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NEARNESS-BASED TOPOLOGY

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ABSTRACT. The aim of this paper is to discuss a kind of topology and relevant notions, based on the concept of a nearness N on a universe. In the first part we pay our attention to the connection between N-topology, N-convergence and N-continuity, as a possible fuzzification of their classical analogies. In second part we present a type of fixed point theorem on an N-topological space. The last part is devoted to an N-partition and corresponding N-factorization of a universe.

1. Introduction

The purpose of this paper is to continue the study of a fuzzification of metric properties based on the concept of a nearness, investigated, for example, already in papers [1], [2] [3], [4] and [5].

In some concrete cases a situation can arise that we are not able to determine the exact distance of objects we are treating, or sometimes it needn't be necessary, even neither purposefully. It can be sufficient to determine which couples of elements are "near" one another, or more precisely, a degree of their "nearness".

In [1] the nearness is introduced as a binary fuzzy relation N on $\mathbb{R}$ with some natural properties, corresponding to the algebraic, topological and lattice structures of real axis.

Any "reasonable" nearness on the set of all real numbers should be evidently compatible with the structure of the real axis. But if we consider nearness on an arbitrary universe X, without any structure, we are no more restricted by any additional requirements and conditions, and a nearness on X should comply only with our intuitive idea of a nearness.

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2. Nearness and Topology

In [3] the nearness N on a universe X was defined as follows.

Definition 1. Let X be a set. A binary fuzzy relation N on X is called a *nearness* on X, if:

(N1) $N(x,x) = 1$, for each $x \in X$,

(N2) $N(x,y) = N(y,x)$, for each $x, y \in X$,

(N3) For each $\varepsilon > 0$ there exists $\delta < 1$ such that

$$N(x,y) > \delta \Longrightarrow \bigl|N(x,z) - N(y,z)\bigr| < \varepsilon\,, \qquad \text{for each} \quad x,y,z \in X\,.$$

The properties (N1) and (N2), it means the reflexivity and the symmetricity of N, are immediate. The property (N3) substitutes, in a sense, the triangular inequality and it has the following meaning: If two points x and y are sufficiently near one another, then the difference of their nearnesses to any other point z is arbitrarily small.

A trivial verification shows that the property (N3) implies the property

(N3') For each $x, y, z \in X$:

$$N(x,y) = 1 \implies N(x,z) = N(y,z)\,.$$

Hence, if N does not distinguish two elements from a universe, if they have the maximal possible degree of nearness, then their nearness to any other element of the universe is of the same degree.

Let us note that there is an interesting connection between the nearnesses and another type of binary fuzzy relations, t-equivalences (see, e.g., [7]). This connection is studied in [3].

Because a nearness is understood as a concept which is a fuzzification of crisp notions like distance, metric, or, in our formulation, rather pseudometric, the following question is natural: What is a connection between pseudometric spaces and spaces endowed with a nearness?

It is a well known fact (see, e.g., [6]), that any pseudometric space (X,ρ) is homeomorphic to a pseudometric space (X,d) with the pseudometric $d(x,y) = \min\bigl(\rho(x,y),1\bigr)$, what means that d is a binary fuzzy relation on X.

If (X,d) is such a pseudometric space, it can be easily proved that the binary fuzzy relation N on X, defined by: $N(x,y) = 1 - d(x,y)$, satisfies properties (N1), (N2) and (N3), i.e., N is a nearness on X. In this manner, we can construct the nearness N on each pseudometric space. As the following example shows, the opposite is not true.

EXAMPLE 1. Let $X = \{a, b, c\}$ be an arbitrary three-point set. Let N be such a reflexive and symmetrical fuzzy relation on X that $N(a,c) = \frac{1}{3}$, $N(a,b) = N(b,c) = \frac{3}{4}$. Then N is evidently a nearness, properties (N1) and (N2) are directly assumed, and (N3) is fulfilled, too.

But despite of that, the binary fuzzy relation $d(x,y) = 1 - N(x,y)$ is not pseudometric, as the triangular inequality fails:

$$d(a,b) + d(b,c) = \frac{1}{2} < \frac{2}{3} = d(a,c)\,.$$

Next example shows one possible construction of nearnesses in the set of all real numbers.

