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WEAKLY ORDERED PARTIAL COMMUTATIVE GROUP OF SELF-ADJOINT LINEAR OPERATORS DENSELY DEFINED ON HILBERT SPACE

Jiří Janda

ABSTRACT. We continue in a direction of describing an algebraic structure of linear operators on infinite-dimensional complex Hilbert space $\mathcal{H}$. In [Paseka, J.– –Janda, J.: *More on $\mathcal{PT}$-symmetry in (generalized) effect algebras and partial groups*, Acta Polytech. **51** (2011), 65–72] there is introduced the notion of a weakly ordered partial commutative group and showed that linear operators on $\mathcal{H}$ with restricted addition possess this structure. In our work, we are investigating the set of self-adjoint linear operators on $\mathcal{H}$ showing that with more restricted addition it also has the structure of a weakly ordered partial commutative group.

Introduction

The notion of an effect algebra was presented by Foulis and Bennett in [4]. The definition was motivated by an attempt to give an algebraic description of positive self-adjoint linear operators between zero and identity operator in a complex Hilbert space $\mathcal{H}$. The notion of generalized effect algebra extends these ideas to unbounded sets of positive linear operators. Recently some new examples of sets of positive linear operators which have the structure of a generalized effect algebra were given by Polakovič and Riečanová in [8], Pulmannová, Riečanová and Zajac in [9] and Paseka and Riečanová [7]. It naturally raises a question about the structure of sets of not only positive linear operators. In [5] Paseka started to investigate a partially ordered commutative group of operators with fixed domain. In [6] Paseka and Janda presented a structure of weakly ordered partial commutative group (shortly a wop-group). They also showed that the set of all linear operators on complex Hilbert space $\mathcal{H}$

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with the usual sum, which is for unbounded operators restricted to the same domain (partial operation $\oplus_{\mathcal{D}}$), possess this structure.

In our work, we continue in this direction and investigate the set of self-adjoint linear operators on infinite dimensional complex space $\mathcal{H}$. This set does not have the structure of a wop-group with originally presented partial operation $\oplus_{\mathcal{D}}$. However, Paseka and Riečanová in [7] defined a partial operation $\oplus_{\mathbf{v}}$ for positive operators as a usual sum, which exists between two unbounded operators only if they have linear combinations which are bounded (and have the same domain). Motivated by this partial operation, we define an operation $\oplus_{\mathbf{u}}$ in a similar way. We prove that the set of self-adjoint linear operators with $\oplus_{\mathbf{u}}$ has the structure of a wop-group. In the second part of this article, we consider the properties of $\oplus_{\mathbf{u}}$, pasting theorems for this operation and relation of presented wop-groups to generalized effect algebras. We show that our operation $\oplus_{\mathbf{u}}$ restricted on positive linear operators $\mathcal{V}(\mathcal{H})$ is the same as the original operation $\oplus_{\mathbf{v}}$ presented in [7].

1. Basic definitions and some known facts

The basic reference for this text is the book by Dvurečenskij and Pulmannová [3], where unexplained terms and notions concerning the subject can be found.

First, we review basic terminology, definitions and statements related to effect algebras and weakly ordered partial commutative groups.

Definition 1 ([4])**.** A partial algebra $(E;+,0,1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $+$ is a partially defined binary operation on E which satisfies the following conditions for any $x,y,z\in E$:

(Ei) : $x+y=y+x$ if $x+y$ is defined,

(Eii) : $(x+y)+z=x+(y+z)$ if one side is defined,

(Eiii) : for every $x\in E$ there exists a unique $y\in E$ such that $x+y=1$ (we put $x'=y$),

(Eiv) : if $1+x$ is defined, then $x=0$.

Definition 2 ([2])**.** A partial algebra $(E;+,0)$ is called a *generalized effect algebra* if $0\in E$ is a distinguished element and $+$ is a partially defined binary operation on E which satisfies the following conditions for any $x,y,z\in E$:

(GEi) : $x+y=y+x$, if one side is defined,

(GEii) : $(x+y)+z=x+(y+z)$, if one side is defined,

(GEiii) : $x+0=x$,

(GEiv) : $x+y=x+z$ implies $y=z$ (cancellation law),

(GEv) : $x+y=0$ implies $x=y=0$.

In every generalized effect algebra E the partial binary operation $\ominus$ and relation $\leq$ can be defined by

(ED): $x \leq y$ and $y \ominus x = z$ if and only if $x + z$ is defined and $x + z = y$.

Then $\leq$ is a partial order on E under which 0 is the least element of E.

Note that every effect algebra satisfies the axioms of a generalized effect algebra. On the other hand, any generalized effect algebra with a greatest element is an effect algebra.

Definition 3 ([6]). A partial algebra $(G; +, 0)$ is called a *commutative partial group* if $0 \in E$ is a distinguished element and $+$ is a partially defined binary operation on E which satisfies the following conditions for any $x, y, z \in E$:

(Gi) : $x + y = y + x$ if $x + y$ is defined,
(Gii) : $(x + y) + z = x + (y + z)$ if both sides are defined,
(Giii) : $x + 0$ is defined and $x + 0 = x$,
(Giv) : for every $x \in E$ there exists a unique $y \in E$ such that $x + y = 0$ (we put $-x = y$),
(Gv) : $x + y = x + z$ implies $y = z$.

A commutative partial group $(G; +, 0)$ is called *weakly ordered* (shortly *wop-group*) with respect to a reflexive and antisymmetric relation $\leq$ on G if $\leq$ is compatible w.r.t. partial addition, i.e., for all $x, y, z \in G$, $x \leq y$ and both $x + z$ and $y + z$ are defined implies $x + z \leq y + z$. We will denote by $Pos(G)$ the set $\{x \in G \mid x \geq 0\}$.

Note that wop-groups equipped with a total operation $+$ such that $\leq$ is an order are exactly partially ordered commutative groups.

Definition 4 ([6]). Let $(G, +, 0)$ be a commutative partial group and let S be subset of G such as:

(Si) : $0 \in S$,
(Sii) : $-x \in S$ for all $x \in S$,
(Siii) : for every $x, y \in S$ such that $x + y$ is defined also $x + y \in S$.

Then we call S a *commutative partial subgroup* of G.

Let G be a wop-group with respect to a weak order $\leq_G$ and let $\leq_S$ be a weak order on a commutative partial subgroup $S \subseteq G$. If for all $x, y \in S$ it holds: $x \leq_S y$ if and only if $x \leq_G y$, we call S a *wop-subgroup* of G.

For a commutative partial group $G = (G, +, 0)$ and a commutative partial subgroup S, let $+_{/S}$ be restriction of $+$ on S. It is clear that for a wop-group $G = (G, +, 0)$ with relation $\leq$, a commutative partial subgroup $S \subseteq G$ with relation $\leq^S$ is wop-subgroup of G if and only if $\leq^S = \leq_{/S}$, where $\leq_{/S}$ is restriction of $\leq$ on S.

Lemma 1 ([6, Lemma 1]). *Let $G = (G, +, 0)$ be a commutative partial group and let S be a commutative partial subgroup of G. Then $(S, +_{/S}, 0)$ is a commutative partial group.*

Let G be a wop-group and let S be a wop-subgroup of G. Then S is a wop--group.

Lemma 2 ([6, Lemma 2]). *Let $G = (G, +, 0)$ be a commutative partial group and S_1, S_2 commutative partial subgroups of G. Then $S = S_1 \cap S_2$ is also a commutative partial subgroup of G.*

Definition 5. Let $(G, +_1, 0)$ and $(G, +_2, 0)$ be wop-groups with respect to weak orders $\leq_1$ and $\leq_2$. We call an operation $+_1$ *the extension* of an operation $+_2$ if for any $x, y \in G$ whenever $x +_2 y$ is defined, then also $x +_1 y$ is defined and $x +_1 y = x +_2 y$. We write $+_2 \subseteq +_1$.

Similarly we call a relation $\leq_1$ *the extension* of $\leq_2$ if for any $x, y \in G$, $x \leq_2 y$ implies $x \leq_1 y$. We write $\leq_2 \subseteq \leq_1$.

