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CHAINS, KITE-TAIL LATTICES AND DENSE CHAINS

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ABSTRACT. The lattices in which for each subset the following properties hold are characterized: if a given element is (strictly) greater (resp. smaller) than the infimum (resp. supremum) of the given subset, if this infimum (resp. supremum) exists, then there always exists an element in this subset that is smaller (resp. greater) than or equal to the given element. These properties are weakened versions of well-known characterizations of chains. Furthermore, this study leads to additional characterizations of dense chains.

1. Introduction

It is well known that a chain can be characterized as a lattice in which for each subset the following properties hold: *if a given element is strictly greater than the infimum* (*resp. strictly smaller than the supremum*) *of the given subset, if this infimum* (*resp. supremum*) *exists, then one can always find an element in this subset that is strictly smaller* (*resp. strictly greater*) *than the given element.* In this paper, we search for the lattices in which weakened versions of these properties hold, namely lattices in which *the element to be found in the given subset is also allowed to coincide with the element considered in the first place.*

In [4], we have introduced a new type of finite lattices, with the descriptive name "kite-tail lattices". Kite-tail lattices are defined as finite vertical sums of the chain $\mathbf{1}$ and the square $\mathbf{M_2}$. We have shown that kite-tail lattices are the only finite lattices in which the weakened properties hold.

In this paper, we will characterize, in a totally different way, all lattices, whether finite or not, in which the weakened properties hold. As an immediate consequence, we will rediscover our former results. Furthermore, these results will lead to new characterizations of dense chains.

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2. A characterization of chains

In this section, we reconstruct the above-mentioned characterization of chains. Consider an arbitrary subset A of an ordered set $(P,\leq)$. The definition of $\inf A$ as the greatest lower bound of A and the definition of $\sup A$ as the smallest upper bound of A can be written as follows:

$$\inf A=\alpha \iff (\forall\beta\in A)(\alpha\leq\beta)\ \wedge\ (\forall\gamma\in P)\big((\forall\beta\in A)(\gamma\leq\beta)\Rightarrow\gamma\leq\alpha\big)\,,$$
$$\sup A=\alpha \iff (\forall\beta\in A)(\beta\leq\alpha)\ \wedge\ (\forall\gamma\in P)\big((\forall\beta\in A)(\beta\leq\gamma)\Rightarrow\alpha\leq\gamma\big)\,.$$

These definitions have to be read with care. If the infimum in the formula '$\inf A=\alpha$' is considered as an operator on the powerset of P, then the formula "$\inf A=\alpha$" can become meaningless if $\inf A$ does not exist. This problem is resolved by interpreting "$\inf A=\alpha$" as "α is the greatest lower bound of A", a statement that can only be true or false, but never meaningless. In order to avoid similar problems in the sequel, we introduce the following notations:

(i) $\mathfrak{I}(P)$ is the subset of $\mathcal{P}(P)$ containing these subsets of P that have a greatest lower bound;

(ii) $\mathfrak{S}(P)$ is the subset of $\mathcal{P}(P)$ containing these subsets of P that have a smallest upper bound.

The formula
$$(\forall\gamma\in P)\big((\forall\beta\in A)(\gamma\leq\beta)\Rightarrow\gamma\leq\alpha\big)$$
is logically equivalent with
$$(\forall\gamma\in P)\big(\neg(\gamma\leq\alpha)\Rightarrow(\exists\beta\in A)\neg(\gamma\leq\beta)\big)\,.$$
When the ordered set $(P,\leq)$ is a chain, the latter formula can be rewritten as
$$(\forall\gamma\in P)\big(\alpha<\gamma\Rightarrow(\exists\beta\in A)(\beta<\gamma)\big)\,.$$
Hence, the definitions of the infimum and supremum of an arbitrary subset A of a chain $(L,\leq)$ can be written as:

$$\inf A=\alpha \iff (\forall\beta\in A)(\alpha\leq\beta)\ \wedge\ (\forall\gamma\in L)\big(\alpha<\gamma\Rightarrow(\exists\beta\in A)(\beta<\gamma)\big)\,,$$
$$\sup A=\alpha \iff (\forall\beta\in A)(\beta\leq\alpha)\ \wedge\ (\forall\gamma\in L)\big(\gamma<\alpha\Rightarrow(\exists\beta\in A)(\gamma<\beta)\big)\,.$$

Proposition 1. *In a chain $(L,\leq)$ the following properties hold:*
$$\big(\forall A\in\mathfrak{I}(L)\big)(\forall\alpha\in L)\big(\inf A<\alpha\Rightarrow(\exists\beta\in A)(\beta<\alpha)\big)\,,$$
$$\big(\forall A\in\mathfrak{S}(L)\big)(\forall\alpha\in L)\big(\alpha<\sup A\Rightarrow(\exists\beta\in A)(\alpha<\beta)\big)\,.$$

Proof. The proof follows immediately from the discussion above. □

The restrictions $\big(\forall A \in \mathfrak{I}(L)\big)$ and $(\forall A \in \mathfrak{S}(L))$ in the foregoing proposition explicitly ensure that $\inf A$ and $\sup A$ effectively exist when occurring in a formula. It is perfectly possible that a chain contains non-trivial subsets that have no greatest lower bound and no smallest upper bound. The subset $]1,2[$ of the chain $\big([0,1\,[\cup]\,1,2\,[\cup]\,2,3]\,,\leq\big)$ is an example of such a subset.

Next, we consider the properties from Proposition 1 for the doubletons of a lattice $(L,\leq)$ with meet and join operation denoted by $\frown$ and $\smile$. This leads to the following simplified properties:

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\frown\beta<\gamma\Rightarrow(\alpha<\gamma\ \vee\ \beta<\gamma)\big)\,,$$
$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma<\alpha\smile\beta\Rightarrow(\gamma<\alpha\ \vee\ \gamma<\beta)\big)\,.$$

The following theorem shows that a lattice $(L,\leq)$ in which at least one of these simplified properties holds, is always a chain. Consequently, the corresponding properties for arbitrary infima and suprema mentioned in Proposition 1 hold in it. This implies that not only the simplified and the more general properties are equivalent, but also that they are mutually equivalent.

Theorem 1. (Characterization of chains). *Consider a lattice $(L,\leq)$. If at least one of the following properties holds:*

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\frown\beta<\gamma\Rightarrow(\alpha<\gamma\ \vee\ \beta<\gamma)\big)\,,$$
$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma<\alpha\smile\beta\Rightarrow(\gamma<\alpha\ \vee\ \gamma<\beta)\big)\,,$$

then $(L,\leq)$ is a chain.

