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SEMIGROUPS OF FUZZY IMPLICATIONS

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ABSTRACT. This paper considers fuzzy implications as a special case of fuzzy relations in $[0,1]$. Under some simple assumptions the sup-min composition of fuzzy implications gives also a fuzzy implication. This leads to an examination of semigroups of fuzzy implications. Contrapositive, selfconjugate and idempotent fuzzy implications are also considered.

1. Introduction

A consideration of fuzzy implications as a special case of fuzzy relations in $[0,1]$ was introduced by Baldwin and Pilsworth [3]. They generated new fuzzy implications as sup-min composition of given ones. We are going to continue this way for the simplest definition of fuzzy implications (cf. [9], Chapter I).

Definition 1. ([1]). A function $I\colon [0,1]^2 \to [0,1]$ is called *fuzzy implication* if it is monotonic with respect to both variables (separately) and fulfils the binary implication truth table:

$$I(0,0) = I(0,1) = I(1,1) = 1\,, \qquad I(1,0) = 0\,. \tag{1}$$

The set of all fuzzy implications is denoted by FI.

By monotonicity

$$I(0,x) = I(x,1) = 1\,, \qquad \text{for} \quad x \in [0,1]\,, \tag{2}$$

I is decreasing with respect to the first variable $(I(1,0) < I(0,0))$, and I is increasing with respect to the second variable $(I(1,0) < I(1,1))$.

Example 1. The most important multivalued implications (cf. [8]) fulfil the above definition:

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$$I_{\mathrm{LK}}(x,y) = \min(1-x+y,1)\,, \qquad \text{(Łukasiewicz [12])}$$

$$I_{\mathrm{GD}}(x,y) = \begin{cases} 1 & \text{if } x \le y\,, \\ y & \text{if } x > y\,, \end{cases} \qquad \text{(Gödel [11])}$$

$$I'_{\mathrm{GD}}(x,y) = \begin{cases} 1 & \text{if } x \le y\,, \\ 1-x & \text{if } x > y\,, \end{cases}$$

$$I_{\mathrm{RC}}(x,y) = 1 - x + xy\,, \qquad \text{(Reichenbach [13])}$$

$$I_{\mathrm{DN}}(x,y) = \max(1-x,y)\,, \qquad \text{(Dienes [4])}$$

$$I_{\mathrm{GG}}(x,y) = \begin{cases} 1 & \text{if } x \le y\,, \\ \frac{y}{x} & \text{if } x > y\,, \end{cases} \qquad \text{(Goguen [10])}$$

$$I'_{\mathrm{GG}}(x,y) = \begin{cases} 1 & \text{if } x \le y\,, \\ \frac{1-x}{1-y} & \text{if } x > y\,, \end{cases}$$

$$I_{\mathrm{RS}}(x,y) = \begin{cases} 1 & \text{if } x \le y\,, \\ 0 & \text{if } x > y\,, \end{cases} \qquad \text{(Rescher [14])}$$

for $x, y \in [0,1]$.

Definition 2. (cf. [15]) The *sup-min composition* of fuzzy relations $I, J \in \mathrm{FI}$ is the relation $I \circ J$,

$$(I \circ J)(x,z) = \bigvee_{y \in [0,1]} \big(I(x,y) \wedge J(y,z)\big)\,, \qquad x, z \in [0,1]\,. \tag{3}$$

The first problem implied by the above definition is the question if the composition of fuzzy implications is a fuzzy implication.

Example 2. Using the Łukasiewicz and Dienes implications we see that

$$(I_{\mathrm{LK}} \circ I_{\mathrm{DN}})(1,0) = (I_{\mathrm{DN}} \circ I_{\mathrm{LK}})(1,0) = \bigvee_{y \in [0,1]} \big(y \wedge (1-y)\big) = 0.5 > 0\,.$$

Therefore, such composition does not fulfil (1). Using Theorems 1–5 from [3] we also have

$$(I_{\mathrm{LK}} \circ I_{\mathrm{LK}})(1,0) = 0.5\,, \qquad (I_{\mathrm{DN}} \circ I_{\mathrm{DN}})(1,0) = 0.5\,, \tag{4}$$

$$I_{\mathrm{GD}} \circ I_{\mathrm{GD}} = I_{\mathrm{GD}}\,, \qquad I_{\mathrm{RS}} \circ I_{\mathrm{RS}} = I_{\mathrm{RS}}\,. \tag{5}$$