Lemma 3. *Let $(G, +_1, 0)$ and $(G, +_2, 0)$ be wop-groups with respect to weak orders $\leq_1$ and $\leq_2$ such that $+_2 \subseteq +_1$ and $\leq_2 \subseteq \leq_1$. Let $S \subseteq G$ be such that $(S, +_{1/S}, 0)$ with $\leq_{1/S}$ is a wop-subgroup of $(G, +_1, 0)$. Then $(S, +_{2/S}, 0)$ with relation $\leq_{2/S}$ forms a wop-subgroup of $(G, +_2, 0)$.*

Proof. Conditions (Si) and (Sii) are clear. For (Siii), let $x, y \in S$ be such that $x +_2 y \in G$. Because $(S, +_{1/S}, 0)$ is a wop-subgroup $x +_1 y \in S$ and by $x +_1 y = x +_2 y$ also $x +_2 y \in S$.

Equality for a weak order holds since we only make its restriction on subset $S \subseteq G$. Namely for $x, y \in S$, $x \leq_2 y$ if and only if $x \leq_{2/S} y$. □

Throughout the following paper, as in [6], we assume that $\mathcal{H}$ is an infinite--dimensional complex Hilbert space, i.e., a linear space with inner product $\langle \cdot , \cdot \rangle$ which is complete in the induced metric. Recall that here for any $x, y \in \mathcal{H}$ we have $\langle x, y \rangle \in \mathbb{C}$ (the set of complex numbers) such that $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathcal{H}$. Moreover, $\overline{\langle x, y \rangle} = \langle y, x \rangle$ and finally $\langle x, x \rangle \geq 0$, where $\langle x, x \rangle = 0$ if and only if $x = 0$. The term *dimension* of $\mathcal{H}$ in the following always means the *Hilbertian dimension* defined as the cardinality of any orthonormal basis of $\mathcal{H}$ (see [1]).

Moreover, we will assume that all considered linear operators A (i.e., linear maps $A \colon D(A) \to \mathcal{H}$) have a domain $D(A)$, a linear subspace dense in $\mathcal{H}$ with respect to metric topology induced by inner product, so $\overline{D(A)} = \mathcal{H}$ (we say that A is *densely defined*). We denote by $\mathcal{D}$ the set of all dense linear subspaces of $\mathcal{H}$. By *positive linear operator* A, (denoted by $A \geq 0$) it means that $\langle Ax, x \rangle \geq 0$ for all $x \in D(A)$.

To every linear operator $A \colon D(A) \to \mathcal{H}$ with $\overline{D(A)} = \mathcal{H}$ there exists the *adjoint operator A^* of A* such that $D(A^*) = \{y \in \mathcal{H} \mid$ there exists $y^* \in \mathcal{H}$ such that $\langle y^*, x\rangle = \langle y, Ax\rangle$ for every $x \in D(A)\}$ and $A^*y = y^*$ for every $y \in D(A^*)$. If $A \subseteq A^*$, then A is called *symmetric*. When $A^* = A$, A is called *self-adjoint* (for more details see [1]).

Recall that linear operator $A\colon D(A) \to \mathcal{H}$ is called a *bounded operator* if there exists a real constant $C \geq 0$ such that $\|Ax\| \leq C\|x\|$ for all $x \in D(A)$ and hence A is an *unbounded operator* if to every $C \in \mathbb{R}$, $C \geq 0$ there exists $x_C \in D(A)$ with $\|Ax_C\| > C\|x_C\|$. By symbol "$\mathbf{0}$" we mean a null operator, it is bounded operator. The set of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For every bounded operator $A\colon D(A) \to \mathcal{H}$ densely defined on $D(A) = D \subset \mathcal{H}$ exists a unique extension B such as $D(B) = \mathcal{H}$ and $Ax = Bx$ for every $x \in D(A)$. We will denote this extension $B = A^{\mathbf{b}}$ (for more details see [1]). The usual sum $+$ of two bounded operators is always bounded, the sum of unbounded and bounded operator is always unbounded and the sum of two unbounded operators can be either bounded or unbounded (see [1]). Bounded and self-adjoint operators are called *Hermitian operators*.

For linear operators we also write $A\colon D(A) \to \mathcal{H}$ and $B\colon D(B) \to \mathcal{H}$, $A \subset B$ if and only if $D(A) \subseteq D(B)$ and $Ax = Bx$ for every $x \in D(A)$.

2. Operations on sets of linear operators

In [6] we presented the following sets with partial operation $\oplus_{\mathcal{D}}$ and proved the following theorems.

Definition 6 ([6])**.** Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \in \mathcal{D}$ dense subspace in $\mathcal{H}$. Let us define the following sets of linear operators densely defined in $\mathcal{H}$:

$$\mathcal{G}r(\mathcal{H}) = \big\{A\colon D(A) \to \mathcal{H} \mid \overline{D(A)} = \mathcal{H} \text{ and } D(A) = \mathcal{H} \text{ if } A \text{ is bounded}\big\},$$
$$\mathcal{G}r_D(\mathcal{H}) = \big\{A \in \mathcal{G}r(\mathcal{H}) \mid D(A) = D \text{ or } A \text{ is bounded}\big\}.$$

Theorem 1 ([6, Theorem 1])**.** *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $+$ be the usual sum and $\oplus_{\mathcal{D}}$ be a partial operation on $\mathcal{G}r(\mathcal{H})$ defined for $A, B \in \mathcal{G}r(\mathcal{H})$ by*

$$A \oplus_{\mathcal{D}} B = \begin{cases} A + B & \text{if } A+B \text{ is unbounded and } (D(A) = D(B) \text{ or} \\ & \text{one out of } A, B \text{ is bounded}), \\ (A+B)^{\mathbf{b}} & \text{if } A+B \text{ is bounded and } D(A) = D(B), \\ \text{undefined} & \text{otherwise} \end{cases}$$

and $\leq_{\mathcal{D}}$ be a relation on $\mathcal{G}r(\mathcal{H})$ defined for $A, B \in \mathcal{G}r(\mathcal{H})$ by $A \leq_{\mathcal{D}} B$ if and only if there is a positive linear operator $C \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathcal{D}} C = B$. Then $\big(\mathcal{G}r(\mathcal{H}); \oplus_{\mathcal{D}}, \mathbf{0}\big)$ is a wop-group with respect to $\leq_{\mathcal{D}}$.

Theorem 2 ([6, Theorem 5]). *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $D \in \mathcal{D}$. Then $\mathcal{G}r_D(\mathcal{H})$ with respect to the relation $\leq_{\mathcal{D}}^{\mathcal{G}r_D(\mathcal{H})}$ defined for $A, B \in \mathcal{G}r_D(\mathcal{H})$ by $A \leq_{\mathcal{D}}^{\mathcal{G}r_D(\mathcal{H})} B$ if and only if there exists positive $C \in \mathcal{G}r_D(\mathcal{H})$ such that $A \oplus_{\mathcal{D}} C = B$ is a wop-subgroup of $\big(\mathcal{G}r(\mathcal{H}); \oplus_{\mathcal{D}}, \mathbf{0}\big)$. Moreover, the induced operation $\oplus_{\mathcal{D}/\mathcal{G}r_D(\mathcal{H})}$ is total and the relation $\leq_{\mathcal{D}}^{\mathcal{G}r_D(\mathcal{H})}$ is a partial order on $\mathcal{G}r_D(\mathcal{H})$, hence $(\mathcal{G}r_D(\mathcal{H}); \oplus_{\mathcal{D}/\mathcal{G}r_D(\mathcal{H})}, \mathbf{0})$ is a partially ordered commutative group.*

Note that for any $A \in \mathcal{G}r_D(\mathcal{H})$, $\lambda A \in \mathcal{G}r_D(\mathcal{H})$ is defined for an arbitrary nonzero $\lambda \in \mathbb{R}$ and $D \in \mathcal{D}$ by $\lambda A(x) = \lambda\big(A(x)\big)$ for all $x \in D$. Whenever $\lambda = 0$, λA is the null operator $\mathbf{0}$ with the domain $\mathcal{H}$.