EXAMPLE 2. In the real case, if $X = \mathbb{R}$, it can be shown (see [2]) that if $b\colon [0,\infty] \to [0,1]$ is such a continuous and decreasing function that $b(0) = 1$, $\lim\limits_{x\to\infty} b(x) = 0$, then the binary fuzzy relation on $\mathbb{R}$:

$$N(x,y) = b\big(|x-y|\big) \qquad \text{for each} \quad x, y \in \mathbb{R}\,.$$

possesses properties (N1), (N2) and (N3), it means, it is a nearness in the sense of the foregoing definition.

The concept of a nearness N in a universe X enables to introduce in a natural way some metric and topological properties.

DEFINITION 2. Let X be a universe and N be a nearness on X. A sequence $\{x_n\}$ of elements of X is called *N-fundamental*, if

$$\lim_{n\to\infty} N(x_n, x_{n+1}) = 1\,.$$

The sequence $\{x_n\}$ is called *N-convergent* and an element $x_0 \in X$ is called its *N-limit* ($x_n \underset{N}{\to} x_0$) if

$$\lim_{n\to\infty} N(x_n, x_0) = 1\,.$$

DEFINITION 3. Let X_1 and X_2 be two universes and N_1 and N_2 be nearnesses on X_1 and X_2, respectively. Let $f\colon X_1 \to X_2$ be a mapping.

The mapping f is said to be *(N_1, N_2)-continuous at a point* $x_0 \in X_1$ if for any sequence $\{x_n\}$ in X_1:

$$\lim_{n\to\infty} N_1(x_n, x_0) = 1 \implies \lim_{n\to\infty} N_2\big(f(x_n), f(x_0)\big) = 1\,.$$

The mapping f is said to be *(N_1, N_2)-continuous* on X_1 if it is (N_1, N_2)-continuous at each $x_0 \in X_1$.

If X is a nonempty set, N a nearness on X, $x_0 \in X$ and $\varepsilon < 1$, let us denote:

$$O_\varepsilon(x_0) = \big\{x \in X\colon N(x, x_0) > \varepsilon\big\}\,.$$

Now denote by $\mathcal{B}$ the system of all $O_\varepsilon(x)$, for all $x \in X$ and $\varepsilon < 1$.

The following theorem, proved in [2], enables to introduce a topology on X:

Theorem 1. *The system $\mathcal{B}$ creates a basis of a topology on X.*

From now on let $\mathcal{T}_N$ denote the nearness-based topology, created on the set X by the basis $\mathcal{B}$ and let $(X, \mathcal{T}_N)$ denote the corresponding topological space.

It easily follows that the space $(X, \mathcal{T}_N)$ always satisfies the first axiom of countability, therefore the topology $\mathcal{T}_N$ can be described in terms of sequences.

Example 3. Let $X = \mathbb{R}$ and let N be a trivial nearness on X:

$$N(x,y) = \begin{cases} 1 & \text{for } x = y\,, \\ 0 & \text{for } x \neq y\,. \end{cases}$$

As it is easy to check, N satisfies the properties (N1), (N2) and (N3). The basis $\mathcal{B}$, derived from the nearness N, consists of all one-point sets, therefore the created topology $\mathcal{T}_N$ is discrete.

Theorem 2. *Let $\{x_n\}$ be a sequence in $(X, \mathcal{T}_N)$. Then*

(1) *$\{x_n\}$ is a fundamental sequence if and only if it is N-fundamental.*

(2) *$\lim\limits_{n\to\infty} x_n = x_0$ if and only if $x_n \underset{N}{\to} x_0$.*

Proof. The standard proof is left to the reader. □

Theorem 3. *Let $(X_1, \mathcal{T}_{N_1})$ and $(X_2, \mathcal{T}_{N_2})$ be two topological spaces. Let $f\colon X_1 \to X_2$ be a mapping. Then*

(1) *f is continuous at $x_0 \in X_1$ if and only if f is (N_1, N_2)-continuous at x_0.*

(2) *f is continuous on X_1 if and only if f is (N_1, N_2)-continuous on X_1.*

Proof.

(1) Let f be continuous at $x_0 \in X_1$ and let $\lim\limits_{n\to\infty} N_1(x_n, x_0) = 1$. We want to show that then also $\lim\limits_{n\to\infty} N_2\big(f(x_n), f(x_0)\big) = 1$, in other words, that for each $\varepsilon < 1$, there exists such a number n_0 that for all natural numbers $n > n_0$ it is $N_2\big(f(x_n), f(x_0)\big) > \varepsilon$.