P r o o f. Assume for instance that the property concerning the join operation holds. In order to show that $(L,\leq)$ is a chain, we have to demonstrate that no two elements are incomparable. We therefore suppose that α and β are two incomparable elements: $\alpha\parallel\beta$. This implies that $\alpha<\alpha\smile\beta$. Furthermore, it holds that $\neg(\alpha<\alpha)$ and $\neg(\alpha<\beta)$. These findings contradict our initial assumption. ☐

Corollary 1. *Consider a lattice $(L,\leq)$, then the following equivalence holds:*

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\frown\beta<\gamma\Rightarrow(\alpha<\gamma\ \vee\ \beta<\gamma)\big)$$
$$\Updownarrow$$
$$\big(\forall A\in\mathfrak{I}(L)\big)(\forall\alpha\in L)\big(\inf A<\alpha\Rightarrow(\exists\beta\in A)(\beta<\alpha)\big)\,.$$

P r o o f. The proof follows immediately from Proposition 1 and Theorem 1. ☐

COROLLARY 2. *Consider a lattice $(L,\leq)$, then the following equivalence holds:*

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma<\alpha\smile\beta\Rightarrow(\gamma<\alpha\ \vee\ \gamma<\beta)\big)$$
$$\Updownarrow$$
$$\big(\forall A\in\mathfrak{S}(L)\big)(\forall\alpha\in L)\big(\alpha<\sup A\Rightarrow(\exists\beta\in A)(\alpha<\beta)\big)\,.$$

Proof. The proof follows immediately from Proposition 1 and Theorem 1. □

COROLLARY 3. *Consider a lattice $(L,\leq)$, then the following equivalence holds:*

$$\big(\forall A\in\mathfrak{I}(L)\big)(\forall\alpha\in L)\big(\inf A<\alpha\Rightarrow(\exists\beta\in A)(\beta<\alpha)\big)$$
$$\Updownarrow$$
$$\big(\forall A\in\mathfrak{S}(L)\big)(\forall\alpha\in L)\big(\alpha<\sup A\Rightarrow(\exists\beta\in A)(\alpha<\beta)\big)\,.$$

Proof. The proof follows immediately from Proposition 1 and Theorem 1. □

When dealing with lattices, the foregoing results allow us to talk about the characterizing property of chains, or at least to talk about the equivalent characterizing properties of chains.

3. Study of the weakened properties

3.1. Background

Quite often, for instance in $(L,\leq)$-fuzzy set theory [3, 6], it turns out to be convenient to be able to invoke the following properties:

$$\big(\forall A\in\mathfrak{I}(L)\big)(\forall\alpha\in L)\big(\inf A<\alpha\Rightarrow(\exists\beta\in A)(\beta\leq\alpha)\big)\,,$$
$$\big(\forall A\in\mathfrak{S}(L)\big)(\forall\alpha\in L)\big(\alpha<\sup A\Rightarrow(\exists\beta\in A)(\alpha\leq\beta)\big)\,.$$

These properties closely resemble the above-mentioned equivalent characterizing properties of chains. They are in fact a little weaker because of the substitution of the strict order $<$ by the order $\leq$. It is therefore immediately clear that they hold in an arbitrary chain.

Apart from our characterization of the finite lattices in which these weakened properties hold [4] (see also Section 4), these properties have, as far as we know, not been studied in lattice theory. The weakened property concerning the supremum, for instance, appears as condition in the important theorems of Negoiță and Ralescu [6] concerning the characterization of $(L,\leq)$-fuzzy sets as generalizations of Gentilhommes "flou sets" [5]. Negoiță and

Ralescu, however, have failed to investigate the lattices in which these properties effectively hold.

Next, we consider these weakened properties for the doubletons of a lattice $(L,\leq)$. This leads to the following properties:

$$\begin{aligned}&\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\frown\beta<\gamma\Rightarrow(\alpha\leq\gamma\ \vee\ \beta\leq\gamma)\big)\,,\\&\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma<\alpha\smile\beta\Rightarrow(\gamma\leq\alpha\ \vee\ \gamma\leq\beta)\big)\,.\end{aligned}$$

We will show that these simplified properties and the properties as formulated for the infimum and supremum are equivalent, and also mutually equivalent. The results in this section will imply in particular that kite-tail lattices are the only finite lattices in which these weakened properties hold.

3.2. The $\mathbf{M_3}$-$\mathbf{N_5}$ theorem

The famous $\mathbf{M_3}$-$\mathbf{N_5}$ theorem of Dedekind and Birkhoff [1, 2] gives a very interesting and practical characterization of modular and distributive lattices. For finite lattices, this theorem allows to verify the modularity and distributivity on a corresponding Hasse diagram.

Definition 1. (Modular lattices) [1, 2] A lattice $(L,\leq)$ is *modular* if and only if

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\leq\gamma\Rightarrow\alpha\smile(\beta\frown\gamma)=\big((\alpha\smile\beta)\frown\gamma\big)\big)\,.$$

Definition 2. (Distributive lattices) [1, 2] A lattice $(L,\leq)$ is *distributive* if and only if

$$\begin{aligned}&\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\frown(\beta\smile\gamma)=(\alpha\frown\beta)\smile(\alpha\frown\gamma)\big)\,,\\&\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\smile(\beta\frown\gamma)=(\alpha\smile\beta)\frown(\alpha\smile\gamma)\big)\,.\end{aligned}$$

Clearly, any distributive lattice is modular.

Two well-known finite lattices, the diamond $\mathbf{M_3}$ and the pentagon $\mathbf{N_5}$ play an important role in the following theorem. Figure 1 depicts Hasse diagrams for these lattices. Note that $\mathbf{M_3}$ is modular, but not distributive, while $\mathbf{N_5}$ is not modular, and hence also not distributive.

Theorem 2. ($\mathbf{M_3}$-$\mathbf{N_5}$ theorem [1, 2])

(i) *A lattice is not modular if and only if it contains a sublattice that is isomorphic to* $\mathbf{N_5}$.

(ii) *A lattice is not distributive if and only if it contains a sublattice that is isomorphic to* $\mathbf{M_3}$ *or* $\mathbf{N_5}$.

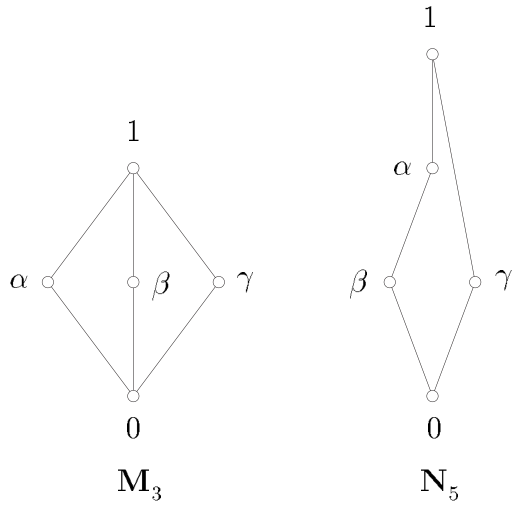

FIGURE 1. Hasse diagrams for the lattices $\mathbf{M_3}$ and $\mathbf{N_5}$.