Lemma 4. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. For $A, B, C \in \mathcal{G}r(\mathcal{H})$, $\lambda \in \mathbb{R}$, following statements are equivalent:*

a) *$A, B, C \in \mathcal{G}r_D(\mathcal{H})$ for some $D \in \mathcal{D}$.*
b) *$A \oplus_{\mathcal{D}} B$, $A \oplus_{\mathcal{D}} C$ and $B \oplus_{\mathcal{D}} C$ are defined.*
c) *$(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ and $(A \oplus_{\mathcal{D}} C)$ are defined.*
d) *$A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C), (A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ and $B \oplus_{\mathcal{D}} (A \oplus_{\mathcal{D}} C)$ are defined and $A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C) = (A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C = B \oplus_{\mathcal{D}} (A \oplus_{\mathcal{D}} C)$.*

Proof. First note, that for an arbitrary $A, B \in \mathcal{G}r(\mathcal{H})$ whenever $A \oplus_{\mathcal{D}} B$ is defined, then $A, B \in \mathcal{G}r_D(\mathcal{H})$ for some $D \in \mathcal{D}$.

(b $\to$ a) Let $A, B, C \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathcal{D}} B$, $A \oplus_{\mathcal{D}} C$ and $B \oplus_{\mathcal{D}} C$ are defined. Let us suppose that A is unbounded and denote $D = D(A)$. Because $(A \oplus_{\mathcal{D}} B)$ and $(A \oplus_{\mathcal{D}} C)$ are defined, then $A, B, C \in \mathcal{G}r_D(\mathcal{H})$. Whenever A is bounded, then since $B \oplus_{\mathcal{D}} C$ is defined, $B, C \in \mathcal{G}r_D(\mathcal{H})$ for some $D \in \mathcal{D}$.

(c $\to$ b) Let $A, B, C \in \mathcal{G}r(\mathcal{H})$ such that $(A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C$ and $(A \oplus_{\mathcal{D}} C)$ are defined.

(α_1) Whenever A is unbounded, we denote $D = D(A)$. Because $(A \oplus_{\mathcal{D}} B)$ and $(A \oplus_{\mathcal{D}} C)$ are defined, then $A, B, C \in \mathcal{G}r_D(\mathcal{H})$, that is $(B \oplus_{\mathcal{D}} C)$ is defined.

(α_2) Let A be bounded. If B or C is bounded, obviously $A, B, C \in \mathcal{G}r_D(\mathcal{H})$ and $(B \oplus_{\mathcal{D}} C)$ is defined. In the case that B and C are unbounded, we have $D = D(B) = D(A + B) = D(C)$, hence $A, B, C \in \mathcal{G}r_D(\mathcal{H})$ and $(B \oplus_{\mathcal{D}} C)$ is defined.

These assertions (d $\to$ c) and (a $\to$ d) are clear. □

Now, we present a slightly weaker operation to describe a structure on the set of linear self-adjoint operators on Hilbert space $\mathcal{H}$. As we stated in the introduction, it is motivated by definition given in [7].

Definition 7. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\oplus_{\mathbf{u}}$ be a partial operation on $\mathcal{G}r(\mathcal{H})$ defined as follows. $A \oplus_{\mathbf{u}} B = A \oplus_{\mathcal{D}} B$ if and only if:

- at least one of $A, B \in \mathcal{G}r(\mathcal{H})$ is bounded or
- $A, B \in \mathcal{G}r(\mathcal{H})$ are both unbounded, $D(A) = D(B)$ and there exists a real number $\lambda_B^A \neq 0$ such that
$$A - \lambda_B^A B \text{ is bounded.}$$

It is clear, that $\oplus_{\mathcal{D}}$ is an extension of $\oplus_{\mathbf{u}}$, i.e., whenever $A \oplus_{\mathbf{u}} B$ is defined, then also $A \oplus_{\mathcal{D}} B$ is defined and $A \oplus_{\mathbf{u}} B = A \oplus_{\mathcal{D}} B$ for all $A, B \in \mathcal{G}r(\mathcal{H})$.

Lemma 5. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $A, B \in \mathcal{G}r(\mathcal{H})$ be unbounded operators, such that $A \oplus_{\mathbf{u}} B$ is defined with $A - \lambda_B^A B$ bounded operator for some $\lambda_B^A \in \mathbb{R}$. Then λ_B^A is unique.*

P r o o f. Let us assume that for unbounded $A, B \in \mathcal{G}r(\mathcal{H})$, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $A - \lambda_1 B$ and $A - \lambda_2 B$ are bounded. Then $(A - \lambda_1 B) - (A - \lambda_2 B)$ is bounded and since $D(A) = D(B) = D(\lambda_1 B) = D(\lambda_2 B)$, we have that $(-\lambda_1 + \lambda_2)B$ is bounded. And because B is unbounded, this is possible only for $\lambda_1 = \lambda_2$. □

Note that both operations $(\oplus_{\mathbf{u}}, \oplus_{\mathcal{D}})$ are always realized by the usual sum $+$ (only restricting $+$ for some pairs of operators) and whenever the result of the usual sum $+$ is a bounded operator on some dense subspace of $\mathcal{H}$, new operations extend it to the operator with domain $\mathcal{H}$ (there is a unique way how to do it).

Let $X \subseteq \mathcal{G}r(\mathcal{H})$ be an arbitrary subset of $\mathcal{G}r(\mathcal{H})$. We will denote by $\leq_{\mathbf{u}}^X$ a relation defined for every $A, B \in X$ by:

$A \leq_{\mathbf{u}}^X B$ if and only if there is a positive linear operator $C \in X$ such that $A \oplus_{\mathbf{u}} C = B$.

It is clear, that for any $A, B \in \mathcal{G}r(\mathcal{H})$, there exists at most one $C \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathcal{D}} C = B$, i.e., if there is such C, then it is unique.

Lemma 6. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. For $A, B, C \in \mathcal{G}r(\mathcal{H})$, $\lambda \in \mathbb{R}$, these statements hold:*

1. *Let $A \oplus_{\mathbf{u}} B$ be defined. Then also $A \oplus_{\mathbf{u}} \lambda B$ is defined for an arbitrary $\lambda \in \mathbb{R}$.*
2. *$\mathcal{G}r(\mathcal{H}) = \big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}}, \mathbf{0}\big)$ is a commutative partial group.*
3. *When A is unbounded and $A \oplus_{\mathbf{u}} B, A \oplus_{\mathbf{u}} C$ are defined, then $B \oplus_{\mathbf{u}} C$ is defined.*
4. *Following statements are equivalent:*
   a) *$A \oplus_{\mathbf{u}} B$, $A \oplus_{\mathbf{u}} C$ and $B \oplus_{\mathbf{u}} C$ are defined.*
   b) *$(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$ and $(A \oplus_{\mathbf{u}} C)$ are defined.*
   c) *$(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$, $A \oplus_{\mathbf{u}} (B \oplus_{\mathbf{u}} C)$ and $(A \oplus_{\mathbf{u}} C) \oplus_{\mathbf{u}} B$ are defined and $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C = A \oplus_{\mathbf{u}} (B \oplus_{\mathbf{u}} C) = (A \oplus_{\mathbf{u}} C) \oplus_{\mathbf{u}} B$.*

P r o o f. (1.) Let $A, B \in \mathcal{G}r(\mathcal{H})$ be unbounded (the remaining cases are evident) linear operators such that $A \oplus_{\mathbf{u}} B$ is defined and $\lambda \neq 0$ be an arbitrary real number ($\lambda = 0$ is clear since $0B = \mathbf{0}$). Then there exists a real nonzero $\lambda_B^A \in \mathbb{R}$ such that $A - \lambda_B^A B$ is bounded. We can define $\lambda_{\lambda B}^A = \frac{\lambda_B^A}{\lambda}$. Clearly $D(B) = D(\lambda B)$ and $A - \lambda_{\lambda B}^A \lambda B = A - \lambda_B^A B$ which is bounded.