We examine in details the composition (3) for fuzzy implications. Since this composition is increasing in $(\mathrm{FI}, \le)$, we have

Lemma 1. *The composition* (3) *of fuzzy implications is monotonic with respect to both variables.*

Lemma 2. *If* $I, J \in \mathrm{FI}$ *then*

$$(I \circ J)(0,0) = (I \circ J)(0,1) = (I \circ J)(1,1) = 1\,. \tag{6}$$

P r o o f. Let $K = I \circ J$ for $I, J \in \mathrm{FI}$. Then from (2) we have

$$\begin{aligned} K(0,0) &= \bigvee_{y\in[0,1]} \big(1 \wedge J(y,0)\big) = \bigvee_{y\in[0,1]} J(y,0) = J(0,0) = 1\,,\\ K(0,1) &= \bigvee_{y\in[0,1]} (1 \wedge 1) = 1\,,\\ K(1,1) &= \bigvee_{y\in[0,1]} \big(I(1,y) \wedge 1\big) = \bigvee_{y\in[0,1]} I(1,y) = I(1,1) = 1\,, \end{aligned}$$

which proves (6). □

In virtue of Lemmas 1, 2, from Definition 1 we have

Theorem 1. *If* $I, J \in \mathrm{FI}$, *then*

$$I \circ J \in \mathrm{FI} \iff (I \circ J)(1,0) = 0\,. \tag{7}$$

The condition (7) can be simply verified. It associates boundary values of considered implications.

Example 3. Let $I, J \in \mathrm{FI}$, $t \in (0,1)$. If $I(1,t) = J(t,0) = 0$, then

$$I(1,y) = 0 \quad \text{for} \quad y \le t\,, \qquad J(y,0) = 0 \quad \text{for} \quad y \ge t\,, \qquad y \in [0,1]\,,$$

and we get

$$(I \circ J)(1,0) = \bigvee_{y\in[0,t]} \big(0 \wedge J(y,0)\big) \vee \bigvee_{y\in[t,1]} \big(I(1,y) \wedge 0\big) = 0\,.$$

Therefore $I \circ J \in \mathrm{FI}$.

2. Semigroups

Now we look for subfamilies of FI closed under composition (3). First we put

Definition 3. An implication $I \in \mathrm{FI}$ is called *left* (*right*) *factor* if

$$I \circ J \in \mathrm{FI} \quad (J \circ I \in \mathrm{FI}) \qquad \text{for all} \quad J \in \mathrm{FI}\,. \tag{8}$$

I is a *central factor* if it is both left and right one. Families of all factors are denoted by FI_L, FI_R and FI_C, respectively.

The above properties have very simple characterization.

Lemma 3. *If* $I \in \mathrm{FI}$ *then* (9)$\Rightarrow$(10) *and* (11)$\Rightarrow$(12), *where*

$$\forall_{x\in[0,1)} I(1,x) = 0\,, \tag{9}$$

$$\forall_{J\in\mathrm{FI}} \forall_{x\in[0,1]} (I \circ J)(1,x) = J(1,x)\,, \tag{10}$$

$$\forall_{x\in(0,1]} I(x,0) = 0\,, \tag{11}$$

$$\forall_{J\in\mathrm{FI}} \forall_{x\in[0,1]} (J \circ I)(x,0) = J(x,0)\,. \tag{12}$$

P r o o f. By a direct application of (3), from (9) we get

$$(I \circ J)(1,z) = \bigvee_{y<1} \big(I(1,y) \wedge J(y,z)\big) \vee \big(I(1,1) \wedge J(1,z)\big) = J(1,z)\,,$$

and from (11) we get

$$(J \circ I)(x,0) = \big(J(x,0) \wedge I(0,0)\big) \vee \bigvee_{y>0} \big(J(x,y) \wedge I(y,0)\big) = J(x,0)$$

for $x,z \in [0,1]$. □

We shall prove that properties (9) and (11) characterize left and right factors. Namely, we have

