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NEARNESS DIFFERENTIABLE FUNCTIONS

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ABSTRACT. Nearness differentiability of crisp functions, based on the notion of fuzzy smooth functions, is introduced in this paper. The main result concerns the corresponding calculus, including chain rule, which is developed.

This paper is a continuation of foregoing papers [K1 – K3], where fuzzy derivatives and fuzzy smooth functions based on the notion of fuzzy nearness relation were studied. The disadvantage of the already introduced fuzzy smooth functions is that the family of these functions is not closed under addition and multiplication. In this paper we have succeeded in getting rid of this disadvantage and we have derived some rules for the differentiation of the multiplication and addition of "nearness differentiable functions", as well as the chain rule. Last two theorems concern the fuzzy derivatives of functions given parametrically and implicitly.

The notion of fuzzy nearness relation has been introduced by the author in [K1] and further developed by V. Janiš in [J2] and by J. Dobrakovová in [D1, D2]. The notion of nearness differentiability, introduced here, is different from fuzzy differentiability, introduced by V. Janiš in [J3].

A slighty different way follows the concept of "fuzzy calculus", given by M. Burgin and A. Šostak in [B-Š] and by M. Burgin in [B] and investigated also by V. Janiš in [J1, J2, J4], where some continuity measures are studied.

Preliminarily, let us repeat some notions from [K2]. Let $\mathcal{N}\colon \mathbb{R}\times\mathbb{R}\to[0,1]$. We say that $\mathcal{N}$ is a **relation of fuzzy nearness** iff the following holds:

(1) $x\mathcal{N}z = 1 \iff qz = x$,

(2) $x\mathcal{N}z = z\mathcal{N}x$,

(3) for each x, z, s, t if $x \le s \le t \le z$ then $s\mathcal{N}t \ge x\mathcal{N}z$,

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(4) for each $x \in \mathbb{R}$, $\lim\limits_{t\to\pm\infty} x\mathcal{N}t = 0$,

(5) for all $\alpha \in]0,1]$ and all $x \in \mathbb{R}$ there exist unique $x_\alpha > x$ and $x_{-\alpha} < x$ such that $x\mathcal{N}x_\alpha = x\mathcal{N}x_{-\alpha} = \alpha$.

EXAMPLE 1. Let us give some examples of fuzzy nearness relations.

(**a**) $x\mathcal{N}y = \max\{0, 1-|x-y|\}$. Hence the fuzzy nearness to x is in this case a triangular fuzzy number.

(**b**) $x\mathcal{N}y = \exp(-|x-y|)$. The fuzzy nearness to x is again an LR-fuzzy number (see, e.g., [M]), but in this case it has an infinite spread.

(**c**) We can have also non-homogeneous fuzzy nearness given, e.g., by

$$x\mathcal{N}y = \begin{cases} \max\{0, 1-|x-y|\} & \text{for } x, y \geq 0, \\ \max\{0, 1-2|x-y|\} & \text{for } x, y \leq 0, \\ \max\{0, 1+2x-y\} & \text{for } x < 0,\ y > 0. \end{cases}$$

In the following $\mathcal{X}$ and $\mathcal{Y}$ will always denote fuzzy nearness relations connected with the x- and y-axes, respectively.

Let $x \in \mathbb{R}$ be any point and $\mathcal{X}$ be a fuzzy nearness relation. Denote by x_α and $x_{-\alpha}$ the points yielding $x_{-\alpha} < x < x_\alpha$ and $x_{-\alpha}\mathcal{X}x = x_\alpha\mathcal{X}x = \alpha$. Further denote

$$f'_\alpha(x) = \frac{f(x_\alpha) - f(x)}{x_\alpha - x}, \qquad f'_{-\alpha}(x) = \frac{f(x_{-\alpha}) - f(x)}{x_{-\alpha} - x},$$

$$f_{M,\alpha}{}'(x) = \max\bigl(f_\alpha{}'(x), f_{-\alpha}{}'(x)\bigr), \qquad f_{m,\alpha}{}'(x) = \min\bigl(f_\alpha{}'(x), f_{-\alpha}{}'(x)\bigr). \quad (1)$$

The *fuzzy α-derivative* of the function f at a point x with respect to $\mathcal{X}$ is defined by $\bigl[f_{m,\alpha}{}'(x),\ f_{M,\alpha}{}'(x)\bigr]$.

