Tatra Mt. Math. Publ. **21** (2001), 31–50

TATRA MOUNTAINS
Mathematical Publications

ON FUZZY CLUSTERINGS WITH VARIABLE BOUNDARY CONDITIONS

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ABSTRACT. Following Zadeh's approach, several authors restricted similarity relations to fuzzy equivalence relations where the transitivity is defined by using an arbitrary fixed t-norm. As we have shown, this approach leads to a very restricted concept of fuzzy partition (fuzzy clustering), which is too special for many applications. Taking into consideration these observations, we defined so-called $\langle a, b, c\rangle$-clusterings, where a, b and c are fixed real numbers from the unit interval characterizing the disjointness, the normality, and the covering property of the given clustering, respectively. These clusterings are more general than partitions generated by fuzzy equivalence relations. But, again by applications, we have recognized that replacing the real numbers a, b and c by fuzzy sets A, B and C on the universe used (interpreted as "variable boundary conditions") is very useful and enlarges the field of applicability of this clustering concept.

1. Introduction

Let U be a non-empty set. Define $\mathbb{FP}(U) =_{\text{def}} \{F\colon F\colon U \to [0,1]\}$, where $[0,1]$ denotes the unit interval of real numbers. A set $\mathfrak{C} \subseteq \mathbb{FP}(U)$ is called fuzzy clustering on U. Let τ be a t-norm on $[0,1]$ and let S be a fuzzy τ-equivalence relation on U, i.e., S fulfills the conditions

1. $\forall x\,\big(x \in U \to S(x,x) = 1\big)$,
2. $\forall x\,\forall y\,\big(x, y \in U \to S(x,y) = S(y,x)\big)$, and
3. $\forall x\,\forall y\,\forall z\,\big(x, y, z \in U \to \tau\big(S(x,y), S(y,z)\big) \le S(x,z)\big)$.

Denote by $\lambda y\, S(x,y)$ the fuzzy set F defined for fixed $x \in U$ by

$$F(y) =_{\text{def}} S(x,y)\,,$$

where $y \in U$.

2000 Mathematics Subject Classification: 03B52, 62H30.
Keywords: fuzzy partitions, fuzzy equivalence relations, $\langle a, b, c\rangle$-clusterings, variable boundary conditions.

This research was supported by the Deutsche Forschungsgemeinschaft as a part of the Collaborative Research Center "Computational Intelligence" (531).

A fuzzy clustering $\mathfrak{C} =_{\text{def}} \{\lambda y\, S(x,y) \colon x \in U\}$ is too special for many applications because $\mathcal{P} =_{\text{def}} \{\text{KER}(C) \colon C \in \mathfrak{C}\}$ is a classical crisp partition of U where $\text{KER}(C) =_{\text{def}} \{x \colon x \in U \wedge C(x) = 1\}$ (see [9], [10], [11], [30], and [31]).

Therefore we introduced and investigated so-called a-b-c-clusterings $\mathfrak{C}$, where a, b and c are fixed real numbers of [0,1] and $\mathfrak{C}$ satisfies the following axioms (see [21]):

ax1. $\forall x\, \forall F \forall G\, \big(x \in U \wedge F, G \in \mathfrak{C} \wedge F(x) > a \wedge G(x) > a \to F = G\big)$,

ax2. $\forall F \big(F \in \mathfrak{C} \to \exists x\, (x \in U \wedge F(x) > b)\big)$, and

ax3. $\forall x\, \big(x \in U \to \exists F\, (F \in \mathfrak{C} \wedge F(x) > c)\big)$.

A variant of this axiom system is considered in [25].

But for developing algebraic foundations of fuzzy information granulation ([7], [22], [23], [24], [33] and [34]), the axiom system above is still too special. Therefore we fix so-called "boundary functions" $A, B, C \colon U \to [0,1]$.

Then we assume that $\mathfrak{C}$ satisfies the following axioms

AX1. $\forall x\, \forall F \forall G\, \big(x \in U \wedge F, G \in \mathfrak{C} \wedge \tau\big(F(x), G(x)\big) > A(x) \to F = G\big)$,

AX2. $\forall F\, \big(F \in \mathfrak{C} \to \exists x\, \big(x \in U \wedge F(x) > B(x)\big)\big)$, and

AX3. $\forall x\, \big(x \in U \to \exists F\, \big(F \in \mathfrak{C} \wedge F(x) > C(x)\big)\big)$.

In the following, we shall discuss how the clusterings satisfying the axioms AX1, AX2 and AX3 can be characterized by binary fuzzy relations analogously to the proofs in the papers [21] and [25].

2. A method for constructing fuzzy clusterings by binary fuzzy relations

For the following we fix a real function

$$\kappa \colon [0,1]^3 \to [0,1]$$

defined for arbitrary $r, s, t \in [0,1]$ as follows

Definition 2.1.

$$\kappa(r,s,t) =_{\text{def}} \begin{cases} 0 & \text{if } s \leq r\,, \\ t & \text{if } s > r\,. \end{cases}$$

Furthermore, we fix functions

$$A \colon U \to [0,1] \qquad \text{and} \qquad B \colon U \to [0,1]$$

used as "boundary conditions".

Now, for an arbitrary binary fuzzy relation S, i.e.,

$$S\colon U \times U \to [0,1]\,,$$

we define a fuzzy clustering $\mathrm{FCLUST}_A(S)$ and for an arbitrary fuzzy clustering

$$\mathfrak{C} \subseteq \mathbb{FP}(U)$$

on U we define a binary fuzzy relation $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ on U as follows

Definition 2.2.