Theorem 2. *Families of factors from Definition 3 have the form*

$$\mathrm{FI}_L = \Big\{I \in \mathrm{FI}\colon \forall_{x\in[0,1)} I(1,x) = 0\Big\}\,, \tag{13}$$

$$\mathrm{FI}_R = \Big\{I \in \mathrm{FI}\colon \forall_{x\in(0,1]} I(x,0) = 0\Big\}\,, \tag{14}$$

$$\mathrm{FI}_C = \mathrm{FI}_L \cap \mathrm{FI}_R\,. \tag{15}$$

P r o o f. Let $I, J \in \mathrm{FI}$. By Lemma 3 we see that (9) implies $(I \circ J)(1,0) = 0$ and from (11) we have $(J \circ I)(1,0) = 0$. Thus, by Theorem 1, we get (8), i.e., (9) implies $(I \in \mathrm{FI}_L)$ and (11) implies $(I \in \mathrm{FI}_R)$. Now, let us assume (8) for certain $I \in \mathrm{FI}$. Putting $J = I_{\mathrm{LK}}$ we see that

$$0 = (I \circ I_{\mathrm{LK}})(1,0) = \bigvee_{y\in[0,1)} \big(I(1,y) \wedge (1-y)\big) \vee 0$$

and therefore we get (9). Similarly,

$$0 = (I_{\mathrm{LK}} \circ I)(1,0) = 0 \vee \bigvee_{y\in(0,1]} \big(y \wedge I(y,0)\big)$$

implies (11). Thus, conditions (9) and (11) are necessary. So we have representations (13) and (14). Finally, by Definition 3, we get (15). □

A comparison of (1) with (13) and (14) leads to

COROLLARY 1. *All implications from families* (13)–(15) *are discontinuous.*

EXAMPLE 4. For implications from Example 1 we have

$$I_{\mathrm{GD}}\,, I_{\mathrm{GG}} \in \mathrm{FI}_{\mathrm{R}}\,, \qquad I'_{\mathrm{GD}}, I'_{\mathrm{GG}} \in \mathrm{FI}_{\mathrm{L}}\,, \qquad I_{\mathrm{RS}} \in \mathrm{FI}_{\mathrm{C}}\,, \tag{16}$$
$$I_{\mathrm{LK}}\,, I_{\mathrm{RC}}\,, I_{\mathrm{DN}} \notin \mathrm{FI}_{\mathrm{L}} \cup \mathrm{FI}_{\mathrm{R}}\,, \qquad I_{\mathrm{GD}}\,, I_{\mathrm{GG}} \notin \mathrm{FI}_{\mathrm{L}}\,, \qquad I'_{\mathrm{GD}}, I'_{\mathrm{GG}} \notin \mathrm{FI}_{\mathrm{R}} \tag{17}$$

because for any $x \in [0,1]$ we get

$$I_{\mathrm{LK}}(x,0) = I_{\mathrm{RC}}(x,0) = I_{\mathrm{DN}}(x,0) = I'_{\mathrm{GD}}(x,0) = I'_{\mathrm{GG}}(x,0) = 1 - x\,,$$
$$I_{\mathrm{LK}}(1,x) = I_{\mathrm{RC}}(1,x) = I_{\mathrm{DN}}(1,x) = I_{\mathrm{GD}}(1,x) = I_{\mathrm{GG}}(1,x) = x\,.$$

Directly, from Lemma 3, we get

LEMMA 4. *Families* (13)–(15) *are closed under composition* (3).

In FI we use partial order of real functions

$$\underset{I,J\in\mathrm{FI}}{\forall}\Big[(I \le J) \iff \underset{x,y\in[0,1]}{\forall} I(x,y) \le J(x,y)\Big]\,. \tag{18}$$

THEOREM 3. $(\mathrm{FI}_{\mathrm{L}}, \circ, \le)$, $(\mathrm{FI}_{\mathrm{R}}, \circ, \le)$, $(\mathrm{FI}_{\mathrm{C}}, \circ, \le)$ *are ordered semigroups of fuzzy implications. They are also complete distributive lattices of fuzzy implications.*

Proof. Since the composition (3) is associative (cf. [15] or [5], Theorem 3.1), then the first part is a direct consequence of Lemma 1 and Lemma 4. By [2] the family $(\mathrm{FI}, \le)$ is a complete, distributive lattice. Now, formulas (13)–(15) describe its sublattices, because the lattice operation do not change constant values. □