The *fuzzy derivative* of the function f at x with respect to $\mathcal{X}$ is denoted by $\frac{df}{d\mathcal{X}}(x)$ and defined by

$$\frac{df}{d\mathcal{X}}(x) = \Bigl[\liminf_{\alpha\to 1-} f_{m,\alpha}{}'(x),\ \limsup_{\alpha\to 1-} f_{M,\alpha}{}'(x)\Bigr]. \qquad (2)$$

I.e., the fuzzy derivative is a (possibly one-point) interval.

Obviously for any fuzzy nearness relations $\mathcal{X}_1$ and $\mathcal{X}_2$ and any function f the following holds

$$\frac{df}{d\mathcal{X}_1}(x) = \frac{df}{d\mathcal{X}_2}(x).$$

We have just defined the notion of fuzzy derivative. This is defined for any real valued function. However, if we want to achieve “reasonable results” on fuzzy derivatives, we have to put some assumptions to the functions in question. That is why we need the following definition of a fuzzy smooth function.

Let $\alpha \in \,]0,1[$. A continuous function f is said to be *α-smooth at a point x* with respect to a given couple $(\mathcal{X};\mathcal{Y})$ iff the following hold

$$\begin{gathered}
\liminf_{z\to x} f'_{-\alpha}(z)\,\mathcal{Y}\,\limsup_{z\to x} {f_\alpha}'(z) \geq \alpha\,,\\
f'_{-\alpha}(x)\,\mathcal{Y}\,\limsup_{z\to x} {f_\alpha}'(z) \geq \alpha\,,\\
f'_{\alpha}(x)\,\mathcal{Y}\,\liminf_{z\to x} {f_{-\alpha}}'(z) \geq \alpha\,,\\
f'_{-\alpha}(x)\,\mathcal{Y}\,{f_\alpha}'(x) \geq \alpha\,.
\end{gathered}$$

A function f is said to be *fuzzy smooth at a point x* with respect to a couple $(\mathcal{X};\mathcal{Y})$ iff there exists an $\alpha_0 \in \,]0,1[$ such that for each $1 > \alpha > \alpha_0$ f is α-smooth at x.

For properties of fuzzy smooth functions see [K2, K3].

Example 2. Let us consider some functions and their possible fuzzy smoothness.

(**a**) Let $f(x) = |x|$. Then for any fuzzy nearness relation $\mathcal{X}$ and any $\alpha \in \,]0,1[$ there holds

$$f'_{-\alpha}(0) = -1 \qquad \text{and} \qquad f'_{\alpha}(0) = 1\,,$$

hence f is not fuzzy smooth at 0 with respect to any couple $(\mathcal{X};\mathcal{Y})$.

(**b**) Let the function f be defined by the following

$$f(x) = \begin{cases} x & \text{for } x \text{ being rationals}\,,\\ -x & \text{for } x \text{ being irrationals}\,. \end{cases}$$

Then for any couple $(\mathcal{X};\mathcal{Y})$ such that

$$x_1 \mathcal{X} x_2 = \max\{0, 1 - |x_1 - x_2|\}\,.$$

f is fuzzy smooth at 0 and its fuzzy derivative is

$$\frac{df}{d\mathcal{X}}(0) = [-1,1]\,.$$

Nearness differentiability. Recall that the fuzzy derivative is independent of the fuzzy nearness relation $\mathcal{Y}$ (and even of $\mathcal{X}$). Hence, we can formulate the following definition.

A function f is *nearness differentiable at a point x* with respect to $\mathcal{X}$ if there exists a fuzzy nearness relation $\mathcal{Y}$ such that f is fuzzy smooth at x with respect to $(\mathcal{X}, \mathcal{Y})$.

Lemma 1. *Let* $\frac{df}{d\mathcal{X}}(x) = \{a\}$. *Then* f *is nearness differentiable at* x *with respect to any fuzzy nearness relation* $\mathcal{X}$.