3.3. Characterization

PROPOSITION 2. *Consider a lattice $(L, \leq)$. If at least one of the following properties holds:*

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha \frown \beta < \gamma \Rightarrow (\alpha \leq \gamma \ \vee \ \beta \leq \gamma)\big)\,,$$

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma < \alpha \smile \beta \Rightarrow (\gamma \leq \alpha \ \vee \ \gamma \leq \beta)\big)\,,$$

then $(L, \leq)$ is a distributive lattice.

Proof. Assume for instance that the property concerning the join operation holds. Suppose that the lattice $(L, \leq)$ is not distributive, then we want to obtain a contradiction. Due to the $\mathbf{M_3}$-$\mathbf{N_5}$ theorem, $(L, \leq)$ contains a sublattice that is isomorphic to $\mathbf{M_3}$ or $\mathbf{N_5}$.

(i) Suppose that the lattice $(L, \leq)$ contains a sublattice that is isomorphic to $\mathbf{M_3}$. Consider the lattice $\mathbf{M_3}$ with Hasse diagram given in Figure 1. It is clear that $\gamma < \alpha \smile \beta = 1$. However, it holds that $\neg(\gamma \leq \alpha)$ and $\neg(\gamma \leq \beta)$, which contradicts our initial assumption.

(ii) Suppose that the lattice $(L, \leq)$ contains a sublattice that is isomorphic to $\mathbf{N_5}$. Consider the lattice $\mathbf{N_5}$ with Hasse diagram given in Figure 1. It is clear that $\alpha < \beta \smile \gamma = 1$. However, it holds that $\neg(\alpha \leq \beta)$ and $\neg(\alpha \leq \gamma)$, which again contradicts our initial assumption.

□

DEFINITION 3. (Covering relation) [2] Consider an ordered set $(P, \leq)$ and $(\alpha,\beta) \in P^2$. We say that α *is covered by* β (or β *covers* α) and write $\alpha \ll \beta$ if and only if

$$\alpha < \beta \ \wedge \ \neg(\exists \gamma \in P)(\alpha < \gamma < \beta)\,.$$

Proposition 3. *Consider a lattice* $(L, \leq)$. *If at least one of the following properties holds:*

$$\big(\forall(\alpha,\beta,\gamma) \in L^3\big)\big(\alpha \frown \beta < \gamma \Rightarrow (\alpha \leq \gamma \ \vee \ \beta \leq \gamma)\big)\,,$$
$$\big(\forall(\alpha,\beta,\gamma) \in L^3\big)\big(\gamma < \alpha \smile \beta \Rightarrow (\gamma \leq \alpha \ \vee \ \gamma \leq \beta)\big)\,,$$

then also the following properties hold:

$$\big(\forall(\alpha,\beta) \in L^2\big)\big(\alpha \parallel \beta \Rightarrow (\alpha \frown \beta \ll \alpha \ \wedge \ \alpha \frown \beta \ll \beta)\big)\,,$$
$$\big(\forall(\alpha,\beta) \in L^2\big)\big(\alpha \parallel \beta \Rightarrow (\alpha \ll \alpha \smile \beta \ \wedge \ \beta \ll \alpha \smile \beta)\big)\,.$$

Proof. Assume again that the property concerning the join operation holds. Consider $(\alpha,\beta) \in L^2$ such that $\alpha \parallel \beta$.

(i) Suppose for instance that $\alpha \smile \beta$ does not cover the element α, i.e.,

$$(\exists \gamma \in L)(\alpha < \gamma < \alpha \smile \beta)\,.$$

Firstly, we show that $\beta \parallel \gamma$.

(a) If $\beta < \gamma$, then $\alpha < \gamma$ implies that $\alpha \smile \beta \leq \gamma$, which contradicts $\gamma < \alpha \smile \beta$.

(b) If on the other hand we have that $\gamma \leq \beta$, then $\alpha < \gamma$ implies that $\alpha < \beta$, which contradicts $\alpha \parallel \beta$.

Hence, we can conclude that $\beta \parallel \gamma$. This means in particular that $\neg(\gamma \leq \beta)$. Moreover, since $\alpha < \gamma$, we also have that $\neg(\gamma \leq \alpha)$.

Since $\gamma < \alpha \smile \beta$, it follows from our initial assumption that $\gamma \leq \alpha \vee \gamma \leq \beta$, which yields a contradiction. This means that $\alpha \smile \beta$ indeed covers α.

(ii) Suppose for instance that α does not cover the element $\alpha \frown \beta$, i.e.,

$$(\exists \gamma \in L)(\alpha \frown \beta < \gamma < \alpha)\,.$$

Firstly, we show that $\beta \parallel \gamma$.

(a) If $\gamma \leq \beta$, then $\gamma < \alpha$ implies that $\gamma \leq \alpha \frown \beta$, which contradicts $\alpha \frown \beta < \gamma$.

(b) If on the other hand we have that $\beta < \gamma$, then $\gamma < \alpha$ implies that $\beta < \alpha$, which contradicts $\alpha \parallel \beta$.

Hence, we can conclude that $\beta \parallel \gamma$.

Secondly, we determine $\beta \smile \gamma$. On the one hand, it follows from $\beta < \alpha \smile \beta$ and $\gamma < \alpha \smile \beta$ that $\beta \smile \gamma \leq \alpha \smile \beta$. On the other hand, if follows from $\beta \parallel \gamma$ that $\beta \smile \gamma > \beta$. The first part of the proof then implies that $\alpha \smile \beta$ covers the element β. From $\beta < \beta \smile \gamma \leq \alpha \smile \beta$ it then follows that $\beta \smile \gamma = \alpha \smile \beta$.

Since $\alpha < \alpha \smile \beta = \beta \smile \gamma$ it follows from our initial assumption that $\alpha \leq \beta \ \vee \ \alpha \leq \gamma$. This contradicts $\alpha \parallel \beta$ and $\gamma < \alpha$. Hence, α effectively covers the element $\alpha \frown \beta$. □

Proposition 4. *Consider a lattice $(L,\leq)$. If at least one of the following properties holds*

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\frown\beta<\gamma\Rightarrow(\alpha\leq\gamma\ \vee\ \beta\leq\gamma)\big)\,,$$
$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma<\alpha\smile\beta\Rightarrow(\gamma\leq\alpha\ \vee\ \gamma\leq\beta)\big)\,,$$

then the cardinality of a maximal subantichain of $(L,\leq)$ is not greater than two.