EXAMPLE 5. Semigroups from Theorem 3 are bounded (as complete) and we can describe their bounds:

$$\min \mathrm{FI}_{\mathrm{L}} = \min \mathrm{FI}_{\mathrm{R}} = \min \mathrm{FI}_{\mathrm{C}} = \min \mathrm{FI} = I_0\,,$$
$$\max \mathrm{FI}_{\mathrm{L}} = I_{\mathrm{L}}\,, \qquad \max \mathrm{FI}_{\mathrm{R}} = I_{\mathrm{R}}\,, \qquad \max \mathrm{FI}_{\mathrm{C}} = I_{\mathrm{C}}\,,$$

where

$$I_0(x,y) = \begin{cases} 1 & \text{if } (x=0) \text{ or } (y=1)\,, \\ 0 & \text{if } (x>0) \text{ and } (y<1)\,; \end{cases} \tag{19}$$

$$I_{\mathrm{L}}(x,y) = \begin{cases} 1 & \text{if } (x<1) \text{ or } (y=1)\,, \\ 0 & \text{if } (x=1) \text{ and } (y<1)\,; \end{cases} \tag{20}$$

$$I_{\mathrm{R}}(x,y) = \begin{cases} 1 & \text{if } (x=0) \text{ or } (y>0)\,, \\ 0 & \text{if } (x>0) \text{ and } (y=0)\,; \end{cases} \tag{21}$$

$$I_{\mathrm{C}}(x,y) = \min(I_{\mathrm{L}}\,, I_{\mathrm{R}}) \tag{22}$$

for $x, y \in [0,1]$.

Lemma 5. *I_{RS} is the neutral element of the composition in* FI.

P r o o f . For arbitrary $I \in \mathrm{FI}$ we have

$$(I \circ I_{\mathrm{RS}})(x,z) = \bigvee_{y \le z} I(x,y) = I(x,z)\,,$$
$$(I_{\mathrm{RS}} \circ I)(x,z) = \bigvee_{y \ge x} I(y,z) = I(x,z)\,,$$

for $x, z \in [0,1]$, which proves that

$$I \circ I_{\mathrm{RS}} = I_{\mathrm{RS}} \circ I = I \qquad \text{for} \quad I \in \mathrm{FI}\,. \tag{23}$$

□

Lemma 6. *For $I \in \mathrm{FI}$ we have*

$$I_0 \circ I = I_0 \Longleftrightarrow I \in \mathrm{FI}_{\mathrm{L}}\,, \tag{24}$$
$$I \circ I_0 = I_0 \Longleftrightarrow I \in \mathrm{FI}_{\mathrm{R}}\,. \tag{25}$$

P r o o f . Let $I \in \mathrm{FI}$, $x \in (0,1]$, $z \in [0,1)$. From (3) we have

$$(I_0 \circ I)(x,z) = \bigvee_{y<1} I_0(x,y) \wedge I(y,z) \vee I_0(x,1) \wedge I(1,z) = I(1,z)\,,$$
$$(I \circ I_0)(x,z) = I(x,0) \wedge I_0(0,z) \vee \bigvee_{y>0} I(x,y) \wedge I_0(y,z) = I(x,0)\,.$$

Thus, by Theorem 2,

$$I_0 \circ I = I_0 \Longleftrightarrow \underset{z<1}{\forall}\, I(1,z) = 0 \Longleftrightarrow I \in \mathrm{FI}_{\mathrm{L}}\,,$$
$$I \circ I_0 = I_0 \Longleftrightarrow \underset{z>0}{\forall}\, I(x,0) = 0 \Longleftrightarrow I \in \mathrm{FI}_{\mathrm{R}}\,,$$

which proves (24) and (25). □

Directly, from Lemma 5 and Lemma 6, we get

Theorem 4. *Semigroups from Theorem 3 have the neutral element I_{RS}. Moreover, I_0 is the zero element in FI_{C}, a right zero in FI_{R}, and a left zero in FI_{L}.*

Since convex combination does not change constant values, then from Theorem 2 we get

Theorem 5. *Families* (13)–(15) *of factors are convex sets of functions.*

3. Conjugacy classes

Now we shall consider a connection between families (13)–(15) and conjugacy classes of fuzzy implications.