1. $\mathrm{FCLUST}_A(S) =_{\mathrm{def}} \{\lambda y\, S(x,y)\colon x \in U \wedge S(x,x) > A(x)\}$,
2. $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y) =_{\mathrm{def}} \sup\{\kappa\big(A(x), F(x), F(y)\big) : F \in \mathfrak{C}\}$.

For characterizing special clusterings $\mathfrak{C} \subseteq \mathbb{FP}(U)$ we define

Definition 2.3.

1. $\mathfrak{C}$ is said to be A-*disjoint* on U
   $=_{\mathrm{def}} \forall x \forall F \forall G\, \big(x \in U \wedge F,\, G \in \mathfrak{C} \wedge F(x) > A(x) \wedge G(x) > A(x) \to F = G\big)$,
2. $\mathfrak{C}$ is said to be B-*normal* on U
   $=_{\mathrm{def}} \forall F\, \big(F \in \mathfrak{C} \to \exists x\, \big(x \in U \wedge F(x) > B(x)\big)\big)$,
3. $\mathrm{FCL}(U, A, B) =_{\mathrm{def}} \{\mathfrak{C}\colon \mathfrak{C} \subseteq \mathbb{FP}(U) \wedge \mathfrak{C}$ is A-disjoint and B-normal on $U\}$.

Remarks.

1. If $\tau = \min$ then a fuzzy clustering $\mathfrak{C}$ is A-disjoint on U if and only if $\mathfrak{C}$ fulfills axiom AX1. The case that τ is an arbitrary t-norm will be shortly discussed in Section 5.
2. Trivially, $\mathfrak{C}$ is B-normal on U if and only if $\mathfrak{C}$ fulfills axiom AX2.

Lemma 2.1. *If $\mathfrak{C}$ is A-disjoint on U then* $\mathrm{FCLUST}_A\big(\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\big) \subseteq \mathfrak{C}$.

P r o o f. Assume

$$G \in \mathrm{FCLUST}_A\big(\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\big)\,, \tag{1}$$

hence by definition of FCLUST_A there exists an $x_0 \in U$ such that

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x_0, x_0) > A(x_0) \qquad \text{and} \tag{2}$$

$$G = \lambda y\ \mathrm{FREL}_{\kappa,D}(\mathfrak{C})(x_0, y)\,. \tag{3}$$

From (2) by definition of $\mathrm{FREL}_{\kappa,A}$ we get that there exists an $G_0 \in \mathfrak{C}$ such that

$$\kappa\big(A(x_0), G_0(x_0), G_0(x_0)\big) > A(x_0)\,, \tag{4}$$

otherwise we would obtain the contradiction

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x_0, x_0) \le A(x_0) \tag{5}$$

to (2). From (4) we obtain

$$G_0(x_0) > A(x_0)\,. \tag{6}$$

Otherwise by Definition 2.1 we would obtain

$$\kappa\big(A(x_0), G_0(x_0), G_0(x_0)\big) = 0\,, \tag{7}$$

hence (7) contradicts to (4).

Now, from (3) we get for every $y \in U$,

$$G(y) = \sup\big\{\kappa\big(A(x_0), F(x_0), F(y)\big) : F \in \mathfrak{C}\big\}\,. \tag{8}$$

By definition of κ we have

$$\forall F\,\big(F \in \mathfrak{C} \wedge F(x_0) \leq A(x_0) \to \kappa\big(A(x_0), F(x_0), f(y)\big) = 0\big)\,, \tag{9}$$

hence

$$\begin{aligned}&\sup\big\{\kappa\big(A(x_0), F(x_0), F(y)\big) : F \in \mathfrak{C}\big\}\\ &\qquad = \sup\big\{\kappa\big(A(x_0), F(x_0), F(y)\big) : F \in \mathfrak{C} \wedge F(x_0) > A(x_0)\big\}\,.\end{aligned} \tag{10}$$

From the assumption that $\mathfrak{C}$ is A-disjoint on U and from (6) we obtain

$$\forall F\,\big(F \in \mathfrak{C} \wedge F(x_0) > A(x_0) \to F = G_0\big), \tag{11}$$

hence for arbitrary $y \in U$,

$$\begin{aligned}&\sup\big\{\kappa\big(A(x_0), F(x_0), F(y)\big) : F \in \mathfrak{C} \wedge F(x_0) > A(x_0)\big\}\\ &\qquad = \kappa\big(A(x_0), G_0(x_0), G(y)\big)\,.\end{aligned} \tag{12}$$

Because of (6) and of Definition 2.1, we have

$$\kappa\big(A(x_0), G_0(x_0), G_0(y)\big) = G_0(y)\,. \tag{13}$$

From (8), (10), (12), and (13) we get for every $y \in U$

$$G(y) = G_0(y)\,, \tag{14}$$

hence

$$G = G_0\,, \tag{15}$$

hence $G \in \mathfrak{C}$ because $G_0 \in \mathfrak{C}$ holds by construction of G_0. $\square$

Now, we are going to prove the inverse inclusion, i.e.,

$$\mathfrak{C} \subseteq \mathrm{FCLUST}_A\big(\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\big)\,.$$

To prove this we need the additional assumptions that $\mathfrak{C}$ is B-normal on U and $A \subseteq B$, i.e.,

$$\forall x\big(x \in U \to A(x) \leq B(x)\big)\,.$$

Lemma 2.2. *If*

(1) $\mathfrak{C}$ *is A-disjoint on U,*

(2) $\mathfrak{C}$ *is B-normal on U, and*

(3) $A \subseteq B$

then $\mathfrak{C} \subseteq \mathrm{FCLUST}_A\big(\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\big)$.