Proof. Assume again that the property concerning the join operation holds. Suppose that L contains three incomparable elements α_1, α_2 and α_3, i.e., $\alpha_1\parallel\alpha_2\parallel\alpha_3\parallel\alpha_1$. Let $\beta_1=\alpha_1\smile\alpha_2$ and $\beta_2=\alpha_2\smile\alpha_3$. Proposition 3 implies that $\alpha_1\ll\beta_1$, $\alpha_2\ll\beta_1$, $\alpha_2\ll\beta_2$ and $\alpha_3\ll\beta_2$.

We will now show that $\beta_1\parallel\beta_2$.

(a) If $\beta_1<\beta_2$, then it follows from $\alpha_2\ll\beta_1$ that $\alpha_2<\beta_1<\beta_2$, which contradicts $\alpha_2\ll\beta_2$.

(b) Similarly in case $\beta_2<\beta_1$.

(c) Finally, if $\beta_1=\beta_2$, then it follows from $\alpha_3\ll\beta_2$ that $\alpha_3<\alpha_1\smile\alpha_2$. Our initial assumption then implies that $\alpha_3\leq\alpha_1\ \vee\ \alpha_3\leq\alpha_2$, which contradicts $\alpha_1\parallel\alpha_3$ and $\alpha_2\parallel\alpha_3$.

Next, we will show that $\alpha_3\parallel\beta_1$.

(a) If $\beta_1\leq\alpha_3$, then it follows that $\alpha_1\leq\alpha_3$, which contradicts $\alpha_1\parallel\alpha_3$.

(b) If $\alpha_3<\beta_1$, then $\alpha_1\smile\alpha_2\smile\alpha_3\leq\beta_1$ and hence $\beta_1\smile\beta_2\leq\beta_1$, or also $\beta_1\smile\beta_2=\beta_1$, which contradicts $\beta_1\parallel\beta_2$.

Now let $\gamma=\alpha_1\smile\alpha_2\smile\alpha_3$ then it follows that $\gamma=\beta_1\smile\beta_2$ and $\gamma=\beta_1\smile\alpha_3$. Since $\beta_1\parallel\beta_2$, it follows from Proposition 3 that $\beta_2\ll\gamma$. Together with $\alpha_3\ll\beta_2$, this means that $\neg(\alpha_3\ll\gamma)$. However, since $\beta_1\parallel\alpha_3$ it also follows from Proposition 3 that $\alpha_3\ll\gamma$, which clearly yields a contradiction. The particular situation in this proof is depicted in Figure 2. □

Corollary 4. *Consider a lattice $(L,\leq)$. If at least one of the following properties holds*

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\alpha\frown\beta<\gamma\Rightarrow(\alpha\leq\gamma\ \vee\ \beta\leq\gamma)\big)\,,$$
$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma<\alpha\smile\beta\Rightarrow(\gamma\leq\alpha\ \vee\ \gamma\leq\beta)\big)\,,$$

then:

(i) *each element of L covers at most two other elements;*

(ii) *each element of L is covered by at most two other elements.*

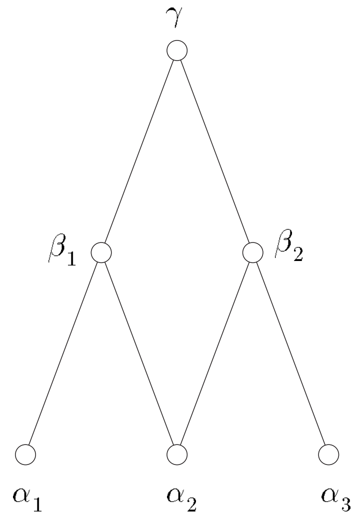

FIGURE 2. Illustration to the proof of Proposition 4

THEOREM 3. *Consider a lattice $(L,\leq)$, then the following equivalence holds:*

$$\big(\forall(\alpha,\beta,\gamma)\in L^3\big)\big(\gamma<\alpha\smile\beta\Rightarrow(\gamma\leq\alpha\ \vee\ \gamma\leq\beta)\big)$$
$$\Updownarrow$$
$$\big(\forall(\alpha,\beta)\in L^2\big)\big(\alpha\parallel\beta\Rightarrow\big(\forall\gamma\in L\setminus\{\alpha,\beta\}\big)(\gamma\leq\alpha\frown\beta\ \vee\ \alpha\smile\beta\leq\gamma)\big)\,.$$

Proof.

(i) We will first show the implication from top to bottom. Consider $(\alpha,\beta,\gamma)\in L^3$ such that $\alpha\parallel\beta$, $\gamma\neq\alpha$ and $\gamma\neq\beta$. From Proposition 4 it follows that $\neg(\alpha\parallel\gamma)$ or $\neg(\beta\parallel\gamma)$. Suppose for instance that $\neg(\alpha\parallel\gamma)$. Further suppose that $\gamma<\alpha$, then we will show that $\gamma\leq\alpha\frown\beta$. Suppose that $\neg(\gamma\leq\alpha\frown\beta)$, then we want to obtain a contradiction. From $\neg(\gamma\leq\alpha\frown\beta)$ it follows that $\alpha\frown\beta<\gamma$ or $\gamma\parallel\alpha\frown\beta$.

In case $\alpha\frown\beta<\gamma$, Proposition 3 leads to a contradiction. Indeed, from $\alpha\parallel\beta$ it follows that $\alpha\frown\beta\ll\alpha$, which contradicts $\alpha\frown\beta<\gamma<\alpha$. Hence, we have to consider the case $\gamma\parallel\alpha\frown\beta$. We will first demonstrate that γ and β are incomparable.

(a) If $\gamma\leq\beta$, then it follows with $\gamma<\alpha$ that $\gamma\leq\alpha\frown\beta$, which contradicts $\gamma\parallel\alpha\frown\beta$.

(b) If on the other hand $\beta<\gamma$, then $\gamma<\alpha$ implies that $\beta<\alpha$, which contradicts $\alpha\parallel\beta$.

Hence, we can conclude that $\beta\parallel\gamma$.

Next, we determine $\beta\smile\gamma$. From $\gamma<\alpha$ it follows with the monotonicity of the join operation that $\beta\smile\gamma\leq\alpha\smile\beta$. We will now show that

$\beta \smile \gamma < \alpha \smile \beta$ leads to a contradiction. In that case, our initial assumption implies that $\beta \smile \gamma \leq \alpha$ or $\beta \smile \gamma \leq \beta$.

(a) If $\beta \smile \gamma \leq \alpha$, then it also follows that $\beta \leq \alpha$, which contradicts $\alpha \parallel \beta$.

