Tatra Mt. Math. Publ. **21** (2001), 105–123

TATRA MOUNTAINS
Mathematical Publications

STATIC AND DYNAMIC COMPOSITE PREFERENCES: A POSSIBLE FORMALIZATION

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ABSTRACT. We define and provide a rationale for what we call a composite fuzzy preference. We define two types of ignorance a decision maker may face, which we refer to as pattern ignorance and listing ignorance. The former relates to the quality of information and the latter relates to the quantity of information. We first consider a definition of static composite preference which combines these two forms of ignorance. We then extend this to a dynamic composite preference in order to describe the temporal evolution of uncertainty. We propose a formalization by which we can represent the convergence of a fuzzy preference to a crisp preference.

1. Introduction

Consider a classical economic decision maker who maximizes utility subject to a budget constraint. For simplicity, suppose there are two infinitely divisible consumption goods which are consumed in quantities x and y and that the price of both goods is taken to be 1. Also, suppose the economic agent derives utility from consumption according to $U(x,y)=\frac{xy}{x+y}$ by spending his income I. The optimal consumption decision is the solution to

$$\sup\left[\frac{xy}{x+y}\right] \qquad \text{subject to } \ x+y\le I\,, \quad \text{and } \ x,y\in\mathbb{R}_+\,. \tag{1}$$

Using the standard Lagrangian approach with $L=\frac{xy}{x+y}-\lambda(x+y-I)$ we solve

$$\frac{\partial L}{\partial x}=\frac{x^2}{(x+y)^2}-\lambda=0\,, \qquad \frac{\partial L}{\partial y}=\frac{y^2}{(x+y)^2}-\lambda=0\,, \qquad \frac{\partial L}{\partial \lambda}=-x-y+I=0 \tag{2}$$

2000 Mathematics Subject Classification: 90A06.
Keywords: composite preferences, convergence.

* The financial help from NSERC (Natural Sciences and Engineering Research Council of Canada) and McGill University in the form of a Max Bell Fellowship is gratefully acknowledged.
† The financial help from SSHRC (Social Sciences and Humanities Research Council of Canada), FCAR (Fonds pour les Chercheurs et l'Aide à la Recherche, Quebec) is also gratefully acknowledged.

for the optimal consumption $x^* = y^* = \frac{1}{2}I$. The optimal utility is then $U^* = U(x^*, y^*) = \frac{1}{4}I$. This agent has clear preferences over consumption bundles (x, y), given by $U(x, y)$, and has a clear optimization procedure to follow, which needs to be performed only once, yielding the optimal consumption for any income level I.

Now, consider the case where the decision maker is restricted to consuming integral quantities of the two goods, that is, x, y must belong to $\mathbb{N}$.[1] The agent's problem is now

$$\sup\left[\frac{xy}{x+y}\right] \qquad \text{subject to} \quad x + y \leq I, \quad \text{and} \quad x, y \in \mathbb{N}. \tag{3}$$

The agent still has clear preferences over consumption bundles given by $U(x, y)$ but the use of a Lagrangian is no longer possible. Without some formal optimization procedure to follow, how should the agent proceed to find the optimal consumption bundle for any income level I?

In complete ignorance of the structure of the problem (2), the diligent decision maker would start by making a primary list $\Xi = \left\{\frac{xy}{x+y}\right\}_{x,y\in\mathbb{N}}$ with the intention of identifying, for each income level I, the largest element U_I of this list such that $x + y \leq I$. However, this list is infinitely long and so, in any finite period of time, it can never be completely listed and neither can all of the optimal elements U_I be identified. The inability to work with the complete list but only with finite sublists is what we shall call the exhaustive listing problem. The decision maker must therefore operate in ignorance of the entire list. We refer to this form of ignorance as listing ignorance. Listing ignorance can be completely eliminated if the agent is able to discern and decisively establish a finite pattern or formula completely describing the optimal elements U_I for each income level I. For instance, in the case of infinitely divisible consumption quantities, the Lagrangian procedure leads to the formula $U_I = \frac{1}{4}I$, making complete knowledge of the entire list unnecessary. However, the process of discerning a pattern and proving its validity is complicated and involves another form of ignorance, which we shall call pattern ignorance.

When approaching this optimization problem, the agent does not know in advance whether or not a formula exists which describes all optimal elements U_I. If a formula does not exist, then the agent would need the entire list to find the sublist of all optimal elements. Since the agent can only work with finite lists he is only able to find a finite number of the optimal elements. The agent would then clearly prefer to have the largest list of optimal elements as possible. The listing preference over these finite lists is: the longer the list, the more preferred it

[1]The issue of indivisibility can have important implications; most of the financial literature assumes the infinite divisibility of assets. However, several papers have shown that such assumption has important consequences for consumption choice.

is. If a formula does exist, the agent still prefers longer lists over the shorter ones since the longer lists offer more information about the underlying pattern and help in its discovery and proof. If the pattern is established, then all lists become unnecessary and hence of equal worth. Thus, in the process of investigating the possible existence of a pattern or formula, there is no reversal of preferences over the lengths of lists: longer lists are always preferable.

As mentioned, the process of uncovering a possible pattern is more complicated. From some sublist Ξ' of the primary list Ξ, the agent may get a hunch about a pattern the optimal elements U_I follow. From the point of view of listing preference, the primary list is preferred to any of its sublists. But, the promise of detecting a pattern makes the sublist more preferable from a pattern detection point of view. In this sense, the pattern preferences should dominate over listing preference (a very nice example of this is given in [11] where the complete solution, or pattern, is found). Upon further examination of Ξ' the agent then may collect more evidence supporting his hunch and consider a further sublist Ξ'' which is more preferred than Ξ'. Suppose that on considering Ξ'' in more detail, the agent's hunch turns out to be invalidated. The sublist Ξ'' is then rejected in favor of Ξ', a reversal of preferences, and it may even be the case that the invalidation of the hunch causes a rejection of Ξ' in favor of Ξ, so that pattern detection must start again from the beginning. The main point is that there are possibly reversals of preference from the pattern detection point of view.

Thus, the overall preferences over sublists of Ξ is a complicated interaction between pattern recognition and the amount of information listed. We model this by constructing a composite preference structure involving both the listing preferences and pattern detection preferences. Until a pattern is decisively established or until every single element has been examined, there is inherent uncertainty or doubt about the validity of any pattern that is being pursued. This suggests that pattern recognition preferences ought to be fuzzy. Since there is doubt about the pattern validity, there will also be doubt about how preferable a longer list is to a shorter list. For example, if we feel we are close to proving a pattern then we do not really care much about longer lists. That is, lists become more and more similar in terms of preferability. If instead we feel we are far from a pattern, then we will strongly prefer long lists to short ones. This suggests that preferences over listing lengths should also be fuzzy. Thus, we shall develop a composite fuzzy preference structure which incorporates these ideas.

2. CPS's (Classical Preference Structures)

A binary relation R on a set of outcomes $\mathcal{O}$ can be viewed simultaneously as a subset of $\mathcal{O} \times \mathcal{O}$ and as a map $R\colon \mathcal{O} \times \mathcal{O} \to \{0,1\}$. For $a, b \in \mathcal{O} \times \mathcal{O}$, if

$R(a,b)=1$ then a and b are related and if $R(a,b)=0$ then they are not. The transpose R^t and complement $\operatorname{co} R$ of R are defined by $R^t(a,b)=R(b,a)$ and $\operatorname{co} R(a,b)=1-R(a,b)$ which in turn determine the sets $R^t=\{(a,b)\colon (b,a)\in R\}$ and $\operatorname{co} R=\{(a,b)\colon (a,b)\notin R\}$.