P r o o f. First, consider the case $\frac{df}{d\mathcal{X}}(x) = \{a\} \neq \pm\infty$. Then there holds

$$\limsup_{\alpha \to 1_-} f'_{M\alpha} = \liminf_{\alpha \to 1_-} f'_{m\alpha} = \alpha\,,$$

hence f is differentiable at x from the classical point of view and due to Lemma 5 in [K3] it is fuzzy smooth and hence also nearness differentiable.

Now, consider the case $\frac{df}{d\mathcal{X}}(x) = \{\infty\}$. Then for any $\mathcal{X}$ and for each y we can find

$$\alpha_y = \inf\Big\{\beta \in]0,1[:\ \ (\forall \alpha \in]\beta, 1[)\big(f'_{m,\alpha}(x) \geq y\big)\Big\}\,.$$

To get the nearness differentiability of f at x w.r.t. $\mathcal{X}$, it is enough to take $\mathcal{Y}$ such that

$$y\,\mathcal{Y}\,f'_{M\alpha_y}(x) \geq \alpha_y\,.$$

The case $\frac{df}{d\mathcal{X}}(x) = \{-\infty\}$ we would treat similarly. □

The following example shows that the property "being nearness differentiable" depends on the choice of the fuzzy nearness relation $\mathcal{X}$.

Example 3. Put $x_0 = 0$ and

$$f(x) = \begin{cases} x \cdot \sin\big(\frac{1}{x}\big) & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \\ -x \cdot \sin\big(\frac{1}{x}\big) & \text{if } x < 0\,. \end{cases}$$

Then any fuzzy nearness relation $\mathcal{X}$ yields $\frac{df_1}{d\mathcal{X}}(0) = [-1, 1]$. If we put

$$x_1 \mathcal{X}_1 x_2 = \begin{cases} 1 - |x_1 - x_2| & \text{if } |x_1 - x_2| < 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

then f_1 is fuzzy smooth at $x_0 = 0$ with respect to any couple $(\mathcal{X}_1; \mathcal{Y})$ (see Example 1 in [K2]) and hence it is nearness differentiable with respect to $\mathcal{X}_1$.

If we take any fuzzy nearness relation $\mathcal{X}_2$ fulfilling the following

$$x\mathcal{X}_2 0 = \begin{cases} 1 - 2x & \text{if } 0 < x < \frac{1}{2}\,, \\ 1 + x & \text{if } 0 > x > -1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

then for $\alpha_k = \big((2k+1)\pi\big)^{-1}$ we have $0_{\alpha_k} = 2\big((2k+1)\pi\big)^{-1}$, $0_{-\alpha_k} = -\big((2k+1)\pi\big)^{-1}$ and hence

$$\big|\, f_{1,\alpha_k}{}'(0) - f_{1,-\alpha_k}{}'(0) \big| = \left| \sin\Big(\frac{1}{2}(2k+1)\pi\Big) - \sin\big((2k+1)\pi\big) \right| = 1\,.$$

Hence f_1 is at $x_0 = 0$ not nearness differentiable with respect to $\mathcal{X}_2$.

Now, we turn our attention to the calculus of nearness differentiable functions. The following theorem states the calculus for functions with finite fuzzy derivatives.

THEOREM 1. *Let $x_0 \in \mathbb{R}$, $\mathcal{X}$ be a fuzzy nearness relation, f and g be nearness differentiable functions at x_0 with respect to $\mathcal{X}$ and k be a real number. If*

$$\frac{df}{d\mathcal{X}}(x_0) = [a_1, b_1] \subset]-\infty, \infty[\qquad \text{and} \qquad \frac{dg}{d\mathcal{X}}(x_0) = [a_2, b_2] \subset]-\infty, \infty[\,,$$

then

(a) *kf is a nearness differentiable function at x_0 w.r.t. $\mathcal{X}$ and*

$$\frac{d(kf)}{d\mathcal{X}}(x_0) = k\frac{df}{d\mathcal{X}}(x_0)\,,$$

(b) *$f+g$ is a nearness differentiable function at x_0 w.r.t. $\mathcal{X}$ and*

$$\frac{d(f+g)}{d\mathcal{X}}(x_0) \subseteq \frac{df}{d\mathcal{X}}(x_0) + \frac{dg}{d\mathcal{X}}(x_0)\,, \tag{3}$$

(c) *$f \cdot g$ is a nearness differentiable function at x_0 w.r.t. $\mathcal{X}$ and*

$$\frac{d(f\cdot g)}{d\mathcal{X}}(x_0) \subseteq g(x_0)\frac{df}{d\mathcal{X}}(x_0) + f(x_0)\frac{dg}{d\mathcal{X}}(x_0)\,. \tag{4}$$

Proof.