(2.) Let $A, B, C \in \mathcal{G}r(\mathcal{H})$. Then (Gi) is valid since $A \oplus_{\mathbf{u}} B$ is defined if and only if $B \oplus_{\mathbf{u}} A$ is defined (by definition when A or B is bounded, for A and B unbounded, let $\lambda_B^A = \frac{-1}{\lambda_A^B}$) and $A \oplus_{\mathbf{u}} B = A \oplus_{\mathcal{D}} B = B \oplus_{\mathcal{D}} A = B \oplus_{\mathbf{u}} A$. Moreover, (Giii) holds because $A \oplus_{\mathbf{u}} \mathbf{0}$ is always defined and $A \oplus_{\mathbf{u}} \mathbf{0} = A$. (Giv) follows from the fact that for every $A \in \mathcal{G}r(\mathcal{H})$ there exists a unique $B \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} B = \mathbf{0}$. We put $-A = B$. The sum for bounded A, B is clearly defined. For unbounded A, B let $\lambda_B^A = -1$ and we have $A + \lambda_B^A(-A) = \mathbf{0}$, hence $A \oplus_{\mathbf{u}} B = \mathbf{0}$. (Gii) is satisfied because $\oplus_{\mathcal{D}}$ is an extension of $\oplus_{\mathbf{u}}$ and $\big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathcal{D}}, \mathbf{0}\big)$ is wop-group. Namely let $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$ and $A \oplus_{\mathbf{u}} (B \oplus_{\mathbf{u}} C)$ be defined, then $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C = (A \oplus_{\mathcal{D}} B) \oplus_{\mathcal{D}} C = A \oplus_{\mathcal{D}} (B \oplus_{\mathcal{D}} C) = A \oplus_{\mathbf{u}} (B \oplus_{\mathbf{u}} C)$. (Gv) also holds since $\oplus_{\mathcal{D}}$ is an extension of $\oplus_{\mathbf{u}}$.

(3.) Let $A, B, C \in \mathcal{G}r(\mathcal{H})$, A be unbounded and $A \oplus_{\mathbf{u}} B$, $A \oplus_{\mathbf{u}} C$ be defined. When B or C is bounded, $B \oplus_{\mathbf{u}} C$ exists. Let B and C be unbounded. Clearly $D(A) = D(B) = D(C)$, i.e., $A, B, C \in \mathcal{G}r_D(\mathcal{H})$ for some $D \in \mathcal{D}$ and $B \oplus_{\mathcal{D}} C$ is defined. There exist nonzero $\lambda_B^A, \lambda_C^A \in \mathbb{R}$ such that $A - \lambda_B^A B$ and $A - \lambda_C^A C$ are bounded operators. Then also $-\big(A - \lambda_C^A C\big)$ is bounded and since $\big(\mathcal{B}(\mathcal{H}), +, \mathbf{0}\big)$ is a group, also $\big(A - \lambda_B^A B\big) - \big(A - \lambda_C^A C\big)$ is bounded. Because $D(A) = D(B) = D\big(\lambda_B^A B\big) = D(C) = D\big(\lambda_C^A C\big)$, we can use commutativity and associativity and we have bounded $-\lambda_B^A B + \lambda_C^A C$, hence $C - \frac{\lambda_B^A}{\lambda_C^A} B$ is bounded and $B \oplus_{\mathbf{u}} C$ defined.

(4.) We start with proving (a $\to$ c).

Let $A, B, C \in \mathcal{G}r(\mathcal{H})$, $A \oplus_{\mathbf{u}} B$, $A \oplus_{\mathbf{u}} C$ and $B \oplus_{\mathbf{u}} C$ are defined. By Lemma 4 there exists $D \in \mathcal{D}$ such that $A, B, C \in \mathcal{G}r_D(\mathcal{H})$. We show that $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$ is defined. There are following cases:

(β_1) Whenever two or more operators from A, B, C are bounded, then statement holds by definition of operation $\oplus_{\mathbf{u}}$.

(β_2) Let A be bounded, B, C be unbounded. Because $B \oplus_{\mathbf{u}} C$ is defined, there exists a nonzero real number λ_C^B such that $B - \lambda_C^B C$ is bounded. Then also $A + \big(B - \lambda_C^B C\big)$ is bounded. Since $D(A + B) = D(B) = D(C) = D\big(\lambda_C^B C\big)$, we can use associativity and we have $(A + B) - \lambda_C^B C$ bounded, hence $(A + B) \oplus_{\mathbf{u}} C = (A \oplus_{\mathcal{D}} B) \oplus_{\mathbf{u}} C$ is defined. As $A \oplus_{\mathbf{u}} B$ exists, $A \oplus_{\mathcal{D}} B = A \oplus_{\mathbf{u}} B$ and $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$ is defined.

The case when B is bounded and A, C are unbounded is an analogy to the previous situation.

The situation when C is bounded and A, B are unbounded is clear.

(β_3) Let $A, B, C \in \mathcal{G}r(\mathcal{H})$ be unbounded. We have $A \oplus_{\mathbf{u}} C$ and $B \oplus_{\mathbf{u}} C$ defined. Hence there exist $\lambda_C^A, \lambda_C^B \in \mathbb{R}$ and bounded $T_1, T_2 \in \mathcal{G}r(\mathcal{H})$ such that $A - \lambda_C^A C = T_1$ and $B - \lambda_C^B C = T_2$. Since we have two bounded operators, we can sum them $\big(A - \lambda_C^A C\big) + \big(B - \lambda_C^B C\big) = T_1 + T_2$. Because $D(A) = D(B) = D(C) = D\big(\lambda_C^A C\big) = D\big(\lambda_C^B C\big)$, we can use commutativity and associativity and we get $(A+B) - \big(\lambda_C^A + \lambda_C^B\big)C = T_1 + T_2$. For $\big(\lambda_C^A + \lambda_C^B\big) \neq 0$, since C is unbounded, $A+B$ has to be unbounded and $A+B = A \oplus_{\mathcal{D}} B$. Hence $(A+B) \oplus_{\mathbf{u}} C = (A \oplus_{\mathcal{D}} B) \oplus_{\mathbf{u}} C$ is defined. When $\big(\lambda_C^A + \lambda_C^B\big) = 0$, $A+B$ is bounded and $(A \oplus_{\mathcal{D}} B) \oplus_{\mathbf{u}} C$ is also defined. As $A \oplus_{\mathbf{u}} B$ exists, $A \oplus_{\mathcal{D}} B = A \oplus_{\mathbf{u}} B$, i.e., $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$ is defined.

We proved the existence of $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$ for an arbitrary $A, B, C \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} B$, $A \oplus_{\mathbf{u}} C$ and $B \oplus_{\mathbf{u}} C$ are defined. Whenever $A \oplus_{\mathbf{u}} B$, $A \oplus_{\mathbf{u}} C$ and $B \oplus_{\mathbf{u}} C$ are defined, we get from commutativity that also $B \oplus_{\mathbf{u}} A$, $C \oplus_{\mathbf{u}} A$ and $C \oplus_{\mathbf{u}} B$ are defined. As above we can prove existence of $A \oplus_{\mathbf{u}} (B \oplus_{\mathbf{u}} C)$ and $(A \oplus_{\mathbf{u}} C) \oplus_{\mathbf{u}} B$. Equality holds because $\oplus_{\mathcal{D}}$ is an extension of $\oplus_{\mathbf{u}}$ and Lemma 4.

The direction (c $\to$ b) is clear. Now, let us show (b $\to$ a).

Let $A, B, C \in \mathcal{G}r(\mathcal{H})$, $(A \oplus_{\mathbf{u}} B) \oplus_{\mathbf{u}} C$, $A \oplus_{\mathbf{u}} C$ be defined. From Lemma 4, we have $A, B, C \in \mathcal{G}r_D(\mathcal{H})$ for some $D \in \mathcal{D}$. For B or C bounded, $B \oplus_{\mathbf{u}} C$ exists from definition. Let B and C be unbounded.

When A is unbounded, we have by (3.) that $B \oplus_{\mathbf{u}} C$ is defined.