P r o o f. Assume

$$G \in \mathfrak{C}. \tag{1}$$

Then by definition of FCLUST_A we have to prove that there is an $x_0 \in U$ such that

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x_0, x_0) > A(x_0) \qquad \text{and} \tag{2}$$

$$G = \lambda y\ \mathrm{FREL}_{\kappa,D}(\mathfrak{C})(x_0, y). \tag{3}$$

Because $\mathfrak{C}$ is B-normal on U there exists an $x_0 \in U$ such that

$$G(x_0) > B(x_0), \tag{4}$$

hence by assumption 3,

$$G(x_0) > A(x_0). \tag{5}$$

From (5) by definition of κ we get

$$\kappa\big(A(x_0), G(x_0), G(x_0)\big) = G(x_0), \tag{6}$$

so by (5)

$$\kappa\big(A(x_0), G(x_0), G(x_0)\big) > A(x_0), \tag{7}$$

therefore because of $G \in \mathfrak{C}$,

$$\sup\big\{\kappa\big(A(x_0), F(x_0), F(x_0)\big) : F \in \mathfrak{C}\big\} > A(x_0), \tag{8}$$

i.e., (2) holds.

Now, to prove (3) we have to show that for every $y \in U$,

$$G(y) = \sup\big\{\kappa\big(D(x_0), F(x_0), F(y)\big) : F \in \mathfrak{C}\big\} \tag{9}$$

holds.

In order to prove (9) it is sufficient to show

$$G(y) \le \sup\big\{\kappa\big(A(x_0), F(x_0), F(y)\big) : F \in \mathfrak{C}\big\} \tag{10}$$

and

$$\sup\big\{\kappa\big(A(x_0), F(x_0), F(y)\big) : F \in \mathfrak{C}\big\} \le G(y). \tag{11}$$

We are going to show (10).

Obviously (10) holds if we have

$$\exists F\ \big(F \in \mathfrak{C} \wedge G(y) \le \kappa\big(A(x_0), F(x_0), F(y)\big)\big). \tag{12}$$

Put $F =_{\mathrm{def}} G$. Then by assumption we have $F \in \mathfrak{C}$. Furthermore, by (5) we have $F(x_0) > A(x_0)$, hence by definition of κ,

$$\kappa\big(A(x_0), F(x_0), F(y)\big) = G(y)\,,$$

so (12) holds.

To prove (11) it is sufficient that for every $F \in \mathfrak{C}$ the inequality

$$\kappa\big(A(x_0), F(x_0), F(y)\big) \le G(y) \tag{13}$$

holds.

Case 1: $F(x_0) \le A(x_0)$. Then by definition of κ we have

$$\kappa\big(A(x_0), F(x_0), F(y)\big) = 0\,,$$

hence (13) holds trivially.

Case 2: $F(x_0) > A(x_0)$. By definition of κ we obtain

$$\kappa\big(A(x_0), F(x_0), F(y)\big) = F(y)\,.$$

Furthermore by (5) we have

$$G(x_0) > A(x_0)\,,$$

hence by A-disjointness of U and $F, G \in \mathfrak{C}$ we get

$$F = G\,,$$

so $\kappa\big(A(x_0), F(x_0), F(y)\big) = G(y)$, therefore (13) holds. □

Theorem 2.3. *If*

1. $\mathfrak{C}$ *is A-disjoint on U,*
2. $\mathfrak{C}$ *is B-normal on U, and*
3. $A \subseteq B$

then $\mathrm{FCLUST}_A\big(\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\big) = \mathfrak{C}$.

P r o o f . Lemma 2.1 and Lemma 2.2. □

Corollary 2.4. *If $A \subseteq B$ then the mapping $\mathrm{FREL}_{\kappa,A}$ is an injection from $\mathrm{FCL}(U, A, B)$ into $\mathbb{FP}(U \times U)$. Furthermore, $\mathrm{FREL}_{\kappa,A}$ is a bijection from $\mathrm{FCL}(U, A, B)$ onto $\big\{\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\colon \mathfrak{C} \in \mathrm{FCL}(U, A, B)\big\}$, and the mapping FCLUST_A is the inversion of the bijection $\mathrm{FREL}_{\kappa,A}$.*

3. On axiomatic characterization of generated binary fuzzy relations

Now we shall tackle the problem how the class

$$\left\{\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\colon \mathfrak{C}\subseteq \mathbb{FP}(U)\wedge \mathfrak{C} \text{ is } A\text{–disjoint and } B\text{–normal on } U\right\}$$

of binary fuzzy relations S on U can be characterized by axioms for S, i.e., without using the mapping $\mathrm{FREL}_{\kappa,A}$ and the concepts of an A-disjoint and a B-normal clustering $\mathfrak{C}$ on U.

Therefore for $S\colon U\times U\to[0,1]$ we define

Definition 3.1.

1. S is said to be *left-side conditioned A-reflexive* on U $=_{\text{def}} \forall x\,\forall y\,\big(x,y\in U\wedge S(x,y)>0\to S(x,x)>A(x)\big)$,
2. S fulfills the *left-side A-replaceability condition* on U $=_{\text{def}} \forall x\,\forall y\,\forall z\,\big(x,y,z\in U\wedge S(x,y)>A(x)\to S(x,z)=S(y,z)\big)$,
3. S is said to be *conditioned B-total* on U $=_{\text{def}} \forall x\,\forall y\,\big(x,y\in U\wedge S(x,y)>0\to \exists z\,(z\in U\wedge S(x,z)>B(z))\big)$,
4. $\mathrm{FRE}(U,A,B) =_{\text{def}} \big\{S\colon S\colon U\times U\to[0,1]\wedge S \text{ fulfills the items } 1,2,3\big\}$.