Let $\varepsilon < 1$. It follows from the assumptions that there exists such a $\delta < 1$ that $f\big(O_\delta(x_0)\big) \subset O_\varepsilon\big(f(x_0)\big)$ and simultaneously there exists such a number n_0 that for all natural numbers $n > n_0$ it is $N_1(x_n, x_0) > \delta$. It means that $x_n \in O_\delta(x_0)$ and therefore $f(x_n) \in O_\varepsilon\big(f(x_0)\big)$.

Now suppose that f is not continuous at $x_0 \in X_1$. Put $\delta_n = 1 - \frac{1}{n}$ for all natural n. There exists such an $\varepsilon < 1$ and a sequence $\{x_n\}$ in X_1, that for each n it is $x_n \in O_{\delta_n}(x_0)$, but $f(x_n) \notin O_\varepsilon\big(f(x_0)\big)$. It means that $\lim\limits_{n\to\infty} N_1(x_n, x_0) = 1$, but $\lim\limits_{n\to\infty} N_2\big(f(x_n), f(x_0)\big) \neq 1$. Thus f is not (N_1, N_2)-continuous at x_0.

(2) Assertion (2) follows from the fact that f is (N_1, N_2)-continuous on X_1 iff for each $x_0 \in X_1$ and for each $\varepsilon < 1$ there exists $\delta < 1$ such that for each $y \in X_1$:

$$N_1(x_0, y) > \delta \Longrightarrow N_2\big(f(x_0), f(y)\big) > \varepsilon .$$

It means $O_\delta(x_0) \subset f^{-1} O_\varepsilon\big(f(x_0)\big)$, thus $f^{-1} O_\varepsilon\big(f(x_0)\big) \in \mathcal{T}_{N_1}$. □

3. N-contractive mappings

There are many different approaches to the fuzzification and generalization of contractive mappings and fixed point theorems. In this section we want to present one of them, based on the concept of a nearness.

Definition 4. Let $(X, \mathcal{T}_N)$ be a topological space. A mapping $f\colon X \to X$ is said to be *N-contractive* if there exists such a constant $\alpha > 1$ that for each $x, y \in X$, $N\big(f(x), f(y)\big) \geq \min\big(1,\ \alpha \cdot N(x, y)\big)$.

A point $x_0 \in X$ is said to be *N-fixed point* of the mapping f, if $N\big(x_0, f(x_0)\big) = 1$.

Theorem 4. *Suppose that $(X, \mathcal{T}_N)$ is a complete topological space and $f\colon X \to X$ is an N-contractive mapping. If there exists a point $x \in X$ such that $N\big(x, f(x)\big) > 0$, then there exists at least one fixed point of f.*

P r o o f. Let $x_1 \in X$ be such a point that $N\big(x_1, f(x_1)\big) > 0$. Put successively $f(x_1) = x_2$, $f(x_2) = x_3$ and so on, for each natural n put $f(x_n) = x_{n+1}$.

Then for each n:

$$\begin{aligned} N(x_n, x_{n+1}) &= N\big(f(x_{n-1})\, f(x_n)\big) \geq \min\big(1,\ \alpha \cdot N(x_{n-1}, x_n)\big) \\ &= \min\Big(1,\ \alpha \cdot N\big(f(x_{n-2}), f(x_{n-1})\big)\Big) \geq \min\big(1,\ \alpha^2 \cdot N(x_{n-2}, x_{n-1})\big) \\ &= \cdots \geq \min\big(1,\ \alpha^{n-1} \cdot N(x_1, x_2)\big) . \end{aligned}$$

Hence $\lim\limits_{n\to\infty} N(x_n, x_{n+1}) = 1$, it means the sequence $\{x_n\}$ is N-fundamental. From the completeness of X and Theorem 1 it follows that there exists such a point $x_0 \in X$ that $\lim\limits_{n\to\infty} N(x_n, x_0) = 1$. Because for each n, $x_n = f(x_{n-1})$, it follows that also $\lim\limits_{n\to\infty} N\big(f(x_n), x_0\big) = 1$. But as can be easily seen, each N-contractive mapping is N-continuous, therefore simultaneously $\lim\limits_{n\to\infty} N\big(f(x_n), f(x_0)\big) = 1$. Applying now (N3) we can conclude that $N\big(x_0, f(x_0)\big) = 1$. □

Theorem 5. *Let $(X, \mathcal{T}_N)$ be a topological space. If an N-contractive mapping $f\colon X \to X$ has two N-fixed points x_0 and y_0, then $N(x_0, y_0) = 1$.*