Let us suppose that A is bounded. Then $A \oplus_{\mathbf{u}} B$ is unbounded and there exists a nonzero $\lambda_C^{(A \oplus_{\mathbf{u}} B)} \in \mathbb{R}$ such that $(A \oplus_{\mathbf{u}} B) - \lambda_C^{(A \oplus_{\mathbf{u}} B)} C = (A+B) - \lambda_C^{(A \oplus_{\mathbf{u}} B)} C$ is bounded. Since $D(A+B) = D(B) = D(C) = D\big(\lambda_C^{(A \oplus_{\mathbf{u}} B)} C\big)$, we can use associativity and $A + \big(B - \lambda_C^{(A \oplus_{\mathbf{u}} B)} C\big)$ is bounded. Hence $B - \lambda_C^{(A \oplus_{\mathbf{u}} B)} C$ is bounded and $B \oplus_{\mathbf{u}} C$ is defined. $\square$

Theorem 3. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\oplus_{\mathbf{u}}$ be a partial operation on $\mathcal{G}r(\mathcal{H})$ defined above. Then $\mathcal{G}r(\mathcal{H}) = \big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}}, \mathbf{0}\big)$ together with relation $\leq_{\mathbf{u}}$ defined for $A, B \in \mathcal{G}r(\mathcal{H})$ by $A \leq_{\mathbf{u}} B$ if and only if there exists positive $C \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} C = B$ is a wop-group.*

P r o o f. By Lemma 6 (2.), $\mathcal{G}r(\mathcal{H})$ is a commutative partial group. Let us verify that the relation $\leq_{\mathbf{u}}$ is a weak order. Reflexivity is clear and antisymmetry immediately follows from the fact that $\leq_{\mathbf{u}} \subseteq \leq_{\mathcal{D}}$. It remains to check compatibility with $\oplus_{\mathbf{u}}$.

Let $A, B, C \in \mathcal{G}r(\mathcal{H})$ such that $A \leq_{\mathbf{u}} B$. Then there exists positive $E \in \mathcal{G}r(\mathcal{H})$ such as $A \oplus_{\mathbf{u}} E = B$. Let $A \oplus_{\mathbf{u}} C$ and $B \oplus_{\mathbf{u}} C$ be defined. Then $B \oplus_{\mathbf{u}} C = (A \oplus_{\mathbf{u}} E) \oplus_{\mathbf{u}} C$ is defined. From Lemma 6 (4.) we get that $(A \oplus_{\mathbf{u}} C) \oplus_{\mathbf{u}} E$ is defined and $B \oplus_{\mathbf{u}} C = (A \oplus_{\mathbf{u}} E) \oplus_{\mathbf{u}} C = (A \oplus_{\mathbf{u}} C) \oplus_{\mathbf{u}} E$. $\square$

Theorem 4. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \in \mathcal{D}$ a dense subspace in $\mathcal{H}$. Then $\mathcal{G}r_D(\mathcal{H}) = \big(\mathcal{G}r_D(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})}, \mathbf{0}\big)$ forms together with relation $\leq_{\mathbf{u}}^{\mathcal{G}r_D(\mathcal{H})}$ a wop-subgroup of $\big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}}, \mathbf{0}\big)$.*

P r o o f. By Theorem 2 and Lemma 3, $\mathcal{G}r_D(\mathcal{H})$ is a commutative partial subgroup of $\mathcal{G}r(\mathcal{H})$. We must show that $\leq_{\mathbf{u}}^{\mathcal{G}r_D(\mathcal{H})} = \leq_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})}$.

Clearly $\leq_{\mathbf{u}}^{\mathcal{G}r_D(\mathcal{H})} \subseteq \leq_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})}$. On the other hand, let $A, B \in \mathcal{G}r_D(\mathcal{H})$ and there exists a positive $C \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} C = B$. We have $A \leq_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})} B$ and since $\leq_{\mathbf{u}} \subseteq \leq_{\mathcal{D}}$ also $A \leq_{\mathcal{D}/\mathcal{G}r_D(\mathcal{H})} B \Leftrightarrow A \leq_{\mathcal{D}}^{\mathcal{G}r_D(\mathcal{H})} B$, from which $C \in \mathcal{G}r_D(\mathcal{H})$. That is for $A, B \in \mathcal{G}r_D(\mathcal{H})$, if $A \leq_{\mathbf{u}} B$ in $\mathcal{G}r(\mathcal{H})$, then also $A \leq_{\mathbf{u}}^{\mathcal{G}r_D(\mathcal{H})} B$ in $\mathcal{G}r_D(\mathcal{H})$. $\square$

Definition 8. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let us define the following sets of linear operators densely defined on $\mathcal{H}$:

$$\begin{aligned} \mathcal{SG}r(\mathcal{H}) &= \big\{A \in \mathcal{G}r(\mathcal{H}) \mid A \subset A^*\big\}, \\ \mathcal{HG}r(\mathcal{H}) &= \big\{A \in \mathcal{G}r(\mathcal{H}) \mid A \subset A^*, D(A) = \mathcal{H}\big\}, \\ \mathcal{V}(\mathcal{H}) &= \big\{A \in \mathcal{G}r(\mathcal{H}) \mid A \geq \mathbf{0}\big\}, \end{aligned}$$

i.e., $\mathcal{SG}r(\mathcal{H})$ is the set of all symmetric operators, $\mathcal{HG}r(\mathcal{H})$ is the set of all Hermitian operators and $\mathcal{V}(\mathcal{H})$ is the set of all positive operators from $\mathcal{G}r(\mathcal{H})$.

From the definition it can be seen that $\mathcal{HG}r(\mathcal{H}) \subseteq \mathcal{SG}r(\mathcal{H}) \subseteq \mathcal{G}r(\mathcal{H})$. It is a well-known fact that every positive operator is symmetric and every positive bounded operator is both self-adjoint and Hermitian (see [1]).

Lemma 7. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Then*

$$\mathcal{SG}r(\mathcal{H}) = \big(\mathcal{SG}r(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{SG}r(\mathcal{H})}, \mathbf{0}\big)$$

forms together with relation $\leq_{\mathbf{u}}^{\mathcal{SG}r(\mathcal{H})}$ a wop-subgroup of $\big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}}, \mathbf{0}\big)$.

P r o o f. By [6, Theorem 3] and Lemma 3, $\mathcal{SG}r(\mathcal{H})$ is a commutative partial subgroup of $\mathcal{G}r(\mathcal{H})$. We must verify that $\leq_{\mathbf{u}}^{\mathcal{SG}r(\mathcal{H})} = \leq_{\mathbf{u}/\mathcal{SG}r(\mathcal{H})}$.

Clearly $\leq_{\mathbf{u}}^{\mathcal{SG}r(\mathcal{H})} \subseteq \leq_{\mathbf{u}/\mathcal{SG}r(\mathcal{H})}$. On the other hand, let $A \oplus_{\mathbf{u}} C = B$, where $A, B \in \mathcal{SG}r(\mathcal{H})$ and $C \in \mathcal{G}r(\mathcal{H})$, C positive. Since C is a positive operator, $C \in \mathcal{SG}r(\mathcal{H})$. That is, for $A, B \in \mathcal{SG}r(\mathcal{H})$, if $A \leq_{\mathbf{u}} B$ in $\mathcal{G}r(\mathcal{H})$, then also $A \leq_{\mathbf{u}}^{\mathcal{SG}r(\mathcal{H})} B$ in $\mathcal{SG}r(\mathcal{H})$. $\square$

3. Self-adjoint operators

Definition 9. Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $D \in \mathcal{D}$. Let

$$\begin{aligned} \mathcal{S}a\mathcal{G}r(\mathcal{H}) &= \big\{A \in \mathcal{G}r(\mathcal{H}) \mid A = A^*\big\}, \\ \mathcal{S}a\mathcal{G}r_D(\mathcal{H}) &= \big\{A \in \mathcal{S}a\mathcal{G}r(\mathcal{H}) \mid D(A) = D \text{ or } A \text{ is bounded}\big\}. \end{aligned}$$

Let us note that if A is self-adjoint operator on $\mathcal{H}$, then λA is self-adjoint for an arbitrary $\lambda \in \mathbb{R}$.