Definition 2.1. [Classical Preference Structures] [12] A classical, or crisp[2], preference structure on $\mathcal{O}$ is a triplet (P,I,J) of binary relations on $\mathcal{O}$ satisfying

(i) P is asymmetric: $P\cap P^t=\emptyset$,

(ii) I is reflexive and symmetric: $\forall a\in\mathcal{O}$, $I(a,a)=1$ and $I=I^t$,

(iii) J is irreflexive and symmetric: $\forall a\in\mathcal{O}$, $J(a,a)=0$ and $J=J^t$,

(iv) $P\cap I=P\cap J=I\cap J=\emptyset$,

(v) (P,I,J) is complete: $P\cup P^t\cup I\cup J=\mathcal{O}\times\mathcal{O}$,

P is the strict preference relation, I is the indifference relation, and J is the incomparability relation on $\mathcal{O}$. The asymmetry of P implies the irreflexivity of P: $(a,b)\in P\cap P^t \iff P(a,b)=P(b,a)=1$. But, $P\cap P^t=\emptyset$ then implies that for all a,b we have $P(a,b)=0$ or $P(b,a)=0$. Hence, $P(a,a)=0$ for all a so P is irreflexive. Also, $\{P,P^t,I,J\}$ is a partition of $\mathcal{O}\times\mathcal{O}$ so the unions in the completeness condition are disjoint: $P\cap P^t=P\cap I=P\cap J=I\cap J=\emptyset$ by definition and $P^t\cap I=(P\cap I)^t=\emptyset^t=\emptyset$ and similarly, $P^t\cap J=\emptyset$. If we define the reflexive binary relation $R=P\cup I$, known as the weak preference relation associated with (P,I,J), the following well known result establishes a one-to-one correspondence between R and (P,I,J).

Theorem 2.2. *If R is a reflexive binary relation on $\mathcal{O}$ then the triplet of binary relations on $\mathcal{O}$ defined by $(P,I,J)=(R\cap\operatorname{co} R^t, R\cap R^t, \operatorname{co} R\cap\operatorname{co} R^t)$ is a CPS on $\mathcal{O}$. Conversely, if (P,I,J) is a CPS on $\mathcal{O}$ and we define a reflexive binary relation R on $\mathcal{O}$ by $R=P\cup I$ then the triplet (P,I,J) can be represented as $(P,I,J)=(R\cap\operatorname{co} R^t, R\cap R^t, \operatorname{co} R\cap\operatorname{co} R^t)$.*

3. CPS's without incomparability

Since $\emptyset$ is a irreflexive and symmetric binary relation on $\mathcal{O}$ we can consider CPS's of the form $(P,I)\equiv(P,I,\emptyset)$ which we refer to as CPS's without incomparability.

Proposition 3.1. *Let P and I be binary relations on $\mathcal{O}$. The pair (P,I) is a CPS without incomparability on $\mathcal{O}$ if and only if the following properties are satisfied:*

[2]The term crisp and classical are synonymous.

(i) P *is asymmetric:* $P \cap P^t = \emptyset$,

(ii) I *is reflexive and symmetric:* $\forall a \in \mathcal{O}$, $I(a,a) = 1$ *and* $I = I^t$,

(iii) $P \cap I = \emptyset$,

(iv) (P, I) *is complete:* $P \cup P^t \cup I = \mathcal{O} \times \mathcal{O}$.

Proof. $(\Rightarrow)$. If $(P, I) \equiv (P, I, \emptyset)$ is a CPS without incomparability then (i)–(iv) above all follow from (i)–(v) of Definition 2.1 with $J = \emptyset$.

$(\Leftarrow)$ Let P, I be binary relations on $\mathcal{O}$ satisfying the above properties. (i) and (ii) of Definition 2.1 are implied directly by (i) and (ii) above. (iii) of Definition 2.1 is satisfied with $J = \emptyset$. Since $P \cap \emptyset = I \cap \emptyset = \emptyset$ holds trivially and $P \cap I = \emptyset$, by (iii) above we have that (iv) of Definition 2.1 is satisfied for $J = \emptyset$. By (iv) above, $P \cup P^t \cup I = \mathcal{O} \times \mathcal{O}$ so $P \cup P^t \cup I \cup \emptyset = P \cup P^t \cup I = \mathcal{O} \times \mathcal{O}$, verifying that (v) of Definition 2.1 holds for $J = \emptyset$. □

Again, P is irreflexive and $\{P, P^t, I\}$ is a partition of $\mathcal{O} \times \mathcal{O}$.

4. (s, ϕ)-FPS's (Fuzzy Preference Structures)

A fuzzy binary relation R on a set of alternatives $\mathcal{O}$ is viewed as a map $R\colon \mathcal{O} \times \mathcal{O} \to [0,1]$ which allows for degrees of relation between the classical extremes of 0 or 1. The transpose R^t is defined by $R^t(a,b) = R(b,a)$. The binary set operations of union and intersection as well as the unary operation of complementation for fuzzy sets and relations are described by t-norms, t-conorms, and involutive negators. A t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$ are increasing, associative and commutative maps $\mathcal{T}, \mathcal{S}\colon [0,1] \times [0,1] \to [0,1]$ such that $\mathcal{T}(1,x) = x$ and $\mathcal{S}(0,x) = x$ for all $x \in [0,1]$. An involutive negator $\mathcal{N}$ is a map $\mathcal{N}\colon [0,1] \to [0,1]$ such that $\mathcal{N}(0) = 1$, $\mathcal{N}(1) = 0$ and $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in [0,1]$. For fuzzy binary relations Q and R, the $\mathcal{T}$-intersection, the $\mathcal{S}$-union, and the $\mathcal{N}$-complement are defined for all $a, b \in \mathcal{O}$ by $(Q \cap_{\mathcal{T}} R)(a,b) = \mathcal{T}\big(Q(a,b), R(a,b)\big)$, $(Q \cup_{\mathcal{S}} R)(a,b) = \mathcal{S}\big(Q(a,b), R(a,b)\big)$, and $(\mathrm{co}_{\mathcal{N}} R)(a,b) = \mathcal{N}\big(R(a,b)\big)$, respectively. Often, we suppress references to the pair (a,b) and write $Q \cap_{\mathcal{T}} R = \mathcal{T}(Q,R)$, $Q \cup_{\mathcal{S}} R = \mathcal{S}(Q,R)$, and $\mathrm{co}_{\mathcal{N}} R = \mathcal{N}(R)$. In order to have a fuzzy analogue of the classical de Morgan laws $X \cap Y = \mathrm{co}(\mathrm{co}\, X \cup \mathrm{co}\, Y)$ and $X \cup Y = \mathrm{co}(\mathrm{co}\, X \cap \mathrm{co}\, Y)$ for sets X and Y one requires that $\mathcal{S}$ be the $\mathcal{N}$-dual t-conorm of $\mathcal{T}$ defined by $\mathcal{T}^{\mathcal{N}}(x,y) = \mathcal{N}\big(\mathcal{T}\big(\mathcal{N}(x), \mathcal{N}(y)\big)\big)$. The triplet of fuzzy set operations $(\mathcal{T}, \mathcal{T}^{\mathcal{N}}, \mathcal{N})$ is called a de Morgan triplet. An involutive negator $\mathcal{N}$ can be completely characterized by a strictly increasing continuous map $\phi\colon [0,1] \to [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$, called a $[0,1]$-automorphism, by $\mathcal{N}(x) \equiv \mathcal{N}_\phi(x) = \phi^{-1}\big(1 - \phi(x)\big)$ [14]. Henceforth, an involutive negator is denoted $\mathcal{N}_\phi$ where ϕ is the characterizing $[0,1]$-automorphism. Also define the

ϕ-transform of a t-norm $\mathcal{T}$ by $\mathcal{T}_\phi(x,y) = \phi^{-1}\big(\mathcal{T}\big(\phi(x),\phi(y)\big)\big)$. Thus, by specifying a t-norm $\mathcal{T}$ and a $[0,1]$-automorphism ϕ one obtains the de Morgan triplet $\big(\mathcal{T}_\phi, (\mathcal{T}_\phi)^{\mathcal{N}_\phi}, \mathcal{N}_\phi\big)$. A particularly important class of t-norms used in the construction of de Morgan triplets is the Frank family $\{\mathcal{T}^s\}_{s\in[0,\infty]}$ defined by

$$\begin{aligned}
\mathcal{T}^0(x,y) &= \lim_{s\to 0} \mathcal{T}^s(x,y) = \min\{x,y\}\,,\\
\mathcal{T}^1(x,y) &= \lim_{s\to 1} \mathcal{T}^s(x,y) = xy\,,\\
\mathcal{T}^\infty(x,y) &= \lim_{s\to \infty} \mathcal{T}^s(x,y) = \max\{x+y-1,0\}\,,\\
\mathcal{T}^s(x,y) &= \log_s\big[1+(s^x-1)(s^y-1)(s-1)^{-1}\big] \qquad \text{for} \quad s\in(0,1)\cup(1,\infty)\,.
\end{aligned}$$

Note that for any $s\in[0,\infty]$ and $x,y\in[0,1]$ we have $\mathcal{T}^s(x,0) = \mathcal{T}^s(0,y) = 0$.