(a) Since f is fuzzy differentiable, there exists a $\mathcal{Y}$ such that f is fuzzy smooth with respect to $(\mathcal{X};\mathcal{Y})$ at x_0. Then, if $k \neq 0$, obviously kf is fuzzy smooth with respect to $(\mathcal{X};\mathcal{Y}_k)$ at x_0, where

$$(k\cdot y_1 \mathcal{Y}_k\ k\cdot y_2 = \alpha) \iff (y_1 \mathcal{Y} y_2 = \alpha)\,.$$

This proves the statement "(a)" for $k \neq 0$. If $k = 0$, the statement is trivial.

(b) We have to find some fuzzy nearness relation $\mathcal{Y}$ such that $f+g$ would be fuzzy smooth at x_0 w.r.t. $(\mathcal{X};\mathcal{Y})$. Denote

$$\begin{aligned} f_\alpha(x_0) &= \inf\Big\{\tau > 0:\ (\forall\beta \in [\alpha, 1[)\big(\tau > f_{M,\beta}{}'(x_0) - f_{m,\beta}{}'(x_0)\big)\Big\}\,,\\ g_\alpha(x_0) &= \inf\Big\{\tau > 0:\ (\forall\beta \in [\alpha, 1[)\big(\tau > g_{M,\beta}{}'(x_0) - g_{m,\beta}{}'(x_0)\big)\Big\}\,, \end{aligned} \tag{5}$$

where $f_{M,\beta}$, $f_{m,\beta}$ and $g_{M,\beta}$, $g_{m,\beta}$ are defined by Formula (1). Define $\mathcal{Y}$ by

$$u\,\mathcal{Y}\,v = \begin{cases} \alpha & \text{if } |u-v| = f_\alpha(x_0) + g_\alpha(x_0)\,,\\ 1 & \text{if } u = v\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

Then $f+g$ is fuzzy smooth at x_0 w.r.t. $(\mathcal{X};\,\mathcal{Y})$, hence it is nearness differentiable. Formula (3) follows from the definition of fuzzy derivative (Formula (2)).

(c) There holds

$$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\,.$$

This implies Formula (4). We have to find some fuzzy nearness relation $\mathcal{Y}$ such that $f\cdot g$ would be fuzzy smooth with respect to $(\mathcal{X};\mathcal{Y})$. For some fixed $\alpha_0<1$ denote

$$G=\sup\{\left|g(x)\right|:\ x\,\mathcal{X}\,x_0\ge\alpha_0\}\,.$$

Define $\mathcal{Y}$ by

$$u\,\mathcal{Y}\,v=\begin{cases}\alpha & \text{if } |u-v|=f_\alpha(x_0)\cdot G+\left|f(x_0)\right|\cdot g_\alpha(x_0)\,,\\ 1 & \text{if } u=v\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

where $f_\alpha(x_0)$ and $g_\alpha(x_0)$ are defined by Formula (5). Then $f\cdot g$ is α-smooth at x_0 w.r.t. $(\mathcal{X};\,\mathcal{Y})$ for all $\alpha\ge\alpha_0$, hence it is fuzzy smooth, end finally we get the nearness differentiability of the product. □

Before formulating Chain Rule we define the multiplication of intervals by

$$[a_1,\,b_1][a_2,\,b_2]=\big[\min\{a_1a_2,\,a_1b_2,\,b_1a_2,\,b_1b_2\};\ \max\{a_1a_2,\,a_1b_2,\,b_1a_2,\,b_1b_2\}\big]\,.$$