Theorem 5. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $\oplus_{\mathbf{u}}$ be a partial operation on $\mathcal{G}r(\mathcal{H})$ defined above. Then $\mathcal{S}a\mathcal{G}r(\mathcal{H}) = \big(\mathcal{S}a\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{S}a\mathcal{G}r(\mathcal{H})}, \mathbf{0}\big)$ forms together with the relation $\leq_{\mathbf{u}}^{\mathcal{S}a\mathcal{G}r(\mathcal{H})}$ a wop-subgroup of $\big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}}, \mathbf{0}\big)$.*

P r o o f. Clearly (Si) and (Sii) are satisfied. We will show that $\mathcal{S}a\mathcal{G}r(\mathcal{H})$ is closed under the operation $\oplus_{\mathbf{u}}$, i.e., for $A, B \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$ if $A \oplus_{\mathbf{u}} B \in \mathcal{G}r(\mathcal{H})$, then $A \oplus_{\mathbf{u}} B \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$.

(γ_1) Let $A, B \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$ be both bounded. Then also $A \oplus_{\mathbf{u}} B = (A + B) \in \mathcal{H}\mathcal{G}r(\mathcal{H}) \subseteq \mathcal{S}a\mathcal{G}r(\mathcal{H})$.

(γ_2) Let A be unbounded and B be bounded. Then according to [1, Proposition 4.1.2] $A^* + B^* = (A + B)^*$, hence $A \oplus_{\mathbf{u}} B = A^* \oplus_{\mathbf{u}} B^* = (A \oplus_{\mathbf{u}} B)^*$.

(γ_3) Now let us assume that $A, B \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$ are both unbounded and $A \oplus_{\mathbf{u}} B \in \mathcal{G}r(\mathcal{H})$. Then $D(A) = D(B)$ and there exists a nonzero real number λ_B^A and a bounded T such that $A - \lambda_B^A B = T$. Since T is a sum of two symmetric operators with the same domain, it is also symmetric (see [6, Theorem 3]). Hence T is self-adjoint. In case that $A \oplus_{\mathbf{u}} B$ is unbounded we have $A \oplus_{\mathbf{u}} B = A + B = (T + \lambda B) + B = T + (1 + \lambda)B$ which is according to (γ_2) self-adjoint. Whenever $A \oplus_{\mathbf{u}} B$ is bounded, then from [6, Theorem 3] $A \oplus_{\mathbf{u}} B = (A + B)^{\mathbf{b}}$ is symmetric, hence self-adjoint.

Now, let us show that $\leq_{\mathbf{u}}^{\mathcal{S}a\mathcal{G}r(\mathcal{H})} = \leq_{\mathbf{u}/\mathcal{S}a\mathcal{G}r(\mathcal{H})}$. Clearly $\leq_{\mathbf{u}}^{\mathcal{S}a\mathcal{G}r(\mathcal{H})} \subseteq \leq_{\mathbf{u}/\mathcal{S}a\mathcal{G}r(\mathcal{H})}$. On the other hand, we have to show, that if for any $A, B \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$ exists positive $C \in \mathcal{G}r(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} C = B$ (i.e., $A \leq_{\mathbf{u}/\mathcal{S}a\mathcal{G}r(\mathcal{H})} B$), then also $C \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$ (i.e., $A \leq_{\mathbf{u}}^{\mathcal{S}a\mathcal{G}r(\mathcal{H})} B$).

(δ_1) In case when $A, B \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$ are both bounded, C is also bounded and self-adjoint.

(δ_2) Let A be bounded and B unbounded. Then C must be unbounded and $C = B \oplus_{\mathbf{u}} (-A)$, hence C is according to (γ_2) also self-adjoint.

(δ_3) Let A be unbounded and B bounded, C must be unbounded and $C = B \oplus_{\mathbf{u}} (-A)$, so C is according to (γ_2) also self-adjoint.

(δ_4) Let A be unbounded and B unbounded.

i) for C bounded, $C = B \oplus_{\mathbf{u}} (-A)$ (defined, $\lambda_B^A = -1$) and with (γ_3) C is self-adjoint, too,

ii) for C unbounded, there exists a bounded $T \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$ such that $A - \lambda_C^A C = T$. Hence $\big(\lambda_C^A C + T\big) + C = B$, so $\big(\lambda_C^A + 1\big)C = B + (-T)$. If $\lambda_C^A = -1$, then $B = T$ which is contradiction B unbounded. Hence for $\lambda_C^A \neq -1$, C is according to (γ_2) self-adjoint. $\square$

Theorem 6. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \in \mathcal{D}$ be a dense subspace in $\mathcal{H}$. Then $\mathcal{S}a\mathcal{G}r_D(\mathcal{H}) = \big(\mathcal{S}a\mathcal{G}r_D(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{S}a\mathcal{G}r_D(\mathcal{H})}, \mathbf{0}\big)$ together with the relation $\leq_{\mathbf{u}}^{\mathcal{S}a\mathcal{G}r_D(\mathcal{H})}$ is a wop-subgroup of $\big(\mathcal{G}r_D(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})}, \mathbf{0}\big)$ and also $\big(\mathcal{S}a\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{S}a\mathcal{G}r(\mathcal{H})}, \mathbf{0}\big)$.*

Proof. Note that $\mathcal{S}a\mathcal{G}r_D(\mathcal{H}) = \mathcal{G}r_D(\mathcal{H}) \cap \mathcal{S}a\mathcal{G}r(\mathcal{H})$. By Lemma 2 $\mathcal{S}a\mathcal{G}r_D(\mathcal{H})$ is partial commutative group. We must verify that $\leq_{\mathbf{u}}^{\mathcal{S}a\mathcal{G}r_D(\mathcal{H})} = \leq_{\mathbf{u}/\mathcal{S}a\mathcal{G}r_D(\mathcal{H})}$.

Clearly $\leq_{\mathbf{u}}^{\mathcal{S}a\mathcal{G}r_D(\mathcal{H})} \subseteq \leq_{\mathbf{u}/\mathcal{S}a\mathcal{G}r_D(\mathcal{H})}$. Let $A, B \in \mathcal{S}a\mathcal{G}r_D(\mathcal{H})$ and $C \in \mathcal{G}r(\mathcal{H})$ positive such as $A \oplus_{\mathbf{u}} C = B$. Since $\mathcal{S}a\mathcal{G}r_D(\mathcal{H}) \subset \mathcal{G}r_D(\mathcal{H})$ and $\mathcal{G}r_D(\mathcal{H})$ is by Theorem 4 a wop-group, we have $A \leq_{\mathbf{u}} B \Leftrightarrow A \leq_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})} B \Leftrightarrow A \leq_{\mathbf{u}}^{\mathcal{G}r_D(\mathcal{H})} B$, i.e., $C \in \mathcal{G}r_D(\mathcal{H})$. By Theorem 5 $\mathcal{S}a\mathcal{G}r(\mathcal{H})$ is a wop-group, hence we can use the same argument for showing that $C \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$. □

4. Pasting of partially ordered commutative subgroups w.r.t. operation $\oplus_{\mathbf{u}}$

In the case of wop-group $\big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathcal{D}}, \mathbf{0}\big)$, $\mathcal{G}r(\mathcal{H})$ is a pasting of its partially ordered commutative subgroups $\big(\mathcal{G}r_D(\mathcal{H}), \oplus_{\mathcal{D}}, \mathbf{0}\big)$ [6, Theorem 7]. For the case of a wop-group $\big(\mathcal{G}r(\mathcal{H}), \oplus_{\mathbf{u}}, \mathbf{0}\big)$, $\big(\mathcal{G}r_D(\mathcal{H}), \oplus_{\mathbf{u}}, \mathbf{0}\big)$ is only a wop-group (wop--subgroup) and does not form a group. In order to find a group analogy, we use the following statement.

Lemma 8. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. For an arbitrary unbounded $A \in \mathcal{G}r(\mathcal{H})$, let us define:*

$$[A]_{\mathbf{u}} = \big\{B \in \mathcal{G}r(\mathcal{H}) \mid (A \oplus_{\mathbf{u}} B) \in \mathcal{G}r(\mathcal{H})\big\}.$$

Then $[A]_{\mathbf{u}} = \big([A]_{\mathbf{u}}, \oplus_{\mathbf{u}/[A]_{\mathbf{u}}}, \mathbf{0}\big)$ with relation $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}}$ forms a wop-subgroup of $\mathcal{G}r_D(\mathcal{H})$, where $D = D(A)$. Moreover, operation $\oplus_{\mathbf{u}/[A]_{\mathbf{u}}}$ is total on $[A]_{\mathbf{u}}$ and $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}}$ is partial order, i.e., $[A]_{\mathbf{u}}$ is partially ordered commutative group.