The resulting family of de Morgan triplets $(\mathcal{T}^s_\phi, (\mathcal{T}^s_\phi)^{\mathcal{N}_\phi}, \mathcal{N}_\phi)$ is parametrized by $s\in[0,\infty]$ and $[0,1]$-automorphisms ϕ. For notational simplicity we shall use $\cap^s_\phi$, $\cup^s_\phi$, and co_ϕ to denote the set operations $\cap_{\mathcal{T}^s_\phi}$, $\cup_{(\mathcal{T}^s_\phi)^{\mathcal{N}_\phi}}$, and $\mathrm{co}_{\mathcal{N}_\phi}$ and we record their forms below:

Lemma 4.1. *For fuzzy binary relations Q and R, a $[0,1]$-automorphism ϕ and $s\in(0,1)\cup(1,\infty)$, the set operations are explicitly given by*

$$\begin{aligned}
&\text{(i)} && Q\cap^0_\phi R = \min\{Q,R\}\,,\\
& && Q\cup^0_\phi R = \max\{Q,R\}\,,\\
&\text{(ii)} && Q\cap^1_\phi R = \phi^{-1}\big(\phi(Q)\,\phi(R)\big)\,,\\
& && Q\cup^1_\phi R = \phi^{-1}\big(\phi(Q)+\phi(R)-\phi(Q)\,\phi(R)\big)\,,\\
&\text{(iii)} && Q\cap^\infty_\phi R = \phi^{-1}\big(\max\{\phi(Q)+\phi(R)-1,0\}\big)\,,\\
& && Q\cup^\infty_\phi R = \phi^{-1}\big(\min\{\phi(Q)+\phi(R),1\}\big)\,,\\
&\text{(iv)} && Q\cap^s_\phi R = \phi^{-1}\big(\log_s\big[1+\big(s^{\phi(Q)}-1\big)\big(s^{\phi(R)}-1\big)(s-1)^{-1}\big]\big)\,,\\
& && Q\cup^s_\phi R = \phi^{-1}\big(1-\log_s\big[1+\big(s^{1-\phi(Q)}-1\big)\big(s^{1-\phi(R)}-1\big)(s-1)^{-1}\big]\big)\,.
\end{aligned}$$

Proof. All follow immediately from the definitions of $\cap^s_\phi$, $\cup^s_\phi$, and $\mathcal{N}_\phi$ along with the fact that ϕ is strictly increasing, order preserving and invertible. □

As has been thoroughly argued in [2], [14], [15], to ensure that the preference modeller has complete freedom in assigning values in $[0,1]$ to the relations in a fuzzy preference structure, one has little choice but to work with the special case $s=\infty$. This special family of de Morgan triplets $\big(\mathcal{T}^\infty_\phi, (\mathcal{T}^\infty_\phi)^{\mathcal{N}_\phi}, \mathcal{N}_\phi\big)$, parametrized by ϕ, is distinguished by referring to their members as Łukasiewicz triplets. The most appropriate fuzzy analogue of a classical preference structure is built upon a Łukasiewicz triplet as follows:

Definition 4.2 [**(s,ϕ)-FPS**]. [14] Let ϕ be a $[0,1]$-automorphism and let $s \in [0,\infty]$. A (s,ϕ)-FPS on $\mathcal{O}$ is a triplet (P,I,J) of fuzzy binary relations on $\mathcal{O}$ satisfying

(i) I is reflexive and symmetric: $\forall a \in \mathcal{O},\ I(a,a)=1$ and $I=I^t$,

(ii) J is irreflexive and symmetric: $\forall a \in \mathcal{O},\ J(a,a)=0$ and $J=J^t$,

(iii) $P \cap_\phi^\infty I = P \cap_\phi^\infty J = \emptyset$,

(iv) $\mathrm{co}_\phi(P \cup_\phi^\infty I) = P^t \cup_\phi^\infty J$,

(v) if $s \in \{0,1,\infty\}$ then $P \cap_\phi^s P^t = I \cap_\phi^{1/s} J$,

if $s \in (0,1)\cup(1,\infty)$ then $s^{\phi(P\cap_\phi^s P^t)} + s^{-\phi(I\cap_\phi^{1/s} J)} = 2$,

where we make the identifications $1/0=\infty$ and $1/\infty = 0$ in condition (v).

As with classical preference structures, the fuzzy binary relation P is the strict preference relation, I is the indifference relation, and J is the incomparability relation on $\mathcal{O}$. For technical reasons, the slightly stronger completeness condition $\mathrm{co}_\phi(P \cup_\phi^\infty I) = P^t \cup_\phi^\infty J$ is used instead of the obvious fuzzy analogue $P \cup_\phi^\infty P^t \cup_\phi^\infty I \cup_\phi^\infty J = \mathcal{O}\times\mathcal{O}$ of classical completeness. In addition, the fuzzy analogues $I \cap_\phi^\infty J = \emptyset$ and $P \cap_\phi^\infty P^t = \emptyset$ of the classical conditions $I \cap J = \emptyset$ and $P \cap P^t = \emptyset$ appear to be missing. However, these three conditions all follow from the definition of a (s,ϕ)-FPS.

Proposition 4.3. *Let (P,I,J) be a (s,ϕ)-FPS. Then*

(i) $\phi(P)+\phi(I) \le 1$ *and* $\phi(P)+\phi(J)\le 1$,

(ii) $\phi(P^t)+\phi(I)\le 1$ *and* $\phi(P^t)+\phi(J)\le 1$,

(iii) $\phi(P)+\phi(P^t)+\phi(I)+\phi(J)=1$,

(iv) $\phi(P^t)+\phi(P)\le 1$ *and* $\phi(I)+\phi(J)\le 1$,

(v) $\phi(P)+\phi(P^t)+\phi(I)\le 1$ *and* $\phi(P)+\phi(P^t)+\phi(J)\le 1$,

(vi) $\phi(P)+\phi(I)+\phi(J)\le 1$ *and* $\phi(P^t)+\phi(I)+\phi(J)\le 1$,

(vii) $P \cup_\phi^\infty P^t \cup_\phi^\infty I \cup_\phi^\infty J = \mathcal{O}\times\mathcal{O}$,

(viii) $P\cap_\phi^\infty P^t = I \cap_\phi^\infty J = P^t \cap_\phi^\infty I = P^t \cap_\phi^\infty J = \emptyset$.

P r o o f . By the definition of $\cap_\phi^s$, $P\cap_\phi^\infty I = \emptyset \iff \phi(P)+\phi(I)\le 1$ and $P\cap_\phi^\infty J=\emptyset \iff \phi(P)+\phi(J)\le 1$, showing (i). Since $P^t\cap_\phi^\infty I = (P\cap_\phi^\infty I)^t = \emptyset^t = \emptyset$ and $P^t\cap_\phi^\infty J = (P\cap_\phi^\infty J)^t = \emptyset^t=\emptyset$, (ii) again follows by definition of $\cap_\phi^s$. For (iii), we use (i) and (ii) to observe that $\mathrm{co}_\phi(P\cup_\phi^\infty I) = \phi^{-1}\big(1-\phi(P\cup_\phi^\infty I)\big) = \phi^{-1}\big(1-\min\{\phi(P)+\phi(I),1\}\big) = \phi^{-1}\big(\max\{1-\phi(P)-\phi(I),0\}\big) = \phi^{-1}\big(1-\phi(P)-\phi(I)\big)$ and $P^t\cup_\phi^\infty J = \phi^{-1}\big(\min\{\phi(P^t)+\phi(J),1\}\big) = \phi^{-1}\big(\phi(P^t)+\phi(J)\big)$. Equating these and applying ϕ yields (iii). (iv), (v) and (vi) and follow trivially from (iii). Suppressing the pairs (a,b), and using (i)–(vi), we have

$$\begin{aligned} P\cup_\phi^\infty P^t\cup_\phi^\infty I\cup_\phi^\infty J &= \phi^{-1}\big(\min\{\phi(P)+\phi(P^t\cup_\phi^\infty I\cup_\phi^\infty J),1\}\big)\\ &= \phi^{-1}\big(\min\{\phi(P)+\min[\phi(P^t)+\phi(I\cup_\phi^\infty J),1],1\}\big)\\ &= \phi^{-1}\big(\min\{\phi(P)+\min[\phi(P^t)+\min\langle\phi(I)+\phi(J),1\rangle,1],1\}\big)\\ &= \phi^{-1}\big(\phi(P)+\phi(P^t)+\phi(I)+\phi(J)\big)\\ &= \phi^{-1}(1)=1 \end{aligned}$$

which proves (vii). (viii) follows directly from (ii) and (iv) and the definition of $\cap_\phi^\infty$. □