THEOREM 2. (Chain Rule for fuzzy derivatives.) *Let $\mathcal{X}$ be any fuzzy nearness relation, g be continuous at a point x and f be continuous at $g(x)$. Denote $g\circ f(x)=f\big(g(x)\big)$. Let $\frac{df}{d\mathcal{X}}\big(g(x)\big)=[a_1,\,b_1]$ and $\frac{dg}{d\mathcal{X}}(x)=[a_2,\,b_2]$. Then the following holds*

(a) *if $a_1,a_2\ne-\infty$, $b_1,b_2\ne\infty$, then*

$$\frac{d(g\circ f)}{d\mathcal{X}}(x)\subseteq\frac{df}{d\mathcal{X}}\big(g(x)\big)\cdot\frac{dg}{d\mathcal{X}}(x)\,.$$

(b) *If $0\notin[a_1,\,b_1]$ and $0\notin[a_2,\,b_2]$, then*

$$\frac{d(g\circ f)}{d\mathcal{X}}(x)\subseteq\frac{df}{d\mathcal{X}}\big(g(x)\big)\cdot\frac{dg}{d\mathcal{X}}(x)\,.$$

Proof. Denote $g(x)=z$. If $g(x)\ne g(y)$ for any $y\ne x$, then

$$\frac{f\big(g(x)\big)-f\big(g(y)\big)}{x-y}=\frac{f\big(g(x)\big)-f\big(g(y)\big)}{g(x)-g(y)}\,\frac{g(x)-g(y)}{x-y}\,,$$

hence

$$\limsup_{y\to x} \frac{f\big(g(y)\big) - f\big(g(x)\big)}{y-x} \le \limsup_{t\to z} \frac{f(t)-f(z)}{t-z} \ \limsup_{y\to x} \frac{g(y)-g(x)}{y-x}$$

and

$$\liminf_{y\to x} \frac{f\big(g(y)\big) - f\big(g(x)\big)}{y-x} \ge \liminf_{t\to z} \frac{f(t)-f(z)}{t-z} \ \limsup_{y\to x} \frac{g(y)-g(x)}{y-x}\,.$$

If $g(x) = g(y)$ then obviously

$$\frac{f\big(g(y)\big) - f\big(g(x)\big)}{y-x} = \frac{f(t)-f(z)}{t-z}\,\frac{g(y)-g(x)}{y-x} = 0\,.$$

This proves statements (a) and (b). □

Corollary to Theorem 2a. *If the fuzzy derivative of g at x equals to $\{0\}$ and $\frac{df}{d\mathcal{X}}\big(g(x)\big) \subset \,]-\infty,\,\infty[$, then $\frac{d(g\circ f)}{d\mathcal{X}}(x) = 0$.*

If g is a strictly monotone and continuous function, then $g(\mathcal{X})$ will denote a fuzzy nearness relation such that

$$g^{-1}(x)\,\mathcal{X}\,g^{-1}(z) = \alpha \iff x\; g(\mathcal{X})\,z = \alpha\,.$$

Theorem 3. (Chain Rule for nearness differentiability.) *Let f and g be continuous functions and $\mathcal{X}$ be some fixed fuzzy nearness relation. Then there holds*

(a) *if g is strictly monotone and nearness differentiable at a point x w.r.t. $\mathcal{X}$, f is nearness differentiable at $g(x)$ w.r.t. $g(\mathcal{X})$ and*

$$0 \notin \frac{dg}{d\mathcal{X}}(x) \subset \,]-\infty,\,\infty[\,,$$

then $g\circ f$ is nearness differentiable at x w.r.t. $\mathcal{X}$.

(b) *If $\frac{dg}{d\mathcal{X}} = \{0\}$ and $\frac{df}{d\mathcal{X}}\big(g(x)\big) \subset \,]-\infty,\infty[$, then $g\circ f$ is nearness differentiable at x w.r.t. $\mathcal{X}$.*

(c) *If $\frac{df}{d\mathcal{X}}\big(g(x)\big) = \{0\}$ and $\frac{dg}{d\mathcal{X}}(x) \subset \,]-\infty,\infty[$, then $g\circ f$ is nearness differentiable at x w.r.t. $\mathcal{X}$.*