Proof. From the definition of $\oplus_{\mathbf{u}}$, we have $[A]_{\mathbf{u}} \subset \mathcal{G}r_D(\mathcal{H})$, where $D \in \mathcal{D}$ is given by $D = D(A)$. (Si) is clear. For (Sii), let $B \in [A]_{\mathbf{u}}$. Then $A \oplus_{\mathbf{u}} B$ exists and by Lemma 6 (1.) also $A \oplus_{\mathbf{u}} (-B)$ is defined, i.e., $-B \in [A]_{\mathbf{u}}$.

We verify (Siii). Let $B, C \in [A]_{\mathbf{u}}$. By Lemma 6 (3.) $B \oplus_{\mathbf{u}} C$ is defined and by Lemma 6 (4.) also $A \oplus_{\mathbf{u}} (B \oplus_{\mathbf{u}} C)$, hence $(B \oplus_{\mathbf{u}} C) \in [A]_{\mathbf{u}}$. We can see that $\oplus_{\mathbf{u}/[A]_{\mathbf{u}}}$ is total on $[A]_{\mathbf{u}}$ and $[A]_{\mathbf{u}}$ is a commutative group.

We show that for any unbounded $B \in [A]_{\mathbf{u}}$ holds $[A]_{\mathbf{u}} = [B]_{\mathbf{u}}$. Let $A, B \in \mathcal{G}r(\mathcal{H})$ be unbounded operators such that $B \in [A]_{\mathbf{u}}$.

$[A]_{\mathbf{u}} \subseteq [B]_{\mathbf{u}}$: Let $C \in [A]_{\mathbf{u}}$. Then because $\oplus_{\mathbf{u}/[A]_{\mathbf{u}}}$ is total on $[A]_{\mathbf{u}}$, $B \oplus_{\mathbf{u}} C$ is defined and $C \in [B]_{\mathbf{u}}$.

$[B]_{\mathbf{u}} \subseteq [A]_{\mathbf{u}}$: Let $D \in [B]_{\mathbf{u}}$. Since $B \oplus_{\mathbf{u}} A$, $B \oplus_{\mathbf{u}} D$ are defined and $\oplus_{\mathbf{u}/[B]_{\mathbf{u}}}$ is total on $[B]_{\mathbf{u}}$, also $A \oplus_{\mathbf{u}} D$ is defined and $D \in [A]_{\mathbf{u}}$. Hence $[A]_{\mathbf{u}} = [B]_{\mathbf{u}}$.

Now, let us show that $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}} = \leq_{\mathbf{u}/[A]_{\mathbf{u}}}$. Clearly $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}} \subseteq \leq_{\mathbf{u}/[A]_{\mathbf{u}}}$. On the other hand, let $E \oplus_{\mathbf{u}} C = B$ where $E, B \in [A]_{\mathbf{u}}$ and $C \in \mathcal{G}r(\mathcal{H})$, C positive. For C bounded, $C \in [A]_{\mathbf{u}}$.

Let C be unbounded, then E or B must be unbounded, too. If E is unbounded, then $[C]_{\mathbf{u}} = [E]_{\mathbf{u}} = [A]_{\mathbf{u}}$, hence $C \in [A]_{\mathbf{u}}$. If B is unbounded and E is bounded, then $B \oplus_{\mathbf{u}} C$ is defined with $\lambda_C^B = 1$. Hence $[C]_{\mathbf{u}} = [B]_{\mathbf{u}} = [A]_{\mathbf{u}}$, i.e., $C \in [A]_{\mathbf{u}}$.

The relation $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}}$ is a partial order because $\leq_{\mathbf{u}}^{\mathcal{G}r_D(\mathcal{H})}$ is a partial order and from previous $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}} = \leq_{\mathbf{u}/[A]_{\mathbf{u}}} = \leq_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})/[A]_{\mathbf{u}}} = \leq_{\mathbf{u}/[A]_{\mathbf{u}}}^{\mathcal{G}r_D(\mathcal{H})}$. □

Lemma 9. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. For an arbitrary unbounded self-adjoint $A \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$, let us define:*

$$[A]_{\mathbf{u}}^{\mathcal{S}a} = \big\{B \in \mathcal{S}a\mathcal{G}r(\mathcal{H}) \mid (A \oplus_{\mathbf{u}} B) \in \mathcal{S}a\mathcal{G}r(\mathcal{H})\big\}.$$

Then $[A]_{\mathbf{u}}^{\mathcal{S}a} = \big([A]_{\mathbf{u}}^{\mathcal{S}a}, \oplus_{\mathbf{u}/[A]_{\mathbf{u}}^{\mathcal{S}a}}, \mathbf{0}\big)$ with relation $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}^{\mathcal{S}a}}$ forms a partially ordered commutative subgroup of $\big([A]_{\mathbf{u}}, \oplus_{\mathbf{u}/[A]_{\mathbf{u}}}, \mathbf{0}\big)$.

P r o o f. Obviously $[A]_{\mathbf{u}}^{\mathcal{S}a} = [A]_{\mathbf{u}} \cap \mathcal{S}a\mathcal{G}r(\mathcal{H})$. Hence according to Lemma 2 $[A]_{\mathbf{u}}^{\mathcal{S}a}$ is a partial commutative subgroup. Since $[A]_{\mathbf{u}}$ is a commutative group, $[A]_{\mathbf{u}}^{\mathcal{S}a}$ is a commutative group, too.

Let us verify that $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}^{\mathcal{S}a}} = \leq_{\mathbf{u}/[A]_{\mathbf{u}}^{\mathcal{S}a}}$. Clearly $\leq_{\mathbf{u}}^{[A]_{\mathbf{u}}^{\mathcal{S}a}} \subseteq \leq_{\mathbf{u}/[A]_{\mathbf{u}}^{\mathcal{S}a}}$. On the other hand, let $E \oplus_{\mathbf{u}} C = B$ where $E, B \in [A]_{\mathbf{u}}^{\mathcal{S}a}$ and $C \in [A]_{\mathbf{u}}$, C is positive. Since $[A]_{\mathbf{u}}^{\mathcal{S}a} \subseteq \mathcal{S}a\mathcal{G}r(\mathcal{H})$ and $\mathcal{S}a\mathcal{G}r(\mathcal{H})$ is a wop-subgroup, $E \leq_{\mathbf{u}/\mathcal{S}a\mathcal{G}r(\mathcal{H})} B$, hence $C \in \mathcal{S}a\mathcal{G}r(\mathcal{H})$. And because $[A]_{\mathbf{u}}^{\mathcal{S}a} = [A]_{\mathbf{u}} \cap \mathcal{S}a\mathcal{G}r(\mathcal{H})$, $C \in [A]_{\mathbf{u}}^{\mathcal{S}a}$. □

Theorem 7 (Pasting theorem)**.** *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $D \in \mathcal{D}$ a dense subspace in $\mathcal{H}$. Then the wop-group $\big(\mathcal{G}r_D(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{G}r_D(\mathcal{H})}, \mathbf{0}\big)$* pastes *their partially ordered commutative subgroups $[A]_{\mathbf{u}}$, where $A \in \mathcal{G}r_D(\mathcal{H})$ is unbounded operator, together over $\mathcal{B}(\mathcal{H})$, i.e., $[A]_{\mathbf{u}} \cap [B]_{\mathbf{u}} = \mathcal{B}(\mathcal{H})$ for every pair of unbounded operators $A, B \in \mathcal{G}r_D(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} B$ is not defined, and*

$$\mathcal{G}r_D(\mathcal{H}) = \bigcup \big\{[A]_{\mathbf{u}} \mid A \textit{ is unbounded in } \mathcal{G}r_D(\mathcal{H})\big\}.$$

P r o o f. From the proof of Lemma 8, we can see that whenever there exists unbounded $C \in \mathcal{G}r_D(\mathcal{H})$ such that $C \in [A]_{\mathbf{u}}$ and $C \in [B]_{\mathbf{u}}$, then $[A]_{\mathbf{u}} = [B]_{\mathbf{u}} = [C]_{\mathbf{u}}$, i.e., for arbitrary $A, B \in \mathcal{G}r_D(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} B$ is not defined if holds $[A]_{\mathbf{u}} \cap [B]_{\mathbf{u}} = \mathcal{B}(\mathcal{H})$.