Lemma 3.1. *For every $\mathfrak{C}\subseteq\mathbb{FP}(U)$, $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ is left-side conditioned A-reflexive on U.*

P r o o f. We have to prove

$$\forall x\,\forall y\,\big(x,y\in U\wedge \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y)>0\to \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,x)>A(x)\big)\,. \tag{1}$$

For $x,y\in U$ assume

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y)>0\,, \tag{2}$$

i.e., by definition of $\mathrm{FREL}_{\kappa,A}$

$$\sup\big\{\kappa\big(A(x),F(x),F(y)\big)\colon F\in\mathfrak{C}\big\}>0\,, \tag{3}$$

hence by definition of sup

$$\text{there exists an } F_0\in\mathfrak{C} \text{ such that } \kappa\big(A(x),F_0(x),F_0(y)\big)>0\,, \tag{4}$$

consequently by definition of κ,

$$F_0(x)>A(x)\,, \tag{5}$$

hence again by definition of κ

$$\kappa\big(A(x),F_0(x),F_0(x)\big)=F_0(x)\,, \tag{6}$$

so by (5)

$$\kappa\big(A(x), F_0(x), F_0(x)\big) > A(x)\,, \tag{7}$$

therefore by $F_0 \in \mathfrak{C}$ and definition of sup,

$$\sup\big\{\kappa\big(A(x), F(x), F(x)\big) : F \in \mathfrak{C}\big\} > A(x)\,, \tag{8}$$

i.e.,

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,x) > A(x)\,, \tag{9}$$

i.e., (1) holds. □

Lemma 3.2. *If $\mathfrak{C}$ is A-disjoint on U then $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ fulfills the left-side A-replaceability.*

Proof. We have to prove

$$\forall x\, \forall y\, \forall z\, \big(x,y,z \in U \wedge \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y) > A(x) \\ \rightarrow \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y) = \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(y,z)\big)\,. \tag{1}$$

Assume

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y) > A(x)\,, \tag{2}$$

i.e.,

$$\sup\big\{\kappa\big(A(x), F(x), F(y)\big) : F \in \mathfrak{C}\big\} > A(x)\,, \tag{3}$$

hence by definition of sup,

$$\text{there exists an } F_0 \in \mathfrak{C} \text{ such that } \kappa\big(A(x), F_0(x), F_0(y)\big) > A(x)\,, \tag{4}$$

therefore

$$\kappa\big(A(x), F_0(x), F_0(y)\big) > 0\,, \tag{5}$$

so by definition of κ (see Definition 2.1),

$$F_0(x) > A(x)\,, \tag{6}$$

so again by definition of κ,

$$\kappa\big(A(x), F_0(x), F_0(y)\big) = F_0(y)\,, \tag{7}$$

therefore by (4)

$$F_0(y) > A(x)\,. \tag{8}$$

By definition of $\mathrm{FREL}_{\kappa,A}$ we have

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,z) = \sup\big\{\kappa\big(A(x), F(x), F(z)\big) : F \in \mathfrak{C}\big\}\,, \tag{9}$$

hence because of $F_0 \in \mathfrak{C}$,

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,z) = \max\Big(\kappa\big(A(x), F_0(x), F_0(z)\big)\,, \\ \sup\big\{\kappa\big(A(x), F(x), F(z)\big) : F \in \mathfrak{C} \wedge F \neq F_0\big\}\Big)\,. \tag{10}$$

Now, by assumption of our lemma we have that $\mathfrak{C}$ is A-disjoint on U, consequently by (6) we get

$$\forall F\left(F \in \mathfrak{C} \wedge F \neq F_0 \rightarrow F(x) \leq A(x)\right), \tag{11}$$

hence by definition of κ

$$\forall F\left(F \in \mathfrak{C} \wedge F \neq F_0 \rightarrow \kappa\big(A(x), F(x), F(z)\big) = 0\right), \tag{12}$$

so

$$\sup\left\{\kappa\big(A(x), F(x), F(z)\big) : F \in \mathfrak{C} \wedge F \neq F_0\right\} = 0\,. \tag{13}$$

Furthermore, by (6) we obtain

$$\kappa\big(A(x), F_0(x), F_0(z)\big) = F_0(z)\,, \tag{14}$$

hence by (10), (13), and (14) we get

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,z) = F_0(z)\,. \tag{15}$$

Analogously, by using (8) we obtain

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(y,z) = F_0(z)\,, \tag{16}$$

so by (15) and (16),

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,z) = \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(y,z)\,. \tag{17}$$

$\square$

LEMMA 3.3. *If $\mathfrak{C}$ is B-normal on U then $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ is conditioned B-total on U.*

P r o o f . We have to prove

$$\begin{aligned}\forall x \forall y\,\big(x, y \in U \wedge \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y) > 0 \rightarrow\\ \exists z\,\big(z \in U \wedge \mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,z) > B(z)\big)\big)\end{aligned} \tag{1}$$

Assume for $x, y \in U$ that

$$\mathrm{FREL}_{\kappa,A}(\mathfrak{C})(x,y) > 0\,, \tag{2}$$

i.e.,

$$\sup\left\{\kappa\big(A(x), F(x), F(y)\big) : F \in \mathfrak{C}\right\} > 0\,, \tag{3}$$

hence

$$\text{there exists an } F_0 \in \mathfrak{C} \text{ such that } \kappa\big(A(x), F_0(x), F_0(y)\big) > 0\,, \tag{4}$$

so by definition of κ

$$F_0(x) > A(x)\,. \tag{5}$$

Because $\mathfrak{C}$ is B-normal on U, we get

$$\text{there exists a } z_0 \in U \text{ such that } F_0(z_0) > B(z_0)\,. \tag{6}$$

By (5) and definition of κ we have

$$\kappa\big(A(x), F_0(x), F_0(z_0)\big) = F_0(z_0)\,, \tag{7}$$

hence by (6),

$$\kappa\big(A(x), F_0(x), F_0(z_0)\big) > B(z_0)\,, \tag{8}$$

therefore because of $F_0 \in \mathfrak{C}$,

$$\sup\big\{\kappa\big(A(x), F(x), F(x_0)\big) \colon F \in \mathfrak{C}\big\} > B(z_0)\,, \tag{9}$$

so, (1) holds for $z =_{\text{def}} z_0$. □

Lemma 3.4. *If S fulfills the left-side A-replaceability condition on U then* $\mathrm{FREL}_{\kappa,A}\big(\mathrm{FCLUST}_A(S)\big) \subseteq S$.