Definition 4. Let $\varPhi$ denote the set of all increasing bijections on $[0,1]$. Fuzzy implications $I, J \in \mathrm{FI}$ are *conjugate* iff

$$\underset{\varphi\in\varPhi}{\exists} I_\varphi = J\,, \tag{26}$$

where

$$I_\varphi(x,y) = \varphi^{-1}\Big(I\big(\varphi(x),\varphi(y)\big)\Big)\,, \qquad x,y\in[0,1]\,. \tag{27}$$

A fuzzy implication $I \in \mathrm{FI}$ is selfconjugate iff

$$\underset{\varphi\in\varPhi}{\forall} I_\varphi = I\,. \tag{28}$$

Example 6. Fuzzy implications I_{GD}, I_{RS}, I_0, I_{L}, I_{R} and I_{C} are selfconjugate (cf. [6]).

Theorem 6. *The operation* (27) *is an interior operation in families* (13)–(15).

Proof. If $I \in \mathrm{FI_L}$, $\varphi\in\varPhi$, $z\in[0,1]$ then by Theorem 2 we have

$$I_\varphi(1,z) = \varphi^{-1}\Big(I\big(\varphi(1),\varphi(z)\big)\Big) = \varphi^{-1}\Big(I\big(1,\varphi(z)\big)\Big) = \varphi^{-1}(0) = 0\,,$$

i.e., $I_\varphi \in \mathrm{FI_L}$. Similar calculation is valid for $I\in\mathrm{FI_R}$, and $I\in\mathrm{FI_C}$ fulfils both properties ($I_\varphi \in \mathrm{FI_L}\cap\mathrm{FI_R}$). □

Corollary 2. *Families* (13)–(15) *with every element I contain its conjugacy class.*

Lemma 7. *The transformation* (27) *is distributive with respect to composition* (3), *i.e.,*

$$(I\circ J)_\varphi = I_\varphi \circ J_\varphi\,, \qquad I,J\in\mathrm{FI}, \quad \varphi\in\varPhi\,. \tag{29}$$

Proof. Let I, $J\in\mathrm{FI}$, $\varphi\in\varPhi$, x, $z\in[0,1]$. We have

$$\begin{aligned}(I\circ J)_\varphi(x,z) &= \varphi^{-1}\Big(\big(I\circ J\big)(\varphi(x),\varphi(z))\Big)\\ &= \varphi^{-1}\Big(\bigvee_y I\big(\varphi(x),y\big)\wedge J\big(y,\varphi(z)\big)\Big)\\ &= \varphi^{-1}\Big(\bigvee_t I\big(\varphi(x),\varphi(t)\big)\wedge J\big(\varphi(t),\varphi(z)\big)\Big)\end{aligned}$$

$$= \bigvee_t \varphi^{-1}\Big(I\big(\varphi(x), \varphi(t)\big)\Big) \wedge \varphi^{-1}\Big(J\big(\varphi(t), \varphi(z)\big)\Big)$$
$$= \bigvee_t I_\varphi(x,t) \wedge J_\varphi(t,z) = (I_\varphi \circ J_\varphi)(x,z),$$

which proves (29). □

Theorem 7. *The composition of selfconjugate fuzzy implications is a selfconjugate function. It is a selfconjugate implication under the condition from Theorem 1.*

P r o o f. If $I, J \in \mathrm{FI}$ are selfconjugate, then directly from (28) and (29) we get

$$(I \circ J)_\varphi = I_\varphi \circ J_\varphi = I \circ J \qquad \text{for} \ \ \varphi \in \Phi,$$

i.e., $I \circ J$ is selfconjugate. □

Directly, from Lemma 7 we also get

Theorem 8. *If a fuzzy implication $I \in \mathrm{FI}$ is idempotent, i.e., $I \circ I = I$, then all conjugate implications are idempotent.*

Remark 1. The composition of conjugate fuzzy implications may be not conjugate with them (may belong to another conjugacy class).

P r o o f. Let $I = J_5$. From [7] we know that $I^2 = J_3 \neq I$ and $J_3^2 = J_3$, where

$$J_3 = I_R, \qquad J_5(x,y) = \begin{cases} 1 & \text{if } (x<1 \text{ and } y>0) \text{ or } (x=0), \\ y & \text{if } x=1, \\ 0 & \text{if } (x>0) \text{ and } (y=0). \end{cases}$$

By Theorem 8, implication I is not conjugate with I^2. As selfconjugate implications (cf. [6]) J_3 and J_5 form singleton conjugacy classes. □

4. Contrapositive implications

Here we discuss the place of contrapositive fuzzy implications in families (13)–(15).