A union is clear. □

Theorem 8 (Pasting theorem for self-adjoint operators). *Let $\mathcal{H}$ be an infinite--dimensional complex Hilbert space and $D \in \mathcal{D}$ a dense subspace in $\mathcal{H}$. Then the wop-group $\mathcal{S}a\mathcal{G}r_D(\mathcal{H})$* pastes *their partially orderer commutative subgroups $[A]_{\mathbf{u}}^{\mathcal{S}a}$, where $A \in \mathcal{S}a\mathcal{G}r_D(\mathcal{H})$ is unbounded operator, together over $\mathcal{H}\mathcal{G}r(\mathcal{H})$, i.e., $[A]_{\mathbf{u}}^{\mathcal{S}a} \cap [B]_{\mathbf{u}}^{\mathcal{S}a} = \mathcal{H}\mathcal{G}r(\mathcal{H})$ for every pair of unbounded operators $A, B \in \mathcal{S}a\mathcal{G}r_D(\mathcal{H})$ such that $A \oplus_{\mathbf{u}} B$ is not defined, and*

$$\mathcal{S}a\mathcal{G}r_D(\mathcal{H}) = \bigcup \big\{ [A]_{\mathbf{u}}^{\mathcal{S}a} \mid A \text{ is unbounded in } \mathcal{S}a\mathcal{G}r_D(\mathcal{H}) \big\}.$$

Proof. Let $A, B \in \mathcal{S}a\mathcal{G}r_D(\mathcal{H})$ for some $D \in \mathcal{D}$ such that $A \oplus_{\mathbf{u}} B$ does not exists. Then $[A]_{\mathbf{u}}^{\mathcal{S}a} \cap [B]_{\mathbf{u}}^{\mathcal{S}a} = [A]_{\mathbf{u}} \cap \mathcal{S}a\mathcal{G}r(\mathcal{H}) \cap [B]_{\mathbf{u}} \cap \mathcal{S}a\mathcal{G}r(\mathcal{H}) = [A]_{\mathbf{u}} \cap [B]_{\mathbf{u}} \cap \mathcal{S}a\mathcal{G}r(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \cap \mathcal{S}a\mathcal{G}r(\mathcal{H}) = \mathcal{H}\mathcal{G}r(\mathcal{H})$.

A union is clear. □

5. Generalized effect algebras

In this section, we go back to the definition of operation $\oplus_{\mathbf{v}}$ on positive linear operators $\mathcal{V}(\mathcal{H})$ given in [7]. The following Theorem 9 tells us that operation $\oplus_{\mathbf{v}}$ is a restriction of operation $\oplus_{\mathbf{u}}$ on positive operators $\mathcal{V}(\mathcal{H})$.

Recall the statement presented in [11, Corollary 6].

Lemma 10 ([11, Corollary 6]). *Let A, B be nonnegative densely defined linear operators having the same domain D. If $A + B$ is bounded, then both A, B are bounded.*

Theorem 9. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\oplus_{\mathbf{v}}$ be defined for $A, B \in \mathcal{V}(\mathcal{H})$ by $A \oplus_{\mathbf{v}} B = A + B$ (the usual sum) if and only if*

1. *either at least one out of A, B is bounded,*
2. *or both A, B are unbounded and there exist positive bounded operators $S_B^A, T_B^A \in \mathcal{B}(\mathcal{H})$ and a positive real number $\lambda_B^A > 0$ such that*

$$A + S_B^A = \lambda_B^A B + T_B^A \quad (i.e., A \oplus_{\mathcal{D}} S_B^A = \lambda_B^A B \oplus_{\mathcal{D}} T_B^A).$$

Then $\oplus_{\mathbf{v}} = \oplus_{\mathbf{u}/\mathcal{V}(\mathcal{H})}$, i.e., $\oplus_{\mathbf{v}}$ is the same operation as restriction of operation $\oplus_{\mathbf{u}}$ on $\mathcal{V}(\mathcal{H})$.

Proof. In an unpublished proof M. Zajac showed that for $A, B \in \mathcal{V}(\mathcal{H})$, $A \oplus_{\mathbf{v}} B$ is defined if and only if there exists positive $\lambda_B^A \in \mathbb{R}$ and bounded T such that $A - \lambda_B^A B = T$. We will follow and slightly extend his proof.

$\oplus_{\mathbf{v}} \subseteq \oplus_{\mathbf{u}/\mathcal{V}(\mathcal{H})}$:

Let $A, B \in \mathcal{V}(\mathcal{H})$ and $A \oplus_{\mathbf{v}} B$ be defined. Then there exist a positive bounded $T_B^A, S_B^A \in \mathcal{V}(\mathcal{H})$ and a positive real number λ_B^A such that $A + S_B^A = \lambda_B^A B + T_B^A$. Then $A - \lambda_B^A B = T_B^A - S_B^A$, hence $A \oplus_{\mathbf{u}} B$ exists.

$\oplus_{\mathbf{u}/\mathcal{V}(\mathcal{H})} \subseteq \oplus_{\mathbf{v}}$:

Let $A, B \in \mathcal{V}(\mathcal{H})$ be unbounded, $T \in \mathcal{B}(\mathcal{H})$ be bounded and λ_B^A be a real nozero number such that $A - \lambda_B^A B = T$ (i.e., $A \oplus_{\mathbf{u}} B$ is defined).

First we show that by these assumptions λ_B^A is positive. Let suppose that λ_B^A is negative. We have positive $\bar{\lambda}_B^A = -\lambda_B^A$ and $A + \bar{\lambda}_B^A B = T$. Both A and $\bar{\lambda}_B^A B$ are positive and unbounded and their sum is positive and bounded, hence it is a contradiction with the Lemma 10.

Since A, B are positive, A, B are symmetric. Then by [6, Theorem 3] also T is symmetric and because T is bounded, T is self-adjoint. Let $m = \|T\|$. Using Schwarz inequality we have $|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = m\langle x, x\rangle$ for all $x \in \mathcal{H}$. Hence $-m\langle x, x\rangle \leq T\langle x, x\rangle \leq m\langle x, x\rangle$ for all $x \in \mathcal{H}$. Let $S_B^A = mI$ and $T_B^A = T + mI$. Clearly $S_B^A, T_B^A \in \mathcal{V}(\mathcal{H})$. And then $A + S_B^A = \big(T + \lambda_B^A B\big) + S_B^A = \lambda_B^A B + (T + mI) = \lambda_B^A B + T_B^A$ (i.e., $A \oplus_{\mathbf{v}} B$ is defined). Hence $\oplus_{\mathbf{u}/\mathcal{V}(\mathcal{H})} \subseteq \oplus_{\mathbf{v}}$. $\square$

P a s e k a and R i e č a n o v á in [7, Theorem 4] showed that $\mathcal{V}(\mathcal{H})$ with the operation $\oplus_{\mathbf{v}}$ forms a generalized effect algebra. Hence we can formulate the following corollary.

Corollary 1. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Then from [7, Theorem 4] and the previous theorem we have that*

$$\Big(Pos\big(\mathcal{G}r(\mathcal{H})\big), \oplus_{\mathbf{u}/Pos(\mathcal{G}r(\mathcal{H}))}, \mathbf{0}\Big) = \big(\mathcal{V}(\mathcal{H}), \oplus_{\mathbf{u}/\mathcal{V}(\mathcal{H})}, \mathbf{0}\big) = \big(\mathcal{V}(\mathcal{H}), \oplus_{\mathbf{v}}, \mathbf{0}\big)$$

is a generalized effect algebra. Moreover, $A \leq_{\mathbf{u}} B$ if and only if $A \leq B$, where $\leq$ is an induced order from the structure of generalized effect algebra.

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Department of Mathematics and Statistics
Faculty of Science
Masaryk University
Kotlářská 2
CZ–611-37 Brno
CZECH REPUBLIC
E-mail: 98599@mail.muni.cz.