It has been shown [14] that in order to establish a $1-1$ correspondence between fuzzy preference structures and weak fuzzy preference relations, the additional characterizing condition (v) in Definition 4.2 must be imposed. This results in the following theorem:

THEOREM 4.4. [14] *Let ϕ be a $[0,1]$-automorphism and let $s\in[0,\infty]$. If R is a reflexive fuzzy binary relation on a $\mathcal{O}$ then the triplet $(P,I,J)=\big(R\cap_\phi^{1/s}\operatorname{co}_\phi R^t,\ R\cap_\phi^s R^t,\ \operatorname{co}_\phi R\cap_\phi^s\operatorname{co}_\phi R^t\big)$ is a (s,ϕ)-FPS on $\mathcal{O}$. Conversely, if (P,I,J) is a (s,ϕ)-FPS on $\mathcal{O}$ and we define a fuzzy reflexive binary relation R on $\mathcal{O}$ by $R=P\cup_\phi^\infty I$ then $(P,I,J)=\big(R\cap_\phi^{1/s}\operatorname{co}_\phi R^t,\ R\cap_\phi^s R^t,\ \operatorname{co}_\phi R\cap_\phi^s\operatorname{co}_\phi R^t\big)$.*

The relation $R=P\cup_\phi^\infty I$ in the prior theorem is the weak preference relation associated with the (s,ϕ)-FPS (P,I,J).

5. (s,ϕ)-FPS's without incomparability

As in the classical case, $J=\emptyset$ is an irreflexive, symmetric fuzzy binary relation on $\mathcal{O}$ so we can consider (s,ϕ)-FPS's without incomparability, that is, (s,ϕ)-FPS's of the form $(P,I)\equiv(P,I,\emptyset)$.

PROPOSITION 5.1. *Let ϕ be a $[0,1]$-automorphism and let $s\in[0,\infty]$. Also, let P and I be fuzzy binary relations on $\mathcal{O}$. The triplet (P,I) is a (s,ϕ)-FPS without incomparability on $\mathcal{O}$ if and only if the following are satisfied:*

(i) *I is reflexive and symmetric: $\forall a\in\mathcal{O}$, $I(a,a)=1$ and $I=I^t$,*

(ii) $P\cap_\phi^\infty I=\emptyset$,

(iii) $\operatorname{co}_\phi(P\cup_\phi^\infty I)=P^t$,

(iv) $P\cap_\phi^s P^t=\emptyset$.

P r o o f. ($\Rightarrow$) Let $(P,I,\emptyset)$ be a (s,ϕ)-FPS. Condition (i) and (ii) above follow directly from Conditions (i) and (iii) of Definition 4.2. The fact that

$P^t \cup_\phi^\infty \emptyset = \phi^{-1}\big(\min\{\phi(P^t) + \phi(\emptyset), 1\}\big) = \phi^{-1}\big(\min\{\phi(P^t), 1\}\big) = P^t$ combined with (iv) of Definition 4.2 yields (iii) above. For (iv) above, observe that $I \cap_\phi^{1/s} \emptyset = \emptyset$ for all $s \in [0, \infty]$. Combining this with (v) of Definition 4.2 we have (iv).

($\Leftarrow$) Let P, I be fuzzy binary relations satisfying the above properties. (i) of Definition 4.2 follows directly from (i) above. Since $\emptyset$ is irreflexive and symmetric, (ii) of Definition 4.2 also holds. (iii) above along with $P \cap_\phi^\infty \emptyset = \emptyset$ gives (iii) of Definition 4.2. $P^t \cup_\phi^\infty \emptyset = P^t$ with (iii) above gives (iv) of Definition 4.2. Since $I \cap_\phi^{1/s} \emptyset = \emptyset$ and $P \cap_\phi^s P^t = \emptyset$ from (iv) above Condition (v) of Definition 4.2 follows. □

Corollary 5.2. *Let ϕ be a $[0,1]$-automorphism and let $s \in [0, \infty]$. If R is a reflexive fuzzy binary relation on $\mathcal{O}$ such that $\mathrm{co}_\phi R \cap_\phi^s \mathrm{co}_\phi R^t = \emptyset$ then the pair $(P, I) = \big(R \cap_\phi^{1/s} \mathrm{co}_\phi R^t, R \cap_\phi^s R^t\big)$ is a (s, ϕ)-FPS on $\mathcal{O}$ without incomparability. Conversely, if (P, I) is a (s, ϕ)-FPS on $\mathcal{O}$ without incomparability and we define a fuzzy reflexive binary relation R on $\mathcal{O}$ by $R = P \cup_\phi^\infty I$ then $(P, I) = \big(R \cap_\phi^{1/s} \mathrm{co}_\phi R^t, R \cap_\phi^s R^t\big)$.*

Proof. Follows immediately from Theorem 4.4 with $J = \emptyset$. □

Corollary 5.3. *Let (P, I) be a (s, ϕ)-FPS without incomparability. Then*

(i) $\phi(P) + \phi(I) \le 1$, $\phi(P^t) + \phi(I) \le 1$ *and* $\phi(P^t) + \phi(P) \le 1$,

(ii) $\phi(P) + \phi(P^t) + \phi(I) = 1$,

(iii) $P \cup_\phi^\infty P^t \cup_\phi^\infty I = \mathcal{O} \times \mathcal{O}$,

(iv) $P \cap_\phi^\infty P^t = P^t \cap_\phi^\infty I = \emptyset$.

Proof. Follows immediately from Proposition 4.3 with $J = \emptyset$. □

6. Pattern and list preferences as FPS's

Due to the uncertainty involved in knowing how long a list needs to be in order to detect a pattern and in the actual detection and verification of a pattern, we model both list and pattern preferences as FPS's: let (P_p, I_p) be a (s_p, ϕ_p)-FPS and (P_l, I_l) a (s_l, ϕ_l)-FPS, both without incomparability. (P_p, I_p) will be our model of pattern preferences and (P_l, I_l) our model of list preferences.

As discussed in the introduction, it appears appropriate to model pattern preferences as reversible so as to allow for the discarding of a hypothesized pattern if disproven. It also appears appropriate to model list preferences as irreversible. These modeling imperatives place certain restrictions on the possible choices of (s_p, ϕ_p) and (s_l, ϕ_l). We formalize this as follows:

ASSUMPTION 6.1. *Strict pattern preferences are reversible:* $\exists\ a,b \in \mathcal{O}$, $P_p(a,b) > 0$ *and* $P_p^t(a,b) > 0$.

ASSUMPTION 6.2. *Strict list preferences are irreversible:* $\forall\ a,b \in \mathcal{O},\ P_l(a,b) > 0 \implies P_l^t(a,b) = 0$, *or, equivalently,* $\forall\ a,b \in \mathcal{O},\ P_l(a,b) = 0$ *or* $P_l^t(a,b) = 0$.

PROPOSITION 6.3. *A strict fuzzy preference relation* P *on* $\mathcal{O}$ *is reversible if and only if* $s = \infty$. *Hence,* P *is irreversible if and only if* $s \in [0,\infty)$.