(b) If on the other hand $\beta \smile \gamma \leq \beta$, then it follows with $\beta \leq \beta \smile \gamma$ that $\beta \smile \gamma = \beta$ and hence $\gamma \leq \beta$, which contradicts $\beta \parallel \gamma$.

We can therefore conclude that $\beta \smile \gamma = \alpha \smile \beta$. Since $\beta \parallel \gamma$, it now follows with Proposition 3 that $\gamma \ll \beta \smile \gamma = \alpha \smile \beta$. This is in disagreement with $\gamma < \alpha < \alpha \smile \beta$ and yields the desired contradiction. Hence, it is indeed so that $\gamma \leq \alpha \frown \beta$.

(ii) Next, we show the implication from bottom to top. Consider $(\alpha, \beta, \gamma) \in L^3$ such that $\gamma < \alpha \smile \beta$. If $\gamma = \alpha$ or $\gamma = \beta$, then of course it holds that $\gamma \leq \alpha \ \vee \ \gamma \leq \beta$. So, assume that $\gamma \neq \alpha$ and $\gamma \neq \beta$.

(a) If $\neg(\alpha \parallel \beta)$, then it holds that $\alpha \smile \beta = \alpha \ \vee \ \alpha \smile \beta = \beta$, and hence also that $\gamma < \alpha \ \vee \ \gamma < \beta$, and certainly $\gamma \leq \alpha \ \vee \ \gamma \leq \beta$.

(b) If $\alpha \parallel \beta$, then our initial assumption implies that $\gamma \leq \alpha \frown \beta \vee \alpha \smile \beta \leq \gamma$. Since $\gamma < \alpha \smile \beta$ this leads to $\gamma \leq \alpha \frown \beta$, and thus $\gamma \leq \alpha \ \wedge \ \gamma \leq \beta$, and certainly $\gamma \leq \alpha \ \vee \ \gamma \leq \beta$. □

Theorem 4. *Consider a lattice $(L, \leq)$, then the following equivalence holds:*

$$\big(\forall(\alpha,\beta,\gamma) \in L^3\big)\big(\alpha \frown \beta < \gamma \Rightarrow (\alpha \leq \gamma \ \vee \ \beta \leq \gamma)\big)$$
$$\Updownarrow$$
$$\big(\forall(\alpha,\beta) \in L^2\big)\big(\alpha \parallel \beta \Rightarrow \big(\forall \gamma \in L \setminus \{\alpha,\beta\}\big)(\gamma \leq \alpha \frown \beta \ \vee \ \alpha \smile \beta \leq \gamma)\big).$$

P r o o f . The proof is similar (dual) to that of Theorem 3. □

Corollary 5. *Consider a lattice $(L, \leq)$, then the following equivalence holds:*

$$\big(\forall(\alpha,\beta,\gamma) \in L^3\big)\big(\alpha \frown \beta < \gamma \Rightarrow (\alpha \leq \gamma \ \vee \ \beta \leq \gamma)\big)$$
$$\Updownarrow$$
$$\big(\forall(\alpha,\beta,\gamma) \in L^3\big)\big(\gamma < \alpha \smile \beta \Rightarrow (\gamma \leq \alpha \ \vee \ \gamma \leq \beta)\big).$$

P r o o f . The proof follows immediately from Theorems 3 and 4. □

Theorems 3 and 4 allow us to conclude that the weakened properties concerning the meet and join operation are equivalent. The additional equivalence with the property

$$\big(\forall(\alpha,\beta) \in L^2\big)\big(\alpha \parallel \beta \Rightarrow \big(\forall \gamma \in L \setminus \{\alpha,\beta\}\big)(\gamma \leq \alpha \frown \beta \ \vee \ \alpha \smile \beta \leq \gamma)\big)$$

gives us a complete insight in the lattices in which these weakened properties hold. For finite lattices, this insight will lead to the kite-tail theorem in the next section.

As for the characterizing properties of chains, the weakened properties concerning the meet and join operation can be equivalently expressed for arbitrary infima and suprema. This is shown in the following propositions.

Proposition 5. *Consider a lattice $(L, \leq)$, then the following equivalence holds:*

$$\big(\forall(\alpha,\beta,\gamma) \in L^3\big)\big(\alpha \frown \beta < \gamma \Rightarrow (\alpha \leq \gamma \ \vee \ \beta \leq \gamma)\big)$$
$$\Updownarrow$$
$$\big(\forall A \in \mathfrak{I}(L)\big)(\forall \alpha \in L)\big(\inf A < \alpha \Rightarrow (\exists \beta \in A)(\beta \leq \alpha)\big)\,.$$

P r o o f . Of course only the implication from top to bottom has to be shown. Consider $A \in \mathfrak{I}(L)$ and $\alpha \in L$ such that $\inf A < \alpha$. Proposition 4 implies that there exists at most one element that is incomparable with α. Hence, we consider the following two cases.

(i) *All elements of L are comparable with α.*
Suppose that $(\exists \beta \in A)(\beta \leq \alpha)$ does not hold, i.e., $(\forall \beta \in A)\big(\neg(\beta \leq \alpha)\big)$, and hence also
$$(\forall \beta \in A)(\alpha < \beta)\,.$$
This implies that $\alpha \leq \inf A$, which yields a contradiction.

(ii) *There exists exactly one element δ that is incomparable with α: $\alpha \parallel \delta$.*
Suppose again that $(\exists \beta \in A)(\beta \leq \alpha)$ does not hold, i.e., $(\forall \beta \in A)(\alpha < \beta \ \vee \ \alpha \parallel \beta)$, or put in another way
$$(\forall \beta \in A)(\alpha < \beta \ \vee \ \beta = \delta)\,.$$
This can be rewritten as
$$(\forall \beta \in A)\big((\alpha < \beta \ \wedge \ \beta \neq \delta) \ \vee \ \beta = \delta\big)\,.$$
If $\alpha < \beta \wedge \beta \neq \delta$, then Theorem 4 implies, since $\alpha \parallel \delta$ and $\beta \in L \backslash \{\alpha, \delta\}$, that $\beta \leq \alpha \frown \delta \ \vee \ \alpha \smile \delta \leq \beta$. In case $\beta \leq \alpha \frown \delta$, then also $\beta \leq \alpha$, which yields a contradiction. This means that
$$(\forall \beta \in A)(\alpha \smile \delta \leq \beta \ \vee \ \beta = \delta)\,.$$
This implies that $\inf A \geq \delta$. The assumption $\inf A < \alpha$ however leads to $\delta < \alpha$, again a contradiction.