P r o o f . We have to prove

$$\forall x\, \forall y\big(x, y \in U \to \mathrm{FREL}_{\kappa,A}\big(\mathrm{FCLUST}_A(S)\big)(x,y) \le S(x,y)\big)\,. \tag{1}$$

By definition of $\mathrm{FREL}_{\kappa,A}$ and FCLUST_A we have

$$\begin{aligned}\mathrm{FREL}_{\kappa,A}\big(\mathrm{FCLUST}_A(S)\big)(x,y) &= \sup\big\{\kappa\big(A(x), F(x), F(y)\big) \colon F \in \mathrm{FCLUST}_A(S)\big\}\\ &= \sup\big\{\kappa\big(A(x), S(z,x), S(z,y)\big) \colon z \in U \wedge S(z,z) > A(z)\big\}\,, \end{aligned}\tag{2}$$

hence

$$\begin{aligned}&\mathrm{FREL}_{\kappa,A}\big(\mathrm{FCLUST}_A(S)\big)(x,y)\\ &\qquad \le \sup\big\{\kappa\big(A(x), S(z,x), S(z,y)\big) \colon x, z \in U \wedge S(z,x) > A(x)\big\}\,. \end{aligned}\tag{3}$$

By definition of κ we have

$$\kappa\big(A(x), S(z,x), S(z,y)\big) = S(z,y) \qquad \text{if} \quad S(z,x) > A(x)\,, \tag{4}$$

so by (3),

$$\mathrm{FREL}_{\kappa,A}\big(\mathrm{FCLUST}_A(S)\big)(x,y) \le \sup\big\{S(z,y) \colon x, z \in U \wedge S(z,x) > A(x)\big\}\,. \tag{5}$$

By assumption of our lemma we have

$$S(z,x) > A(x) \to S(z,y) = S(x,y)\,, \tag{6}$$

therefore

$$\sup\big\{S(z,y) \colon x, z \in U \wedge S(z,x) > A(x)\big\} = S(x,y)\,. \tag{7}$$

Finally, (5) and (6) imply (1). □

Lemma 3.5. *If S fulfills the left-side A-replaceability on U then* $\mathrm{FCLUST}_A(S)$ *is A-disjoint on U.*

Proof. We have to prove

$$\forall z\,\forall F\,\forall G\,\bigl(z \in U \wedge F, G \in \mathrm{FCLUST}_A(S) \wedge \\ F(z) > A(z) \wedge G(z) > A(z) \to F = G\bigr). \quad (1)$$

Assume for $z \in U$,

$$F \in \mathrm{FCLUST}_A(S)\,, \quad (2)$$
$$G \in \mathrm{FCLUST}_A(S)\,, \quad (3)$$
$$F(z) > A(z)\,, \qquad \text{and} \quad (4)$$
$$G(z) > A(z)\,. \quad (5)$$

From (2) and (3) we get that there are elements $x, y \in U$ such that

$$\forall w\,\bigl(w \in U \to F(w) = S(x,w)\bigr)\,, \quad (6)$$
$$\forall w\,\bigl(w \in U \to G(w) = S(y,w)\bigr)\,, \quad (7)$$
$$S(x,z) > A(z)\,, \qquad \text{and} \quad (8)$$
$$S(y,z) > A(z)\,. \quad (9)$$

By assumption of our lemma for arbitrary $w \in U$ from (8) we get

$$S(x,w) = S(z,w) \quad (10)$$

and, furthermore, for arbitrary $w \in U$ from (9) we analogously get

$$S(y,w) = S(z,w)\,, \quad (11)$$

therefore (10) and (11) imply

$$S(x,w) = S(y,w)\,, \quad (12)$$

hence by (6) and (7),

$$F(w) = G(w)\,, \quad (13)$$

i.e., (1) holds. □

Lemma 3.6. *If S is left-side conditioned A-reflexive on U then* $S \subseteq \mathrm{FREL}_{\kappa,A}\bigl(\mathrm{FCLUST}_A(S)\bigr)$.

Proof. We have to prove

$$\forall x\,\forall y\,\bigl(x, y \in U \to S(x,y) \le \mathrm{FREL}_{\kappa,A}\bigl(\mathrm{FCLUST}_A(S)\bigr)(x,y)\bigr)\,. \quad (1)$$

If we have
$$S(x,y) = 0 \tag{2}$$
then (1) holds trivially.

