership from $t_1, \dots, t_n$ must differ from 1.

In this point of view, Eke [2] introduces too many notions and notations for such systems. A description of fuzzy point solution can be based on classical notions concerning conventional systems (6) (cf. Theorem 6).

Classical methods from linear algebra are not adequate for fuzzy point arithmetic from Theorems 3 and 4, as we show in Corollaries 3 and 4. In particular all considerations on Gaussian algorithm with fuzzy points in [2] are useless. It seems that our results can be treated as an interesting example of calculations with singular fuzzy numbers.

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Received September 10, 2000

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