$\square$

Proposition 6. *Consider a lattice $(L, \leq)$, then the following equivalence holds:*

$$\big(\forall(\alpha,\beta,\gamma) \in L^3\big)\big(\gamma < \alpha \smile \beta \Rightarrow (\gamma \leq \alpha \ \vee \ \gamma \leq \beta)\big)$$
$$\Updownarrow$$
$$\big(\forall A \in \mathfrak{S}(L)\big)(\forall \alpha \in L)\big(\alpha < \sup A \Rightarrow (\exists \beta \in A)(\alpha \leq \beta)\big)\,.$$

P r o o f . The proof is similar (dual) to that of Proposition 5. $\square$

COROLLARY 6. *Consider a lattice $(L,\leq)$, then the following equivalence holds:*

$$\big(\forall A\in\mathfrak{I}(L)\big)(\forall\alpha\in L)\big(\inf A<\alpha\Rightarrow(\exists\beta\in A)(\beta\leq\alpha)\big)$$
$$\Updownarrow$$
$$\big(\forall A\in\mathfrak{S}(L)\big)(\forall\alpha\in L)\big(\alpha<\sup A\Rightarrow(\exists\beta\in A)(\alpha\leq\beta)\big)\,.$$

Theorems 3 and 4, and Propositions 5 and 6 give us a complete insight in the lattices in which the weakened properties concerning the infimum and supremum hold. Let us consider a few examples.

EXAMPLE 1. In the chain $([0,1],\leq)$ the weakened properties concerning the infimum and supremum hold in an evident way. This is no longer the case for the product lattice $\big([0,1]^2,\leq\big)$. Indeed, $\big([0,1]^2,\leq\big)$ contains a lot of incomparable elements, as for instance (a,b) and (b,a), with $a<b$. If the weakened properties held, then Proposition 3 would imply that for instance (b,b) covers both (a,b) and (b,a). This contradicts the known fact that the covering relation of $\big([0,1]^2,\leq\big)$ is the empty relation.

The foregoing negative example can be generalized to the following proposition.

PROPOSITION 7. *Consider a lattice $(L,\leq)$ with $\#L\geq 3$, then in the product lattice $(L^2,\leq)$ the following property does not hold:*

$$\big(\forall(\lambda,\mu)\in(L^2)^2\big)\big(\lambda\parallel\mu\Rightarrow\big(\forall\nu\in L^2\setminus\{\lambda,\mu\}\big)(\nu\leq\lambda\frown\mu\ \vee\ \lambda\smile\mu\leq\nu)\big)\,.$$

Proof.

(i) If $(L,\leq)$ is a chain, then we choose $(\alpha,\beta,\gamma)\in L^3$ such that $\alpha<\beta<\gamma$. Let $\lambda=(\alpha,\gamma)$ and $\mu=(\gamma,\alpha)$, then it holds that $\lambda\parallel\mu$. Now put $\nu=(\beta,\beta)$, then it holds that
$$\lambda\frown\mu=(\alpha,\alpha)<\nu<\lambda\smile\mu=(\gamma,\gamma)\,.$$

(ii) If $(L,\leq)$ is not a chain, then we choose $(\alpha,\beta)\in L^2$ such that $\alpha\parallel\beta$. Let $\lambda=(\alpha,\beta)$ and $\mu=(\beta,\alpha)$, then it holds that $\lambda\parallel\mu$. Now put $\nu=(\alpha,\alpha\frown\beta)$, then it holds that
$$\lambda\frown\mu=(\alpha\frown\beta,\alpha\frown\beta)<\nu<\lambda\smile\mu=(\alpha\smile\beta,\alpha\smile\beta)\,.$$

□

EXAMPLE 2. In this example, we construct a non-finite complete lattice in which the weakened properties concerning the infimum and supremum hold. Consider the complete lattice $([-1,2]^2,\leq)$ and the following three subsets

$$L_1=\big\{(x,x)\colon x\in[-1,0]\big\}\,,$$
$$L_2=\big\{(0,1),(1,0)\big\}\,,$$
$$L_3=\big\{(x,x)\colon x\in[1,2]\big\}\,.$$

One easily verifies that $(L_1 \cup L_2 \cup L_3, \leq)$ is a complete lattice in which the desired properties hold. The set $L' = L_1 \cup L_2 \cup L_3$ is depicted in Figure 3.

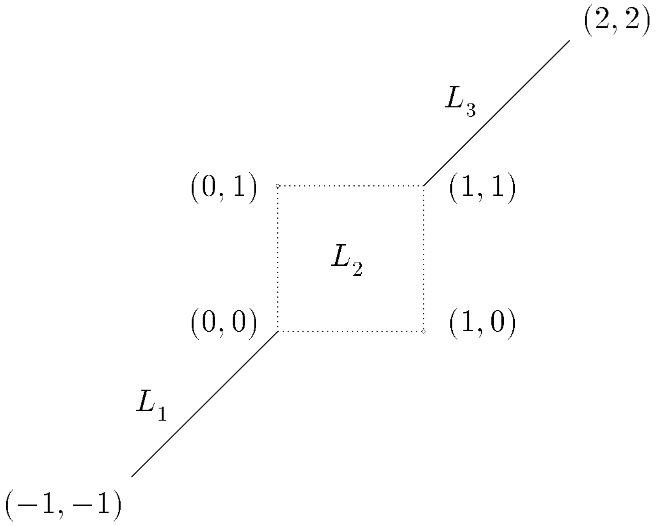

FIGURE 3. The set L' from Example 2.

4. Kite-tail lattices and their characterization

In this section, we recall the definition of a recently introduced type of finite lattices, called kite-tail lattices. Kite-tail lattices are finite vertical sums of the simple lattices $\mathbf{1}$ (2-element chain) and $\mathbf{M_2}$ (4-element lattice with 2 incomparable elements).

DEFINITION 4. (Linear sum) [2] Consider two disjoint ordered sets $(P, \leq)$ and $(Q, \leq')$. The *linear sum* $P \oplus Q$ of P and Q is defined as the set $P \cup Q$ equipped with the order relation $\preccurlyeq$, defined by

$$\alpha \preccurlyeq \beta \Leftrightarrow \big((\alpha,\beta) \in P^2 \ \wedge\ \alpha \leq \beta\big) \ \vee\ \big((\alpha,\beta) \in Q^2 \ \wedge\ \alpha \leq' \beta\big) \ \vee\ (\alpha,\beta) \in P \times Q\,.$$

For finite ordered sets $(P, \leq)$ and $(Q, \leq')$, the Hasse diagram for $P \oplus Q$ can be obtained by placing the Hasse diagram for P directly below the Hasse diagram for Q and then adding a line segment from each maximal element of P to each minimal element of Q.