Proof. Since $\phi(0) = 0$ and ϕ is order preserving, P is reversible if and only if there is a pair (a,b) such that $\phi\big(P(a,b)\big) > 0$ and $\phi\big(P^t(a,b)\big) > 0$. The set of restrictions on the P and P^t of a (s,ϕ)-FPS (P,I) without incomparability are listed in Proposition 5.1 and Corollary 5.3. It is clear that conditions (i), (ii), and (iii) of Proposition 5.1 and all conditions of Corollary 5.3 do not rule out the possibility of reversibility. The key condition then is the characterizing condition (iv) of Proposition 5.1: $\big(P \cap_\phi^s P^t\big)(a,b) = 0$ for all a,b, which we consider by cases. For $s = 0$: $\big(P \cap_\phi^0 P^t\big)(a,b) = \min\{P(a,b), P^t(a,b)\} = 0 \iff P(a,b) = 0$ or $P^t(a,b) = 0$. Thus, the $s = 0$ case is irreversible. For $s = 1$: $\big(P \cap_\phi^1 P^t\big)(a,b) = \phi^{-1}\big(\phi\big(P(a,b)\big)\phi\big(P^t(a,b)\big)\big) = 0 \iff \phi\big(P(a,b)\big)\phi\big(P^t(a,b)\big) = 0 \iff \phi\big(P(a,b)\big) = 0$ or $\phi\big(P^t(a,b)\big) = 0$. Thus, the $s = 1$ case also irreversible. For $s = \infty$, the characterizing condition is not an additional condition; it follows from the definition (see Corollary 5.3). The characterizing condition in this case is $\phi\big(P(a,b)\big) + \phi\big(P^t(a,b)\big) \leq 1$, which does not rule out reversibility. Finally, for $s \in (0,1) \cup (1,\infty)$, we have $(P \cap_\phi^s P^t)(a,b) = \phi^{-1}\big(\log_s\big[1 + \big(s^{\phi(P(a,b))} - 1\big)\big(s^{\phi(P^t(a,b))} - 1\big)(s-1)^{-1}\big]\big) = 0$ $\iff \big(s^{\phi(P(a,b))} - 1\big)\big(s^{\phi(P^t(a,b))} - 1\big) = 0 \iff \phi\big(P(a,b)\big) = 0$ or $\phi\big(P^t(a,b)\big) = 0$ and so we again have irreversibility. □

Thus, in selecting appropriate FPS's for pattern and listing preferences, (P_p, I_p) must be a (∞, ϕ_p)-FPS and (P_l, I_l) must be a (s_l, ϕ_l)-FPS for some $s_l \in [0,\infty)$.

7. Pattern and list preferences as a composite FPS

We are interested in constructing a new (s_c, ϕ_c)-FPS without incomparability, (P_c, I_c), which incorporates both of the more fundamental preference structures (P_p, I_p) and (P_l, I_l), and to determine appropriate $s_c \in [0,1]$ and $[0,1]$-automorphisms ϕ_c. In addition, we would like to incorporate the idea that pattern preferences dominate list preferences. To achieve this, we first consider a composite of two CPS's by constructing a composite weak preference relation.

We then fuzzify this weak preference relation and, after verifying that the resulting binary relation is reflexive, we use Theorem 4.4 to construct an FPS.

Suppose for the moment that (P_l, I_l) and (P_p, I_p) are CPS's without incomparability. Since $\{P_p, P_p^t, I_p\}$ and $\{P_l, P_l^t, I_l\}$ are both partitions of $\mathcal{O} \times \mathcal{O}$ we can construct a finer partition of $\mathcal{O} \times \mathcal{O}$ as follows:

$$\begin{aligned}\mathcal{O} \times \mathcal{O} = [P_p \cap P_l] &\cup [P_p \cap P_l^t] \cup [P_p \cap O_l] \\ &\cup [P_p^t \cap P_l] \cup [P_p^t \cap P_l^t] \cup [P_p^t \cap O_l] \\ &\cup [I_p \cap P_l] \cup [I_p \cap P_l^t] \cup [I_p \cap O_l] \,.\end{aligned}$$

The composite P_c of P_l and P_p will involve some assignment of the above 9 sets. Assume that P_p dominates P_l in the sense that if P_p and P_l give opposite orderings then the ordering given by P_p will be the one assigned to P_c. Also, assume composite indifference results only when there is indifference with both CPS's. This leads to the following natural choice for the composite CPS (P_c, I_c):

$$P_c = [P_p \cap P_l] \cup [P_p \cap P_l^t] \cup [P_p \cap O_l] \cup [I_p \cap P_l] \,, \qquad I_c = [I_p \cap O_l] \,.$$

PROPOSITION 7.1. *Using the weak relations $R_p = P_p \cup I_p$ and $R_l = P_l \cup I_l$ we have*

(i) $P_c = P_p \cup [I_p \cap P_l]$,

(ii) (P_c, I_c) *is a CPS without incomparability,*

(iii) $R_c \equiv P_c \cup I_c = P_p \cup [R_p \cap R_l]$.

Proof. For (i), $P_c = [P_p \cap P_l] \cup [P_p \cap P_l^t] \cup [P_p \cap O_l] \cup [I_p \cap P_l] = [P_p \cap (P_l \cup P_l^t \cup I_l)] \cup [I_p \cap P_l] = P_p \cup [I_p \cap P_l]$. For (ii), we verify conditions of Proposition 3.1:

$$\begin{aligned}P_c \cap P_c^t &= \big(P_p \cup [I_p \cap P_l]\big) \cap \big(P_p^t \cup [I_p \cap P_l^t]\big) \\ &= (P_p \cap P_p^t) \cup (I_p \cap P_p^t \cap P_l) \cup (P_p \cap I_p \cap P_l^t) \cup (I_p \cap P_l \cap I_p \cap P_l^t) \\ &= \emptyset\end{aligned}$$

since $P_p \cap P_p^t = I_p \cap P_p^t = I_p \cap P_p = P_l \cap P_l^t = \emptyset$. Now, using the symmetry of I_p and I_l $(a, b) \in I = I_p \cap I_l \iff (a, b) \in I_p$ and $(a, b) \in I_l \iff (b, a) \in I_p$ and $(b, a) \in I_l \iff (b, a) \in I = I_p \cap I_l$ which shows that I is symmetric. Also, for all a, $(a, a) \in I_p$ and $(a, a) \in I_l$ hence $(a, a) \in I = I_p \cap I_l$ implying that I is reflexive. We have $P \cap I = \big(P_p \cup [I_p \cap P_l]\big) \cap (I_p \cap I_l) = (P_p \cap I_p \cap I_l) \cup \big([I_p \cap P_l] \cap (I_p \cap I_l)\big) = \emptyset$ since $P_p \cap I_p = P_l \cap I_l = \emptyset$. Finally, completeness holds by construction. However, this is also easy to see directly: $\big(P_p \cup [I_p \cap P_l]\big) \cup \big(P_p^t \cup [I_p \cap P_l^t]\big) \cup (I_p \cap I_l) = P_p \cup P_p^t \cup \big[I_p \cap (P_l \cup P_l^t \cup I_l)\big] = P_p \cup P_p^t \cup I_p = \mathcal{O} \times \mathcal{O}$. For (iii), we have $R_c = P_c \cup I_c = \big(P_p \cup [I_p \cap P_l]\big) \cup (I_p \cap I_l) = P_p \cup \big(I_p \cap (P_l \cup I_l)\big) =$

$P_p \cup (I_p \cap R_l) = (P_p \cup I_p) \cap (P_p \cup R_l) = R_p \cap (P_p \cup R_l) = (R_p \cap P_p) \cup (R_p \cap R_l) = P_p \cup (R_p \cap R_l)$. $\square$

We propose to fuzzify the weak relation $R_c = P_c \cup (R_p \cap R_l)$ and then use R_c to construct a composite FPS. Now, consider (P_p, I_p) as a (∞, ϕ_p)-FPS and (P_l, I_l) as a (s_l, ϕ_l)-FPS where $s_l \in [0, \infty)$. From Theorem 4.4 we have the associated weak preference relations $R_p = P_p \cup_{\phi_p}^{\infty} I_p$ and $R_l = P_l \cup_{\phi_l}^{\infty} I_l$. Define the relation $R_c = P_p \cup_{\zeta}^{u} (R_p \cap_{\xi}^{v} R_l)$ where $u, v \in [0, \infty]$ and ζ, ξ are both $[0,1]$-automorphisms.