P r o o f. Let $N\big(x_0, f(x_0)\big) = 1$ and $N\big(y_0, f(y_0)\big) = 1$. Then it follows from (N3') that

$$N(x_0, y_0) = N\big(y_0, f(x_0)\big) = N\big(f(x_0), f(y_0)\big) .$$

By the contractivity of f, there exists a constant $\alpha > 1$ such that

$$N(x_0, y_0) = N\big(f(x_0), f(y_0)\big) \geq \min\big(1,\ \alpha \cdot N(x_0, y_0)\big)\,.$$

But it evidently implies $N(x_0, y_0) = 1$. □

4. N-partition and N-factor space

Let N be a nearness on a universe X with properties (N1), (N2) and (N3). As the following example shows, if N doesn't distinguish some points, corresponding topological space $(X, \mathcal{T}_N)$ need not be T_0-space.

EXAMPLE 4. Let $X = \{a, b, c\}$ be arbitrary three-point set and let N be a binary fuzzy relation on X which is reflexive, symmetrical and $N(a,b) = 1$, $N(a,c) = N(b,c) = \frac{1}{2}$. It is easy to check that N is a nearness and N-based topology $\mathcal{T} = \big\{\emptyset, X, \{a,b\}, \{c\}\big\}$.

Suppose now that X is any set endowed with a nearness N.

Let us denote for each $x \in X$ by $[x]$ the set of all points in X which are indistinguishable with x:

$$[x] \underset{\text{def}}{=} \big\{y \in X\colon N(x,y) = 1\big\}, \qquad \text{for each} \quad x \in X\,.$$

THEOREM 6. *Let X be a nonempty set and N be a nearness on X. The family $\mathcal{X} = \big\{[x], x \in X\big\}$ is a partition of X.*

Proof. It is trivial that $X = \bigcup\limits_{x \in X} [x]$.

Let $x, y, z \in X$ and let $z \in [x] \cap [y]$. It means that $N(x,z) = N(y,z) = 1$ and applying (N3') we get $N(x,y) = 1$, thus $[x]$ and $[y]$ coincide. □

Define now a binary fuzzy relation $\mathcal{N}$ on $\mathcal{X}$ as follows:

$$\mathcal{N}\big([x],[y]\big) \underset{\text{def}}{=} N(x,y), \qquad \text{for each} \quad [x],[y] \in \mathcal{X}\,.$$

This definition is correct, what is a straightforward consequence of (N3').

THEOREM 7. *Let $(X, \mathcal{T}_N)$ be a topological space and let $\mathcal{X}$ and $\mathcal{N}$ be defined as above. Then $\mathcal{N}$ is a nearness on $\mathcal{X}$, the topological space $(\mathcal{X}, \mathcal{T}_{\mathcal{N}})$ is a Hausdorff space, and the projection $p\colon\ X \to \mathcal{X}$, defined by $p(x) = [x]$ for each $x \in X$, is nearness-preserving.*

Proof. Properties (N1), (N2), (N3) for $\mathcal{N}$ follow directly from the definition.

From definitions it also simply follows that if $[x],[y] \in \mathcal{X}, [x] \neq [y]$, then $\mathcal{N}([x],[y]) < 1$, therefore $[x]$ and $[y]$ have disjoint neighbourhoods. The details are left to the reader. □

The correspondence between topological spaces $(X, \mathcal{T}_N)$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{N}})$, established by the previous theorem, can be utilized as follows:

By any N-contractive mapping f on a space $(X, \mathcal{T}_N)$, there is induced an $\mathcal{N}$-contractive mapping $\mathcal{F}$ on $(\mathcal{X}, \mathcal{T}_{\mathcal{N}})$ by the formula:

$$\mathcal{F}([x]) = \{y \in X : N(f(x), y) = 1\}, \qquad \text{for each} \quad [x] \in \mathcal{X}.$$

Summarizing now Theorems 4, 5 and 7, we can state the following theorem, which is a simple consequence of them.

THEOREM 8. *Let $(X, \mathcal{T}_N)$ be a complete topological space, f be an N-contractive mapping on X, and let a mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ be defined as above. Then $\mathcal{F}$ is an $\mathcal{N}$-contractive mapping (with the same constant α) and there exists just one $[x_0] \in \mathcal{X}$ such that $\mathcal{F}([x_0]) = [x_0]$.*

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