(d) *If $\frac{df}{d\mathcal{X}}\big(g(x)\big) = \{a\} \notin \{-\infty, 0, \infty\}$ and g is nearness differentiable at x w.r.t. $\mathcal{X}$, then $g\circ f$ is nearness differentiable at x w.r.t. $\mathcal{X}$.*

Proof. (**a**) Because of the strict monotonicity of g we have

$$\frac{f\big(g(y)\big) - f\big(g(x)\big)}{y-x} = \frac{f\big(g(y)\big) - f\big(g(x)\big)}{g(y)-g(x)}\,\frac{g(y)-g(x)}{y-x}\,.$$

f is nearness differentiable at $g(x)$, hence there exists a fuzzy nearness relation $\mathcal{Y}_f$ such that f is fuzzy smooth at $g(x)$ w.r.t. $(g(\mathcal{X});\,\mathcal{Y}_f)$. Fix some $\alpha_0 < 1$ and denote

$$G = \sup\biggl\{ \Bigl| \frac{g(z)-g(x)}{z-x} \Bigr| \;:\; z \in [x_{-\alpha_0},\, x_{\alpha_0}] \setminus \{x\} \biggr\}.$$

Define $\mathcal{Y}$ by

$$u \cdot G\,\mathcal{Y}\,v \cdot G = u\,\mathcal{Y}_f\,v\,.$$

Then $g \circ f$ is α_0-smooth at x w.r.t. $(\mathcal{X};\,\mathcal{Y})$, hence it is fuzzy smooth and hence nearness differentiable w.r.t. $\mathcal{X}$.

Statements (**b**) and (**c**) are implied by Corollary to Theorem 2a and Lemma 1.

(**d**) There holds

$$\frac{df}{d\mathcal{X}}\bigl(g(x)\bigr) = \{a\} \notin \{-\infty, 0, \infty\}\,.$$

Fix some $\alpha_0 < 1$ and put

$$F = \sup\biggl\{ \biggl| \frac{f(z)-f\bigl(g(x)\bigr)}{z-g(x)} \biggr| \;:\; z \in [t_{-\alpha_0},\, t_{\alpha_0}] \setminus \{g(x)\} \biggr\},$$

where $t_{-\alpha_0}\, g(\mathcal{X})\, g(x) = g(x)\, g(\mathcal{X}) t_{\alpha_0} = \alpha_0$. The nearness differentiability of g at x w.r.t. $\mathcal{X}$ implies the existence of $\mathcal{Y}_g$ such that g is fuzzy smooth at x w.r.t. $(\mathcal{X}, \mathcal{Y}_g)$. This implies that $g \circ f$ is fuzzy smooth at x w.r.t. $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{Y}$ is defined by

$$(u \cdot F)\,\mathcal{Y}\,v \cdot G = u\,\mathcal{Y}_g\,v\,.$$

Consequently we get the nearness differentiability of $g \circ f$ at x w.r.t. $\mathcal{X}$. $\square$

Similarly to the fuzzy derivative we can define also partial fuzzy derivative. For the partial derivatives of functions of two variables we will use the following notation

$$\frac{\partial f}{\partial \mathcal{X}}(x,\, y)\,; \qquad \frac{\partial f}{\partial \mathcal{Y}}(x,\, y)\,,$$

and in the notation of the partial α-derivatives we will use the denominators $\partial \mathcal{X}_\alpha$, or $\partial \mathcal{Y}_\alpha$.

Before formulating the results concerning functions given parametrically and implicitly yet one more notation is needed. Let $\mathcal{Z}$ be a fuzzy nearness relation. Then we denote

$$z \in_{\mathcal{Z}_\alpha} [a,\, b] \iff \bigl(z \in [a,\, b] \text{ or } \ z\mathcal{Z}a \geq \alpha \ \text{ or } \ z\mathcal{Z}b \geq \alpha\bigr)\,.$$

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a strictly monotone function. Then we denote $\varphi(\mathcal{X})$ the fuzzy nearness relation defined by

$$\tilde{t}\,\varphi(\mathcal{X})\,t = \varphi^{-1}(\tilde{t})\,\mathcal{X}\,\varphi^{-1}(t)\,.$$