PROPOSITION 7.2. *$R_c = P_p \cup_{\zeta}^{u} (R_p \cap_{\xi}^{v} R_l)$ is reflexive.*

Proof.

$$\begin{aligned}
R_c(a,a) &= \left[P_p \cup_{\zeta}^{u} (R_p \cap_{\xi}^{v} R_l)\right](a,a) \\
&= \mathcal{N}_{\zeta}\big(\mathcal{T}_{\zeta}^{u}\big(\mathcal{N}_{\zeta}\big(P_p(a,a)\big), \mathcal{N}_{\zeta}\big([R_p \cap_{\xi}^{v} R_l](a,a)\big)\big)\big) \\
&= \mathcal{N}_{\zeta}\big(\mathcal{T}_{\zeta}^{u}\big(\mathcal{N}_{\zeta}\big(P_p(a,a)\big), \mathcal{N}_{\zeta}\big(\mathcal{T}_{\xi}^{v}\big(R_p(a,a), R_l(a,a)\big)\big)\big)\big) \\
&= \mathcal{N}_{\zeta}\big(\mathcal{T}_{\zeta}^{u}\big(\mathcal{N}_{\zeta}(0), \mathcal{N}_{\zeta}\big(\mathcal{T}_{\xi}^{v}(1,1)\big)\big)\big) \\
&= \mathcal{N}_{\zeta}\big(\mathcal{T}_{\zeta}^{u}\big(1, \mathcal{N}_{\zeta}(1)\big)\big) \\
&= \mathcal{N}_{\zeta}\big(\mathcal{T}_{\zeta}^{u}(1,0)\big) \\
&= \mathcal{N}_{\zeta}(0) \\
&= 1\,.
\end{aligned}$$

$\square$

DEFINITION 7.3. (Composite Pattern & Listing FPS.) For the pattern preference structure (P_p, I_p) without incomparability we use a (∞, ϕ_p)-FPS. For the listing preference structure (P_l, I_l) we use a (s_l, ϕ_l)-FPS where $s_l \in [0, \infty)$. We define for any $u, v \in [0, \infty]$ and $[0,1]$-automorphisms ζ, ξ the composite weak preference relation by $R_c = P_p \cup_{\zeta}^{u} (R_p \cap_{\xi}^{v} R_l)$. And, for any $w \in [0, \infty]$ and $[0,1]$-automorphism ψ we obtain, using Theorem 4.4, the (s, ψ)-FPS (P_c, I_c, J_c) determined by

$$\begin{aligned}
P_c &= R_c \cap_{\psi}^{1/s} \operatorname{co}_{\psi} R_c^t\,, \\
I_c &= R_c \cap_{\psi}^{s} R_c^t\,, \\
J_c &= \operatorname{co}_{\psi} R_c \cap_{\psi}^{s} \operatorname{co}_{\psi} R_c^t\,.
\end{aligned}$$

Thus, our composite FPS has many parameters: $s_l \in [0, \infty)$, $u, v, s \in [0, \infty]$ and $[0,1]$-automorphisms $\phi_p, \phi_l, \zeta, \xi, \psi$. Numerical experiments suggest that a wide variety of behaviors are possible for (P_c, I_c, J_c); which parameters are most appropriate is currently under investigation.

An alternative method of combining both pattern and listing preferences is the following: define the weak relation instead by $R'_c = R_p \cup^{s_c}_{\phi_c} R_l$ for some $s_c \in [0, \infty]$ and $[0, 1]$-automorphism ϕ_c.

Lemma 7.4. *R'_c is reflexive.*

Proof. Using the reflexivity of R_p and R_l we have for all $a \in \mathcal{O}$

$$\begin{aligned} R'_c(a,a) &= (R_p \cup^{s_c}_{\phi_c} R_l)(a,a) = \mathcal{N}_{\phi_c}\big(\mathcal{T}^{s_c}_{\phi_c}\big(\mathcal{N}_{\phi_c}\big(R_p(a,a)\big), \mathcal{N}_{\phi_c}\big(R_l(a,a)\big)\big)\big) \\ &= \mathcal{N}_{\phi_c}\big(\mathcal{T}^{s_c}_{\phi_c}\big(\mathcal{N}_{\phi_c}(1), \mathcal{N}_{\phi_c}(1)\big)\big) = \mathcal{N}_{\phi_c}\big(\mathcal{T}^{s_c}_{\phi_c}(0,0)\big) = \mathcal{N}_{\phi_c}(0) = 1\,. \end{aligned}$$

□

From this reflexive binary relation R'_c, one can, using Theorem 4.4, construct a (s, ϕ)-FPS for any $s \in [0, \infty]$ and $[0, 1]$-automorphism ϕ as before. This alternative construction involves the parameters $s_l \in [0, \infty)$, $s, s_c \in [0, \infty]$, and $[0, 1]$-automorphisms $\phi_p, \phi_l, \phi, \phi_c$. However, this composite preference is "symmetric" in the pattern and list preferences and so will not reflect pattern dominance over list dominance.

8. Dynamic evolution of composite preferences

8.1 Convergence from fuzzy to crisp preferences

The investigation of lists for patterns is a dynamic process. One would expect that over time the uncertainty about what lists to consider and whether or not there is a pattern at all diminishes over time. At the start of this dynamic process, we imagine the decision maker has absolutely no information about the problem; he is in a state of complete ignorance which we could model, for example, by a (∞, ϕ)-FPS without incomparability where $\phi(x) < x$ for $x \in (0, 1)$ and binary relations satisfying $P_c(a, b) = P^t_c(a, b) = \frac{1}{2}$ and $I(a, b) = 0$ for all a, b. As time passes and uncertainty is reduced, the fuzzy binary relations take on values progressively closer to 0 or 1. Thus, we are interested in having the composite fuzzy preferences converge to crisp preferences. Also, we would like this convergence to take place within the particular FPS we have chosen for our composite preferences.

Suppose our composite preferences are, at all times, a (s, ϕ)-FPS where s and ϕ are fixed. Since we are working with dynamic preferences we must indicate the time dependence of the fuzzy binary relations. Also, to lighten notation, we drop the subscript "c". Thus, $\{(P_t, I_t)\}_{t \in \mathbb{R}_+}$ is our notation for time dependent composite preferences; for each $t \in \mathbb{R}_+$, (P_t, I_t) is a (s, ϕ)-FPS. We desire complete freedom in specifying the limiting CPS, which we have.

PROPOSITION 8.1. *Let (P, I) be any CPS without incomparability. Then, for any $s \in [0, \infty]$ and $[0,1]$-automorphism ϕ, (P, I) is a (s, ϕ)-FPS without incomparability.*

Proof. I is reflexive and symmetric so (i) of Proposition 5.1 holds. Since (P, I) is a CPS, P and I take on the values 0 and 1 only. Hence $P \cap_{\phi}^{\infty} I = \mathcal{T}_{\phi}^{\infty}(P, I) = \phi^{-1}\big(\mathcal{T}^{\infty}(\phi(P), \phi(I))\big) = 0$ if either or both of P and I are 0; if $P = I = 1$ then it is 1. This is exactly the behaviour of $P \cap I$. Hence, using (iii) of Proposition 3.1, $P \cap_{\phi}^{\infty} I = P \cap I = \emptyset$, proving (ii) of Proposition 5.1. In a similar fashion, $\mathrm{co}_{\phi}(P \cup_{\phi}^{\infty} I) = \mathrm{co}(P \cup I)$. By (iv) of Proposition 3.1 we have that $\mathrm{co}(P \cup I) = P^t$ hence $\mathrm{co}_{\phi}(P \cup_{\phi}^{\infty} I) = P^t$, proving (iii) of Proposition 5.1. For (iv) of Proposition 5.1 use the fact that $P \cap P^t = \emptyset$ by (i) of Proposition 3.1 so P and P^t cannot simultaneously be 1. Hence, $P \cap_{\phi}^{s} P^t = \mathcal{T}_{\phi}^{s}(P, P^t) = 0$ since at least one of the two arguments is zero. □

8.2 Aggregate composite preferences

We are interested in modelling a large number N of decision makers, each in the process of investigating economic data for patterns. Let $(P_t^1, I_t^1), \dots, (P_t^N, I_t^N)$, $t \in \mathbb{R}_+$, be N (s, ϕ)-FPS's of the N agents. For this group of agents we can consider the mean values of these relations:

$$\bar{P}_t = \frac{1}{N} \sum_{i=1}^{N} P_t^i \,, \qquad \bar{I}_t = \frac{1}{N} \sum_{i=1}^{N} I_t^i \,.$$

As $t \to \infty$, all agents either learn of the pattern, through their own discovery or sharing of information, or become thoroughly acquainted with the infinite list. Let $\bar{P}_{\infty} = \lim_{t \to \infty} \bar{P}_t$ and $\bar{I}_{\infty} = \lim_{t \to \infty} \bar{I}_t$. Whatever the case, we must have $\bar{P}_{\infty}$ and $\bar{I}_{\infty}$ in $\{0, 1\}$ for convergence to a crisp structure. Rather than work with the individual preferences, we use a probability density function on $[0, 1]$ and treat individuals statistical ensemble, which reflects the erratic behaviour in the initial stages of the process. Thus, let $g_t(x)$ be a time dependent probability density function on the interval $[0, 1]$ such that

$$\int_0^1 x g_t(x)\, dx = \bar{P}_t \,, \quad \lim_{t \to \infty} \int_0^1 x g_t(x)\, dx \in \{0, 1\} \,, \quad \lim_{t \to \infty} \int_0^1 (x - \bar{P})^2 g_t(x)\, dx = 0 \,.$$

That is, the mean of the distribution is the sample preference mean, which converges to 0 or 1 and the variance tends to zero. The same could be done for the indifference relation.