Definition 5. (cf. [9]). A fuzzy implication $I \in \mathrm{FI}$ is called *contrapositive* iff $I' = I$, where

$$I'(x,y) = I(1-y, 1-x), \qquad x,y \in [0,1]. \tag{30}$$

Theorem 9. *For operation* (30) *we have*

$$\mathrm{FI}'_{\mathrm{L}} = \mathrm{FI}_{\mathrm{R}}\,, \quad \mathrm{FI}'_{\mathrm{R}} = \mathrm{FI}_{\mathrm{L}}\,, \quad \mathrm{FI}'_{\mathrm{C}} = \mathrm{FI}_{\mathrm{C}}\,. \tag{31}$$

Proof. Let $I \in \mathrm{FI}_{\mathrm{L}}$, $x \in [0,1]$. We calculate

$$I'(x,0) = I(1-0, 1-x) = I(1, 1-x) = 0\,,$$

and $I' \in \mathrm{FI}$ by Theorem 2. Similarly, we get the rest of (31). □

Corollary 3. *Contrapositive factors belongs to* FI_{C}.

Since $\min(I, I')$ is contrapositive for $I \in \mathrm{FI}$ (cf. [2]), then by (31) we have

Corollary 4. *If* $I \in \mathrm{FI}_L \cup \mathrm{FI}_R$, *then* $\min(I, I') \in \mathrm{FI}_C$.

Example 7. The implication $I_{\mathrm{RS}} \in \mathrm{FI}_{\mathrm{C}}$ is contrapositive. However, not every element of FI_{C} is contrapositive. Let $I = L_{\mathrm{GG}} \wedge I'_{\mathrm{GD}}$, we have $I \in \mathrm{FI}_{\mathrm{C}}$, but $I' = I'_{\mathrm{GG}} \wedge I_{\mathrm{GD}} \neq I$.

Lemma 8. *The transformation* (30) *has the property*

$$(I \circ J)' = J' \circ I' \qquad \text{for } I, J \in \mathrm{FI}\,. \tag{33}$$

Proof. Let $IJ \in \mathrm{FI}$, $x, z \in [0,1]$. We have

$$\begin{aligned} (I \circ J)'(x,z) &= (I \circ J)(1-z, 1-x) = \bigvee_y I(1-z, y) \wedge J(y, 1-x) \\ &= \bigvee_t I(1-z, 1-t) \wedge J(1-t, 1-x) \\ &= \bigvee_t I'(t,z) \wedge J'(x,t) = \bigvee_t J'(x,t) \wedge I'(t,z) \\ &= (J' \circ I')(x,z)\,, \end{aligned}$$

which proves (32). □

As an immediate consequence of the above lemma we get

Theorem 10. *The composition* (3) *of commuting contrapositive implications is contrapositive.*

Corollary 5. *Powers of contrapositive implication* $I \in \mathrm{FI}_{\mathrm{C}}$ *are contrapositive.*

Corollary 6. *If fuzzy implication* I *is idempotent, then* I' *is idempotent.*

Fuzzy implications conjugate with a contrapositive implication may not be contrapositive. However, we have

THEOREM 11. *Let*

$$\Phi_1 = \{\varphi \in \Phi \colon \varphi(1-x) = 1 - \varphi(x)\,,\ x \in [0,1]\}\,.$$

If $I \in \mathrm{FI}$ *is contrapositive and* $\varphi \in \Phi_1$, *then* I_φ *is contrapositive.*

Proof. Let $I \in \mathrm{FI}$, $\varphi \in \Phi_1$. If I is contrapositive then

$$\begin{aligned} I_\varphi(1-y, 1-x) &= \varphi^{-1}\Big(I\big(\varphi(1-y), \varphi(1-x)\big)\Big) \\ &= \varphi^{-1}\Big(I\big(1-\varphi(y), 1-\varphi(x)\big)\Big) = \varphi^{-1}\Big(I\big(\varphi(x), \varphi(y)\big)\Big) \\ &= I_\varphi(x,y)\,, \end{aligned}$$

i.e., $(I_\varphi)' = I_\varphi$. □

5. Conclusion

We see that the composition of fuzzy implications is a way of construction new implications. In particular we obtain new semigroups of implications (Theorem 3), new idempotent implications (Theorem 8), and new contrapositive implications (Theorem 10). A rich list of new fuzzy implications generated by those from Example 1 is presented in [2]. Families of conjugate fuzzy implications were intensively examined by authors in [1].

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