THEOREM 4. *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a continuous, strictly monotone function and $\psi\colon \mathbb{R} \to \mathbb{R}$ and $F\colon \mathbb{R}^2 \to \mathbb{R}$ be continuos functions. Fix some $t \in \mathbb{R}$. Denote $\mathcal{T}$, $\mathcal{X}$, $\mathcal{Y}$ some fuzzy nearness relations, connected with the variables t, x, y, respectively, and further denote*

$$x = \varphi(t)\,, \quad y = \psi(t)\,, \quad f(t) = F\big(\varphi(t),\, \psi(t)\big)\,.$$

Assume the fuzzy derivatives $\frac{\partial F}{\partial X}(x,\, y)$, $\frac{d\varphi}{d\mathcal{T}}(t)$, $\frac{\partial F}{\partial Y}(x,\, y)$, $\frac{d\psi}{d\mathcal{T}}(t)$ to be finite (i.e., to be subsets of $]-\infty,\, \infty[$). Then if there exists a fuzzy nearness relation $\mathcal{Z}$ such that

$$F_{-\alpha,y}{}'\big(\varphi(t_{-\alpha}),\, y\big) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x,\, y) \quad \& \quad F_{\alpha,y}{}'\big(\varphi(t_{\alpha}),\, y\big) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x,\, y)$$

(using the straightforward modification of the notation of Formulas (1)), then

$$\frac{df}{d\mathcal{T}}(t) \subseteq \frac{\partial F}{\partial X}(x,\, y)\frac{d\varphi}{d\mathcal{T}}(t) + \frac{\partial F}{\partial Y}(x,\, y)\frac{d\psi}{d\mathcal{T}}(t)\,.$$

P r o o f.

$$\frac{f(t)-f(s)}{t-s} = \frac{F\big(\varphi(t),\, \psi(t)\big) - F\big(\varphi(t),\, \psi(s)\big)}{t-s} + \frac{F\big(\varphi(t),\, \psi(s)\big) - F\big(\varphi(s,\, \psi(s)\big)}{s-t}\,.$$

Now we can use the Chain Rule (Theorem 2) and we get the assertion in question. □

THEOREM 5. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that*

$$F(x,\, y) = c$$

defines a function $\psi\colon \mathbb{R} \to \mathbb{R}$ ($\psi(x) = y$). Denote by $\mathcal{X}$ and $\mathcal{Y}$ some fuzzy nearness relations connected with the variables x and y, respectively. Let the fuzzy derivatives $\frac{\partial F}{\partial X}(x,\, y)$, $\frac{\partial F}{\partial Y}(x,\, y)$, $\frac{d\psi}{d\mathcal{X}}(x)$ be finite (i.e., to be subsets of $]-\infty,\, \infty[$). Assume that there is a fuzzy nearness relation $\mathcal{Z}$ such that

$$F_{-\alpha,y}{}'(x_{-\alpha},\, y) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x,\, y) \quad \& \quad F_{\alpha,y}{}'(x_{\alpha},\, y) \in_{\mathcal{Z}_\alpha} \frac{\partial F}{\partial \mathcal{Y}_\alpha}(x,\, y)\,.$$

Then

$$-\frac{\partial F}{\partial \mathcal{X}}(x,\, y) \subseteq \frac{\partial F}{\partial \mathcal{Y}}(x,\, y)\frac{d\psi}{d\mathcal{X}}(x)\,.$$

P r o o f. There holds

$$0 = \frac{F\big(x,\, \psi(x)\big) - F\big(s,\, \psi(s)\big)}{x-s} = \frac{F\big(x,\, \psi(x)\big) - F\big(s,\, \psi(x)\big)}{x-s} + \frac{F\big(s,\, \psi(x)\big) - F\big(s,\, \psi(s)\big)}{x-s}\,,$$

hence, for $y(s) \neq y(x)$

$$-\frac{F\big(x,\, \psi(x)\big) - F\big(s,\, \psi(s)\big)}{x-s} = \frac{F\big(s,\, \psi(s)\big) - F\big(s,\, \psi(x)\big)}{s-x}\,.$$

The assertion in question is implied by the Chain Rule (Theorem 2). □

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