So, we can assume
$$S(x,y) > 0\,, \tag{3}$$
hence by assumption of our lemma
$$S(x,x) > A(x)\,. \tag{4}$$
By definition of $\mathrm{FREL}_{\kappa,A}$ and FCLUST_A we have
$$\mathrm{FREL}_{\kappa,A}\big(\mathrm{FCLUST}_A(S)\big)(x,y) = \\ \sup\big\{\kappa\big(A(x),S(z,x),S(z,y)\big) : z \in U \wedge S(z,z) > A(z)\big\}\,. \tag{5}$$
Because of (4) we get
$$\kappa\big(A(x),S(x,x),S(x,y)\big) \in \\ \big\{\kappa\big(A(x),S(z,x),S(z,y)\big) : z \in U \wedge S(z,z) > A(z)\big\} \tag{6}$$
and
$$\kappa\big(A(x),S(x,x),S(x,y)\big) = S(x,y)\,, \tag{7}$$
hence
$$S(x,y) \in \big\{\kappa\big(A(x),S(z,x),S(z,y)\big) : z \in U \wedge S(z,z) > A(z)\big\}\,, \tag{8a}$$
so
$$S(x,y) \le \sup\big\{\kappa\big(A(x),S(z,x),S(z,y)\big) : z \in U \wedge S(z,z) > A(z)\big\}\,, \tag{9}$$
i.e., (1) holds. □

Lemma 3.7. *If S is conditioned B-total on U then $\mathrm{FCLUST}_A(S)$ is B-normal on U.*

Proof. We have to prove
$$\forall F\,\big(F \in \mathrm{FCLUST}_A(S) \to \exists x\,\big(x \in U \wedge F(x) > B(x)\big)\big)\,. \tag{1}$$
Assume
$$F \in \mathrm{FCLUST}_A(S)\,. \tag{2}$$
By definition of FCLUST_A there exists an $x_0 \in U$ such that
$$\forall y\,\big(y \in U \to F(y) = S(x_0,y)\big)\,, \qquad \text{and} \tag{3}$$
$$S(x_0,x_0) > A(x_0)\,. \tag{4}$$

From (4) we obtain

$$S(x_0, x_0) > 0\,, \tag{5}$$

so by assumption of our lemma, there exists a $z_0 \in U$ such that

$$S(x_0, z_0) > B(z_0)\,. \tag{6}$$

From (3) for $y =_{\text{def}} z_0$ we obtain

$$F(z_0) = S(x_0, z_0)\,, \tag{7}$$

hence by (6),

$$F(z_0) > B(z_0)\,, \tag{8}$$

so, by putting $x =_{\text{def}} z_0$ we get that (1) holds. □

Theorem 3.8. *If*

1. *S fulfills the left-side A-replaceability condition on U and*
2. *S is left-side conditioned A-reflexive on U*

then $\mathrm{FREL}_{\kappa,A}(\mathrm{FCLUST}_A(S)) = S$.

Proof. By Lemma 3.4 and Lemma 3.6. □

Theorem 3.9. *If*

1. *S fulfills the left-side A-replaceability condition on U and*
2. *S is conditioned B-total on U*

then $\mathrm{FCLUST}_A(S)$ *is A-disjoint and B-normal on U.*

Proof. By Lemma 3.5 and Lemma 3.7. □

Corollary 3.10. *For arbitrary fixed functions $A\colon U \to [0,1]$ and $B\colon U \to [0,1]$, FCLUST_A is an injection from $\mathrm{FRE}(U,A,B)$ into $\mathrm{FCL}(U,A,B)$.*

Corollary 3.11. *For arbitrary fixed functions $A\colon U \to [0,1]$ and $B\colon U \to [0,1]$, if $A \subseteq B$ then*

1. *$\mathrm{FREL}_{\kappa,A}$ is a bijection from $\mathrm{FCL}(U,A,B)$ onto $\mathrm{FRE}(U,A,B)$, and*
2. *FCLUST_A is the inversion of $\mathrm{FREL}_{\kappa,A}$ with respect to $\mathrm{FRE}(U,A,B)$ and $\mathrm{FCL}(U,A,B)$.*

4. Incorporating the covering property

In this section we shall incorporate the function $C\colon U \to [0,1]$ in order to characterize special fuzzy clusterings on U and special binary fuzzy relations on U.

To this end for arbitrary $\mathfrak{C} \subseteq \mathbb{FP}(U)$ and arbitrary $S\colon U \times U \to [0,1]$ we define

Definition 4.1.

1. $\mathfrak{C}$ is said to be a C-*covering* of U
   $=_{\text{def}} \forall x \left(x \in U \to \exists F \left(F \in \mathfrak{C} \wedge F(x) > C(x)\right)\right),$
2. $\mathrm{FCL}(U, A, B, C)$
   $=_{\text{def}} \left\{\mathfrak{C}\colon \mathfrak{C} \in \mathrm{FCL}(U, A, B) \wedge \mathfrak{C} \text{ is a } C\text{-covering of } U\right\},$
3. S is said to be C-*surjective* on U
   $=_{\text{def}} \forall y \left(y \in U \to \exists x \left(x \in U \wedge S(x, y) > C(y)\right)\right),$
4. $\mathrm{FRE}(U, A, B, C)$
   $=_{\text{def}} \left\{S\colon S \in \mathrm{FRE}(U, A, B) \wedge S \text{ is } C\text{-surjective on } U\right\}.$

Lemma 4.1. *If*

1. $\mathfrak{C}$ *is a* C-*covering of* U *and*
2. $A \subseteq C$

then $\mathrm{FREL}_{\kappa, A}(\mathfrak{C})$ *is* C-*surjective on* U.