8.3 The Chapman-Kolmogorov equation

We now provide for a random walk on the level of a composite fuzzy preference, with the important property that once such level of fuzzy composite preference reaches one of the endpoints of $[0,1]$, the probability of the level of the fuzzy composite preference to go down to a lower level once it is very close to the 1 endpoint is very low. Similarly, the probability of the level of the fuzzy composite preference to go up once it is very close to the 0 endpoint is also very low. Hence, such random walk mimics the convergence over time of the fuzzy composite preference to a crisp preference.

For the general interval: $[-r,r]$[3] we obtain, for the probability of an upward movement: $p(\uparrow r) = 1 - \frac{1}{2}\left[1 - \frac{\eta}{r}\right]$; $\eta \in [-r,r]$ and for the probability of a downward movement: $p(\downarrow r) = 1 - \frac{1}{2}\left[1 + \frac{\eta}{r}\right]$; $\eta \in [-r,r]$ It is immediate that $\eta \in [-r,r]$: $p(\uparrow r)_{\eta=-r} = p(\downarrow r)_{\eta=r} = 0$. This is a nice property as it shows that once a state coincides with one of the endpoints the probability of separating from that endpoint is nil. We then have $\forall \eta \in [-r,r],\ \eta{*}{*} > \eta{*}$: $p(\downarrow r)_{\eta=\eta*} > p(\downarrow r)_{\eta=\eta**}$. Also, $\forall \eta \in [-r,r],\ \eta{*}{*} > \eta{*}$: $p(\uparrow r)_{\eta=\eta*} < p(\uparrow r)_{\eta=\eta**}$

These two simple facts indicate that the closer the level of a variable to any endpoint, the lower the likelihood that this variable will separate from those endpoints. Also, the further away the level of such a variable is from any endpoint, the higher the likelihood of it not being attracted to such endpoints. Given that we have been working with a general interval $[-r,r]$ we can, of course, convert this into the desired $[0,1]$ interval. The following is then immediate: $\eta \in [-r,r] \Rightarrow \eta + r \in [0,2r] \Rightarrow \frac{\eta+r}{2r} \in [0,1]$. Transition probabilities for movement from a lower level to higher level are then given by $\forall r* \in [-r,r]$; $\forall a \in \mathbb{R}_0^+$: $t(r* - a, r*) = 1 - \left[\frac{r - r* + a}{2r}\right]$.[4] Transition probabilities for movement from a higher level to a lower level are $\forall r* \in [-r,r]$; $\forall a \in \mathbb{R}_0^+$: $t(r* + a, r*) = 1 - \left[\frac{r + r* + a}{2r}\right]$.

We can now define the Chapman-Kolmogorov equation as follows:

$$p(r, s+1) = t(r-1, r)p(r-1, s) + t(r+1, r)p(r+1, s).$$

This equation defines the probability of occurrence of r at time $s+1$ as the sum of two products. The first product consists of the transition probability for a movement from level $r-1$ to r multiplied with the probability of occurrence of level $r-1$ at time s. The second product concerns the transition probability from moving from a level $r+1$ to a level r with the probability of occurrence of level $r+1$ at time s. This equation will be important in the next section.

[3] In order to lighten notation we convert $\bar{P}$ to r. Also we indicate a probability by a small p.

[4] Remark we denote r with a '$*$' to indicate it is a level, while r here stands for the value of the endpoint. Where there is no confusion possible we simply use r to denote a level.

8.4 The evolution of $\bar{P}$

In this section we investigate the time evolution of the mean fuzzy composite preference.

Proposition 8.2.

$$\frac{\partial r}{\partial t} = \frac{t(r, r+1)}{\Delta t} - \frac{t(r, r-1)}{\Delta t}.$$

This proposition indicates that the difference between the transition probability of a level increase per unit of time minus the transition probability of a level decrease per unit of time is equal to the slope of the function which relates the level, r with time. This is one of the main propositions in this paper.

Proof. We write first that: $\frac{\partial r}{\partial t} = f\big(r(t)\big)$, which indicates that the evolution of the mean composite preference is a function of the mean composite preference. The Chapman-Kolmogorov equation was: $p(r, s+1) = t(r-1, r)p(r-1, s) + t(r+1, r)p(r+1, s)$. For small Δt let $t = s\Delta t$; t is therefore small and s exceeds t. Consider $\frac{\partial p(r,t)}{\partial t} \approx \frac{\Delta p(r, t=s\Delta t)}{\Delta(s\Delta t)}$ which can be re-written as: $\frac{\Delta p(r,t)}{\Delta(t)} = \frac{p(r,s+1)-p(r,s)}{\Delta t}$. Substituting into the Chapman-Kolmogorov equation, we obtain: $\frac{\Delta p(r,t)}{\Delta(t)} = \big[\frac{t(r-1,r)}{\Delta t}p(r-1,s) + \frac{t(r+1,r)}{\Delta t}p(r+1,s)\big] - \frac{p(r,s)}{\Delta t}$. We can write: $\frac{\Delta p(r,t)}{\Delta(t)} \approx \frac{\partial p(r,t)}{\partial t}$, as $t \ll s$. Hence, we obtain then: $\frac{\partial p(r,t)}{\partial t} = \big[\frac{t(r-1,r)}{\Delta t}p(r-1,t) + \frac{t(r+1,r)}{\Delta t}p(r+1,t)\big] - \frac{p(r,t)}{\Delta t}$. We now consider: $\frac{\partial p(r',t)}{\partial t} = \int \big[t(r,r')p(r,t) - t(r',r)p(r',t)\big]\,dr$. This indicates the 'net' probability change which is equal to the 'arriving' r' (through the transition probability $t(r,r')$) minus the probability 'leaving' from r'. Using the Kramers-Moyal expansion one can show that the latest equation can be written as: $\frac{\partial p(r,t)}{\partial t} = -\frac{\partial}{\partial r}\alpha_1(r)p(r,t) + \frac{1}{2}\frac{\partial^2}{\partial r^2}\alpha_2(r)p(r,t)$. It now occurs that the terms $\alpha_1(r)$ and $\alpha_2(r)$ are: $\alpha_1(r) = \frac{t(r,r+1)}{\Delta t} - \frac{t(r,r-1)}{\Delta t}$ and $\alpha_2(r) = \frac{t(r,r+1)}{\Delta t} + \frac{t(r,r-1)}{\Delta t}$. The main issue becomes now in showing this is indeed true. We use $\frac{\partial p(r',t)}{\partial t} = \int \big[t(r,r')p(r,t) - t(r',r)p(r',t)\big]\,dr*$ and we define: $\beta(1,r) = t(r+1,r)$ and after substitution of this definition into the former equation, we obtain: $\frac{\partial P(r,t)}{\partial t} = \int \beta(1, r-1)P(r-1) - \beta(1,r)P(r)\,dr$ and which can be approximately be re-written as: $\frac{\partial p(r,t)}{\partial t} = \sum \beta(1, r-1)p(r-1) - \beta(1,r)p(r)$. Recall our re-worked Chapman equation: $\frac{\partial p(r,t)}{\partial t} = \big[\frac{t(r-1,r)}{\Delta t}p(r-1,t) + \frac{t(r+1,r)}{\Delta t}p(r+1,t)\big] - \frac{p(r,t)}{\Delta t}$. Now define $\frac{t(\cdot,\cdot)}{\Delta t} = \widehat{t}(\cdot,\cdot)$. Consider the constraint: $t(r, r-1) + t(r, r+1) = 1$ and this indicates that there must be a transition from one return state to a new return state. The return state can not stay put. The key of the proof resides now in recognizing that: $\widehat{t}(r, r+1)p(r,t) + \widehat{t}(r, r-1)p(r,t) \approx \sum \beta(1,r)p(r)$.