DEFINITION 5. (Vertical sum) [2] Consider an upper bounded ordered set $(P, \leq)$ with the greatest element 1 and a lower bounded ordered set $(Q, \leq')$ with the smallest element 0'. The *vertical sum* $P \mathbin{\overline{\oplus}} Q$ of P and Q is defined as the linear sum $P \oplus Q$ in which the elements 1 and 0' have been identified.

Definition 6. (Kite-tail lattices) [4] A finite lattice $(L, \leq)$ is a *kite-tail lattice* if and only if it is isomorphic to a vertical sum with as only terms the lattices $\mathbf{1}$ and $\mathbf{M_2}$.

Example 3. In Figure 4, we have drawn the Hasse diagrams for the following kite-tail lattices

$$\mathbf{1} \mathbin{\overline{\oplus}} \mathbf{1} \mathbin{\overline{\oplus}} \mathbf{1} \mathbin{\overline{\oplus}} \mathbf{1}\,,$$
$$\mathbf{1} \mathbin{\overline{\oplus}} \mathbf{1} \mathbin{\overline{\oplus}} \mathbf{M_2} \mathbin{\overline{\oplus}} \mathbf{1} \mathbin{\overline{\oplus}} \mathbf{M_2} \mathbin{\overline{\oplus}} \mathbf{M_2}\,,$$
$$\mathbf{M_2} \mathbin{\overline{\oplus}} \mathbf{M_2} \mathbin{\overline{\oplus}} \mathbf{M_2} \mathbin{\overline{\oplus}} \mathbf{M_2} \mathbin{\overline{\oplus}} \mathbf{M_2}\,.$$

These examples explain the origin of the somewhat naive name 'kite-tail lattice'.

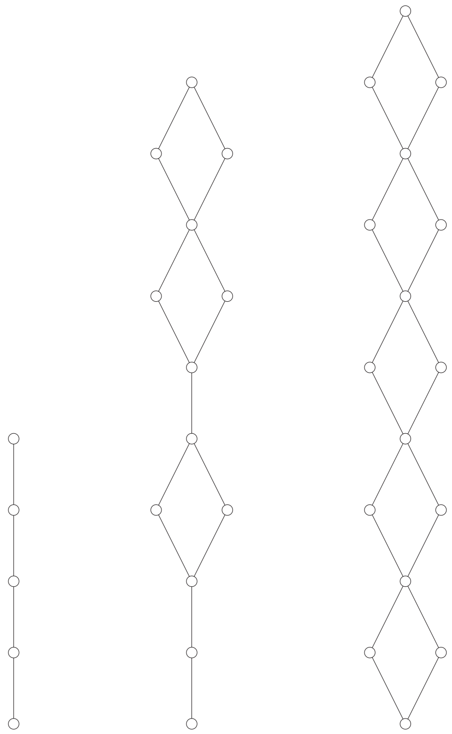

Figure 4

For finite lattices, the insight gained by Theorems 3 and 4 can be translated into the following theorem.

Theorem 5. (Kite-tail theorem [4]) *Consider a finite lattice $(L, \leq)$, then the following equivalence holds:*

$$(L, \leq) \text{ is a kite-tail lattice}$$
$$\Updownarrow$$
$$\big(\forall(\alpha,\beta) \in L^2\big)\big(\alpha \parallel \beta \Rightarrow \big(\forall\gamma \in L \setminus \{\alpha,\beta\}\big)(\gamma \leq \alpha \frown \beta \ \vee\ \alpha \smile \beta \leq \gamma)\big)\,.$$

P r o o f . The proof follows immediately from the definition of a kite-tail lattice. □

In Section 3, we have shown that the property in Theorem 5 is equivalent with the weakened properties concerning the infimum and supremum. When dealing with finite lattices, these results allow us to talk about the characterizing property of kite-tail lattices, or at least to talk about the equivalent characterizing properties of kite-tail lattices.

Similarly as in Theorem 5, we can conclude that the non-finite lattices in which the characterizing property of kite-tail lattices holds, can be seen as chains, possibly interrupted by an arbitrary number of insertions of the finite lattice $\mathbf{M_2}$.

5. Dense chains and their characterization

In this section, the notations $\downarrow \alpha$ and $\uparrow \alpha$ stand for the following sets $\downarrow \alpha = \{\beta\colon \beta \in L \ \wedge \ \beta \le \alpha\}$ and $\uparrow \alpha = \{\beta\colon \beta \in L \ \wedge \ \beta \ge \alpha\}$ [2]. Further, we define $\leftarrow, \alpha[\ = \downarrow \alpha \setminus \{\alpha\}$ and $]\alpha, \rightarrow\ = \uparrow \alpha \setminus \{\alpha\}$.

Sometimes the weakened properties concerning the infimum and supremum are required jointly with one of the following ones [3]:

$$\big(\forall A \in \mathfrak{I}(L)\big)(\forall \alpha \in L)(]\alpha, \rightarrow\ \subseteq A \Rightarrow \alpha \ge \inf A)\,,$$
$$\big(\forall A \in \mathfrak{S}(L)\big)(\forall \alpha \in L)(\leftarrow, \alpha[\ \subseteq A \Rightarrow \alpha \le \sup A)\,.$$

One easily verifies that these properties cannot hold in a finite lattice, excluding kite-tail lattices from our discussion. Indeed, consider a finite lattice $(L, \le)$, then the order relation is completely determined by the covering relation $\ll$. Suppose that $L \ne \{0_L\}$ and consider an element α of L that covers 0_L, i.e., $0_L \ll \alpha$, then $\leftarrow, \alpha[\ = \{0_L\}$. Let $A = \{0_L\}$ then $\sup A = 0_L$, and it holds that $\leftarrow, \alpha[\ \subseteq A$, but also $\neg(\alpha \le 0_L)$. The condition

$$\big(\forall A \in \mathfrak{S}(L)\big)(\forall \alpha \in L)(\leftarrow, \alpha[\ \subseteq A \Rightarrow \alpha \le \sup A)$$

is therefore not fulfilled.

Consequently, it only makes sense to investigate which non-finite lattices, already satisfying the characterizing property of kite-tail lattices, also satisfy the additional properties. As we will show further on, it will be required that the covering relation is the empty relation.

Definition 7. (Dense ordered sets) An ordered set $(P, \le)$ is *dense* if and only if its covering relation $\ll$ is the empty relation.

THEOREM 6. *Consider a lattice $(L,\leq)$, then the following equivalence holds:*

$$\left\{\begin{array}{l}\big(\forall A\in\mathfrak{S}(L)\big)(\forall\alpha\in L)\big(\alpha<\sup A\Rightarrow(\exists\beta\in A)(\alpha\leq\beta)\big)\,,\\ \quad\big(\forall A\in\mathfrak{S}(L)\big)(\forall\alpha\in L)(\leftarrow,\alpha[\;\subseteq A\Rightarrow\alpha\leq\sup A)\end{array}\right.$$
$$\Updownarrow$$
$$(L,\leq)\ \textit{is a dense chain.}$$

Proof.