Proof. Let

$$y \text{ be an arbitrary element from } U\,. \tag{1}$$

Then we have to prove

$$\exists x \left(x \in U \wedge \mathrm{FREL}_{\kappa, A}(\mathfrak{C})(x, y) > C(y)\right), \tag{2}$$

i.e., by definition of $\mathrm{FREL}_{\kappa, A}$ we have to show

$$\exists x \left(x \in U \wedge \sup\left\{\kappa\left(A(x), F(x), F(y)\right) : F \in \mathfrak{C}\right\} > C(y)\right). \tag{3}$$

Obviously, it is sufficient to prove

$$\exists\, x\, \exists F \left(x \in U \wedge F \in \mathfrak{C} \wedge \kappa\left(A(x), F(x), F(y)\right) > C(y)\right). \tag{4}$$

Now, by (1) and assumption 1 of our lemma we have that

$$\exists F_0\left(F_0 \in \mathfrak{C} \wedge F_0(y) > C(y)\right). \tag{5}$$

Because of assumption 2 of our lemma we have

$$\forall z\left(z \in U \to C(z) \geq A(z)\right), \tag{6}$$

hence by (5)

$$F_0(y) > A(y)\,, \tag{7}$$

so by definition of κ,

$$\kappa\left(A(y), F_0(y), F_0(y)\right) = F(y)\,, \tag{8}$$

therefore by (5)

$$\kappa\left(A(y), F_0(y), F_0(y)\right) > C(y)\,. \tag{9}$$

Now, in order to prove (4), we put

$$x =_{\text{def}} y \qquad \text{and} \qquad F =_{\text{def}} F_0\,. \tag{10}$$

□

Lemma 4.2. *If S is a C-surjective binary fuzzy relation on U then $\mathrm{FCLUST}_A(S)$ is a C-covering of U.*

P r o o f . Assume $y \in U$. Then by assumption of our lemma

$$\exists x \left(x \in U \wedge S(x,y) > C(y) \right) . \tag{1}$$

Define

$$F =_{\mathrm{def}} \lambda y \, S(x,y) \, , \tag{2}$$

hence

$$S(x,y) = F(y) > C(y) \, , \tag{3}$$

i.e., Lemma 4.2 holds. □

Theorem 4.3. *If*

1. *$\mathfrak{C}$ is A-disjoint on U,*
2. *$\mathfrak{C}$ is B-normal on U,*
3. *$\mathfrak{C}$ is a C-covering of U,*
4. *$A \subseteq B$, and*
5. *$A \subseteq C$*

then

1. $\mathrm{FCLUST}_A\left(\mathrm{FREL}_{\kappa,A}(\mathfrak{C})\right) = \mathfrak{C}$,
2. $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ *is left-side conditioned A-reflexive on U,*
3. $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ *fulfills the left-side A-replaceability condition on U,*
4. $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ *is conditioned B-total on U, and*
5. $\mathrm{FREL}_{\kappa,A}(\mathfrak{C})$ *is a C-surjective binary fuzzy relation on U.*

P r o o f . Theorem 2.3, Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 4.1. □

Corollary 4.4. *For every $A \colon U \to [0,1]$, $B \colon U \to [0,1]$, $C \colon U \to [0,1]$, if $A \subseteq B$ and $A \subseteq C$ then $\mathrm{FREL}_{\kappa,A}$ is an injection from $\mathrm{FCL}(U,A,B,C)$ into $\mathrm{FRE}(U,A,B,C)$.*

Theorem 4.5. *For every $A \colon U \to [0,1]$, $B \colon U \to [0,1]$, and $S \colon U \times U \to [0,1]$, if*

1. *S is left-side conditioned A-reflexive on U,*
2. *S fulfills the left-side A-replaceability condition on U,*
3. *S is conditioned B-total on U, and*
4. *S is C-surjective on U*

then

1. $\mathrm{FREL}_{\kappa,A}(\mathrm{FCLUST}_A(S)) = S$,
2. $\mathrm{FCLUST}_A(S)$ *is A-disjoint on U,*
3. $\mathrm{FCLUST}_A(S)$ *is B-normal on U, and*
4. $\mathrm{FCLUST}_A(S)$ *is a C-covering of U.*

P r o o f . Theorem 3.8, Lemma 3.5, Lemma 3.7 and Lemma 4.2. □

Corollary 4.6. *For every fixed function $A\colon U \to [0,1]$, $B\colon U \to [0,1]$ and $C\colon U \to [0,1]$, if $A \subseteq B$ and $A \subseteq C$ then*

1. $\mathrm{FREL}_{\kappa,A}$ *is a bijection from* $\mathrm{FCL}(U,A,B,C)$ *onto* $\mathrm{FRE}(U,A,B,C)$ *and*
2. FCLUST_A *is the inversion of* $\mathrm{FREL}_{\kappa,A}$ *with respect to* $\mathrm{FRE}(U,A,B,C)$ *and* $\mathrm{FCL}(U,A,B,C)$.

5. Conclusions and further investigations

For an arbitrary fuzzy clustering $\mathfrak{C} \subseteq \mathbb{FP}(U)$ we have the following

Proposition 5.1. *$\mathfrak{C}$ is A-disjoint on U if and only if $\mathfrak{C}$ fulfills the following condition*

$$\forall x\,\forall F\,\forall G\,\big(x \in U \wedge F, G \in \mathfrak{C} \wedge \min\big(F(x), G(x)\big) > A(x) \to F = G\big)\,.$$

Now, we fix an arbitrary function

$$\delta\colon [0,1] \times [0,1] \to [0,1]\,,$$

and define

Definition 5.1. $\mathfrak{C}$ is said to be δ-A-disjoint on U
$=_{\mathrm{def}} \forall x\,\forall F\,\forall G\,\big(x \in U \wedge F, G \in \mathfrak{C} \wedge \delta\big(F(x), G(x)\big) > A(x) \to F = G\big)$.