Observe the following: $\int d1 \sum_{n=1}^{\infty} \frac{(-1)^n}{f(n)} \frac{\partial^n}{\partial r^n}[\beta(1,r)P(r)] = \sum_{n=1}^{\infty} \frac{(-1)^n}{f(n)} \frac{\partial^n}{\partial r^n} [\alpha_n(r)P(r)]$.[5] Now also remark that: $\alpha_n(r) = \int d1\beta(1,r) = \sum \beta(1,r)$. Substituting the very last equation into the RHS of the before last equation we obtain: $\sum_{n=1}^{\infty} \frac{(-1)^n}{f(n)} \cdot \frac{\partial^n}{\partial r^n}[\alpha_n(r)p(r)] = \sum_{n=1}^{\infty} \frac{(-1)^n}{f(n)} \cdot \frac{\partial^n}{\partial r^n}[\sum \beta(1,r)p(r)]$. The LHS of the above equality is equal to $\int d1. \sum_{n=1}^{\infty} \frac{(-1)^n}{f(n)} . \frac{\partial^n}{\partial r^n}[\beta(1,r).P(r)]$. We also remark that: $\sum_{n=1}^{\infty} \frac{(-1)^n}{f(n)} \cdot \int d1 \frac{\partial^n}{\partial r^n}[\beta(1,r) \cdot P(r)] = \sum_{n=1}^{\infty} \frac{(-1)^n}{f(n)} . \frac{\partial^n}{\partial r^n}[\sum \beta(1,r)P(r)]$.

This proves that we have identified the unknowns: $\alpha_1(r)$ and $\alpha_2(r)$. The reason is now straightforward. We know, $\widehat{t}\,(r,r+1)p(r,t) + \widehat{t}\,(r,r-1)p(r,t) \approx \sum \beta(1,r)p(r)$, and given $\alpha_n(r) = \int d1\, \beta(1,r) = \sum \beta(1,r)$, this proves that: $\alpha_1(r) = \widehat{t}\,(r,r+1) - \widehat{t}\,(r,r-1) \wedge \alpha_2(r) = \widehat{t}\,(r,r+1) + \widehat{t}\,(r,r-1)$.

The final objective now becomes to show that, given: $\frac{\partial r}{\partial t} = f(r(t))$, in fact: $\frac{\partial r}{\partial t} = \alpha_1(r) = \widehat{t}\,(r,r+1) - \widehat{t}\,(r,r-1)$ holds. This last equation is extremely important for the further continuation of the paper. Proving this is actually quite straightforward. We re-consider from before: $\frac{\partial p(r,t)}{\partial t} = -\frac{\partial}{\partial r}\alpha_1(r) \cdot p(r,t) + \frac{1}{2}\frac{\partial^2}{\partial r^2}\alpha_2(r) \cdot p(r,t)$. Let us be interested only in: $\frac{\partial p(r,t)}{\partial t} = -\frac{\partial}{\partial r}\alpha_1(r) \cdot p(r,t)$, and we define: $p(r,t) = \delta[r - r(t)]$, which indicates the probability is managed by the Dirac-delta function. In other words, the path followed by r is deterministic. In other words, this means, that the path of the mean composite fuzzy preferences, $\bar{P}_C$, is deterministic and in fact the mean composite preferences are the mean of an infinitely high peaked Dirac-delta function. Now, we want to show that: $\frac{\partial p(r,t)}{\partial t} = -\frac{d}{dr}(f(r(t)) \cdot p)$. The proof is immediate. Simply consider that as we need to find an expression for $\frac{\partial p(r,t)}{\partial t}$, and as we know that $p(r,t) = \delta[r - r(t)]$ then everything boils down to using the chain rule: $\frac{dp}{dr(t)} \cdot \frac{dr(t)}{dt}[\delta[r - r(t)]] = -\frac{dp}{dr(t)}\delta[r - r(t)] \cdot \frac{dr(t)}{dt}$ and substituting $\frac{dr(t)}{dt}$ with $\frac{\partial r}{\partial t} = f(r(t))$ yields then the needed result.[6] Finally, we obtain:

$$\frac{\partial p(r,t)}{\partial t} = -\frac{d}{dr}(f(r(t)) \cdot p) = -\frac{d}{dr}(\alpha_1(r) \cdot p) \Rightarrow \alpha_1(r) = f(r(t)) \Leftrightarrow \frac{\partial r}{\partial t} = \alpha_1(r)$$

and hence it follows then:

$$\frac{\partial r}{\partial t} = \alpha_1(r) = \widehat{t}\,(r,r+1) - \widehat{t}\,(r,r-1).$$

□

[5] Remark that $f(\cdot)$ indicates the factorial of the argument. This equality can be found in general format in [7].

[6] By definition, $\delta[r - r(t)] = 1$ if $r = r(t)$.

8.5 Solution of $\frac{\partial r}{\partial t} = \widehat{t}\,(r, r+1) - \widehat{t}\,(r, r-1)$

Implicit in the formulation we obtained in Proposition 8.2 is the fact that the mean composite fuzzy preference follows a deterministic path. When further building up our model, we add stochasticity around those mean values by modulating the variance of the time-dependent distribution. We claim that participants must have a learning ability. If they start out at the 'most confusing' level where: $\bar{P}_0 = 0.5$, participants, must ON AVERAGE, be less confused over time because of some learning abilities, and hence they then converge to crisp preferences.

We are now interested in solving $\frac{\partial r}{\partial t} = \widehat{t}\,(r, r+1) - \widehat{t}\,(r, r-1)$. Let us re-write: $\widehat{t}\,(r, r+1) - \widehat{t}\,(r, r-1) = \frac{t(r,r+1)}{\Delta t} - \frac{t(r,r-1)}{\Delta t}$ and this is of course true by definition. Using prior facts we can then substitute to obtain: $\frac{\partial r}{\partial t} = \frac{1-\left[\frac{r-r*+1}{2r}\right]}{\Delta t} - \frac{1-\left[\frac{r+r*+1}{2r}\right]}{\Delta t}$ and this can be simplified to: $\frac{r*}{r}\frac{1}{\Delta t}$. Hence, $\frac{\partial r}{\partial t} = \frac{r*}{r}.\frac{1}{\Delta t}$ and this simple differential equation can be solved by observing that the integrating factor is simply: $\theta(t) = e^{-\int 1/(r\Delta t)dt}$. We obtain then, after multiplication by the integrating factor: $e^{-\frac{t}{r\Delta t}}\frac{\partial r}{\partial t} - \frac{r*}{r\Delta t}e^{-\frac{t}{r\Delta t}} = 0 \Leftrightarrow \frac{\partial}{\partial t}\left(e^{-\frac{t}{r\Delta t}}r*\right) = 0$ and hence the solution becomes then: $e^{-\frac{t}{r\Delta t}}r* = C \Rightarrow r* = Ce^{\frac{t}{r\Delta t}} = Ce^{\frac{D\cdot t}{r}}$. We obtain hence an exponential function in terms of time. We can now re-work this a little more as we need to position the evolution of the mean composite fuzzy preference within $[0,1]$. The mean length of this interval is $1/2$. Hence, $\bar{r} = 1/2$. Furthermore, as at $t = 0$ we need to obtain the midpoint of the interval $[0,1]$ then: $\mathrm{C}e^{2Dt}\big|_{t=0} = 1/2 \Leftrightarrow C = 1/2$. Hence, the formulation for the exponential function for $[0,1]$ is then: $r* = 1/2e^{2Dt}$. The mirror image of this function, symmetrical around the midpoint of $[0,1]$ is then: $r** = -1/2e^{2Dt} + 1$. The requirement for obtaining a 'convergence' to a crisp preference of '1' is then for $r*$: $1/2e^{2Dt} = 1 \Leftrightarrow t = \ln 2/2D$. This same value for t will yield '0' in the function for $r**$.

Acknowledgement. We thank the seminar participants at the Fifth International Conference on Fuzzy Set Theory and Its Applications, FSTA 2000, for their constructive thoughts about this paper. Personal thanks to Dr. L. Switzer for his ongoing encouragement.

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Received May 2, 2000

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