(i) We will first show the implication from top to bottom, starting with the proof that $(L,\leq)$ is a chain. We therefore suppose that α and β are incomparable, and try to obtain a contradiction. With Propositions 3 and 6 and Theorem 3, it follows from the first assumption that $\alpha\frown\beta\lll\alpha$ and $\leftarrow,\alpha[\;=\,\downarrow(\alpha\frown\beta)$. Now let $A=\,\downarrow(\alpha\frown\beta)$, then $\sup A=\alpha\frown\beta$ and it holds that $\leftarrow,\alpha[\;\subseteq A$, but also $\neg(\alpha\leq\alpha\frown\beta)$. This violates the second assumption.

Next, we will show that the chain $(L,\leq)$ is dense. Consider $(\alpha,\beta)\in L^2$ such that $\alpha\lll\beta$. Since $(L,\leq)$ is a chain, it holds that $\leftarrow,\beta[\;=\,\downarrow\alpha$. Now let $A=\,\downarrow\alpha$, then $\sup A=\alpha$ and it holds that $\leftarrow,\beta[\;\subseteq A$, but also $\neg(\beta\leq\alpha)$. This again violates the second assumption.

(ii) The implication from bottom to top is also quite simple. The first property follows from Proposition 1. For the second property, consider $A\in\mathfrak{S}(L)$ and $\alpha\in L$ such that $\leftarrow,\alpha[\;\subseteq A$. We distinguish two cases.

(a) If $\leftarrow,\alpha[\;=A$, then we show that $\alpha=\sup A$. Let $\sup\leftarrow,\alpha[\;=\beta$, then of course $\beta\leq\alpha$. Now suppose that $\beta<\alpha$, then it follows that $\beta\lll\alpha$. Indeed, if there existed γ such that $\beta<\gamma<\alpha$, then it would follow that $\sup\leftarrow,\alpha[\;\geq\gamma>\beta$, clearly a contradiction. The covering $\beta\lll\alpha$, however, contradicts the assumption that $(L,\leq)$ is dense. Consequently, it holds that $\alpha=\beta$.

(b) If $\leftarrow,\alpha[\;\subset A$, then we show that $\alpha\leq\sup A$. Consider $\beta\in A\setminus\leftarrow,\alpha[$, then it follows that $\neg(\beta<\alpha)$. Since $(L,\leq)$ is a chain, the latter implies that $\alpha\leq\beta$. Obviously, it then holds that $\sup A\geq\beta\geq\alpha$. □

THEOREM 7. *Consider a lattice $(L,\leq)$, then the following equivalence holds:*

$$\left\{\begin{array}{l}\big(\forall A\in\mathfrak{I}(L)\big)(\forall\alpha\in L)\big(\inf A<\alpha\Rightarrow(\exists\beta\in A)(\beta\leq\alpha)\big)\,,\\ \quad\big(\forall A\in\mathfrak{I}(L)\big)(\forall\alpha\in L)(]\alpha,\rightarrow\;\subseteq A\Rightarrow\alpha\geq\inf A)\end{array}\right.$$
$$\Updownarrow$$
$$(L,\leq)\ \textit{is a dense chain.}$$

Proof. The proof is similar (dual) to that of Theorem 6. □

Corollary 7. *Consider a lattice $(L, \leq)$ that satisfies*

$$\big(\forall(\alpha,\beta) \in L^2\big)\big(\alpha \parallel \beta \Rightarrow \big(\forall\gamma \in L \setminus \{\alpha,\beta\}\big)(\gamma \leq \alpha \frown \beta \ \vee \ \alpha \smile \beta \leq \gamma)\big)\,,$$

then the following equivalence holds:

$$\big(\forall A \in \mathfrak{I}(L)\big)(\forall\alpha \in L)(]\alpha, \rightarrow \subseteq A \Rightarrow \alpha \geq \inf A)$$
$$\Updownarrow$$
$$\big(\forall A \in \mathfrak{S}(L)\big)(\forall\alpha \in L)(\leftarrow, \alpha[\ \subseteq A \Rightarrow \alpha \leq \sup A)\,.$$

P r o o f . The proof follows from Theorems 3, 4, 6 and 7. □

In the following example, we will show that the property

$$\big(\forall A \in \mathfrak{S}(L)\big)(\forall\alpha \in L)(\leftarrow, \alpha[\ \subseteq A \Rightarrow \alpha \leq \sup A)$$

does not allow to conclude that the lattice $(L, \leq)$ is dense, even not when it is complete.

Example 4. Consider the unit square $\big([0,1]^2, \leq\big)$ and the following two subsets

$$L_1 = \big[(0,0), (0,1)\big]\,,$$
$$L_2 = \big\{(x,x) \colon x \in [0,1]\big\}\,.$$

Now consider the set $L = L_1 \cup L_2$ depicted in Figure 5. We equip this set with the order relation $\preccurlyeq$ defined by

$$x \preccurlyeq y \iff x \leq y \ \wedge \ (x,y) \in \Big(L_1 \times \big(L_1 \cup \big\{(1,1)\big\}\big)\Big) \cup L_2^2\,.$$

One easily verifies that $(L, \preccurlyeq)$ is a complete lattice that satisfies the property

$$\big(\forall A \in \mathcal{P}(L)\big)(\forall\alpha \in L)(\leftarrow, \alpha[\ \subseteq A \Rightarrow \alpha \leq \sup A)\,.$$

Let for instance $\alpha = 1_L = (1,1)$ then $\leftarrow, \alpha[\ = L \setminus \big\{(1,1)\big\}$ and it holds that $\sup \leftarrow, \alpha[\ = (1,1)$. The complete lattice $(L, \preccurlyeq)$, however, is not dense since $(0,1) \ll (1,1)$.

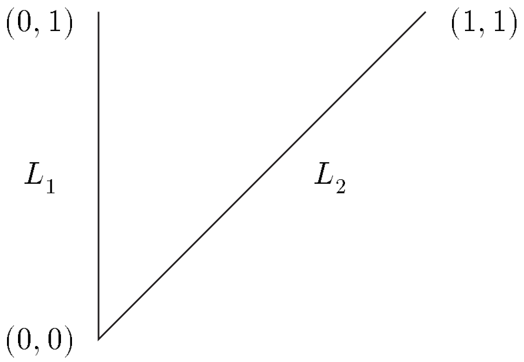

Figure 5. The set L from Example 4

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