Proposition 5.2. *If*

1. $\forall r\big(r \in [0,1] \to \delta(r,1) = r \wedge \delta(1,r) = r\big)$,
2. *δ is monotone, i.e.,* $\forall r\,\forall r'\,\forall s\,\forall s'\,\big(r, r', s, s' \in [0,1] \wedge r \le r' \wedge s \le s' \to \delta(r,s) \le \delta(r',s')\big)$, *and*
3. *$\mathfrak{C}$ is A-disjoint on U*

then $\mathfrak{C}$ is δ-A-disjoint on U.

P r o o f . Obviously, it is sufficient to prove

$$\forall r\,\forall s\,\forall t\,\big(r, s, t \in [0,1] \wedge \delta(r,s) > t \to r > t \wedge s > t\big)\,. \tag{1}$$

In order to show (1) we assume

$$r \leq t \ \vee \ s \leq t\,. \tag{2}$$

Case 1: $r \leq t$.
Then by assumption 2 of our proposition we obtain

$$\delta(r,s) \leq \delta(t,s) \leq \delta(t,1)\,, \tag{3}$$

furthermore, by assumption 1,

$$\delta(t,1) = t\,, \tag{4}$$

hence by (3) and (4) we get the contradiction $\delta(r,s) \leq t$.

Case 2: $s \leq t$.
Analogously. □

Corollary 5.3. *If δ is a t-norm and $\mathfrak{C}$ is A-disjoint, then $\mathfrak{C}$ is δ-A-disjoint, i.e., $\mathfrak{C}$ fulfills axiom AX1.*

Concluding remarks.

1. If we try to generalize the investigations and results of the Sections 2, 3, and 4 to δ-A-disjoint clusterings then we are faced with serious difficulties. In a following paper we shall discuss this complex of problems in detail.

2. For some applications, for instance, to granulation theory, we have to modify the Definitions 2.1, 2.2, 2.3, 3.1, 4.1, and 5.1 as follows (see [25]).

Definition 2.1'.

$$\kappa'(r,s,t) =_{\text{def}} \begin{cases} 0 & \text{if } s < r\,, \\ t & \text{if } s \geq r\,. \end{cases}$$

Definition 2.2'.

1. $\text{FCLUST'}_A(S) =_{\text{def}} \{\lambda y\, S(x,y) \colon x \in U \wedge S(x,x) \geq A(x)\}$,
2. $\text{FREL}'_{\kappa',A}(\mathfrak{C})(x,y) =_{\text{def}} \sup\{\kappa'(A(x),F(x),F(y)) : F \in \mathfrak{C}\}$.

Definition 2.3'.

1. $\mathfrak{C}$ is said to be strongly A-disjoint on U
   $=_{\text{def}} \forall x\, \forall F \forall G\, (x \in U \wedge F,G \in \mathfrak{C} \wedge F(x) \geq A(x) \wedge G(x) \geq A(x) \rightarrow F = G)$,
2. $\mathfrak{C}$ is said to be weakly B-normal on U
   $=_{\text{def}} \forall F(F \in \mathfrak{C} \rightarrow \exists x\, (x \in U \wedge F(x) \geq B(x)))$,
3. $\text{FCL}'(U,A,B) =_{\text{def}} \{\mathfrak{C} \colon \mathfrak{C} \subseteq \mathbb{FP}(U) \wedge \mathfrak{C}$ is strongly A-disjoint and weakly B-normal on $U\}$.

Definition 3.1'.

1. S is said to be left-side conditioned weakly A-reflexive on U
   $=_{\text{def}} \forall x\, \forall y\, \big(x, y \in U \wedge S(x,y) > 0 \to S(x,x) \geq A(x)\big)$,
2. S fulfills the strong left-side A-replaceability condition on U
   $=_{\text{def}} \forall x\, \forall y\, \forall z\, \big(x, y, z \in U \wedge S(x,y) \geq A(x) \to S(x,z) = S(y,z)\big)$,
3. S is said to be conditioned weakly B-total on U
   $=_{\text{def}} \forall x\, \forall y\, \big(x, y \in U \wedge S(x,y) > 0 \to \exists z\, \big(z \in U \wedge S(x,z) \geq B(z)\big)\big)$,
4. $\mathrm{FRE}'(U, A, B) =_{\text{def}} \big\{S \colon S \colon U \times U \to [0,1] \wedge S$ fulfills the items 1, 2, 3$\big\}$.

Definition 4.1'.

1. $\mathfrak{C}$ is said to be a weak C-covering of U
   $=_{\text{def}} \forall x\, \big(x \in U \to \exists F\, \big(F \in \mathfrak{C} \wedge F(x) \geq C(x)\big)\big)$,
2. $\mathrm{FCL}'(U, A, B, C) =_{\text{def}} \big\{\mathfrak{C} \colon \mathfrak{C} \in \mathrm{FCL}'(U, A, B) \wedge \mathfrak{C}$ is a weak C-covering of $U\big\}$,
3. S is said to be weakly C-surjective on U
   $=_{\text{def}} \forall y\, \big(y \in U \to \exists x\, \big(x \in U \wedge S(x,y) \geq C(y)\big)\big)$,
4. $\mathrm{FRE}'(U, A, B, C)$
   $=_{\text{def}} \big\{S \colon S \in \mathrm{FRE}'(U, A, B) \wedge S$ is weakly C-surjective on $U\big\}$.

Definition 5.1'. $\mathfrak{C}$ is said to be strongly δ-A-disjoint on U
$=_{\text{def}} \forall x\, \forall F\, \forall G\, \big(x \in U \wedge F, G \in \mathfrak{C} \wedge \delta\big(F(x), G(x)\big) \geq A(x) \to F = G\big)$.

This modified approach will be investigated in a forthcoming second paper.

Acknowledgments. The author would like to thank Claus-Peter Alberts for his help in preparing the manuscript.

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Received April 19, 2000

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