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ON λ-ASYMPTOTICALLY WIJSMAN GENERALIZED STATISTICAL CONVERGENCE OF SEQUENCES OF SETS

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ABSTRACT. The concept of Wijsman statistical convergence was defined by [Nuray, F.—Rhoades, B. E.: *Statistical convergence of sequences of sets*, Fasc. Math. **49** (2012), 1–9]. In this paper we present three definitions which are a natural combination of the definition of asymptotic equivalence, statistical convergence, generalized statistical convergence and Wijsman convergence. In addition, we also present asymptotically equivalent sequences of sets in sense of Wijsman and study some properties of this concept.

1. Introduction

Marouf presents definitions for asymptotically equivalent and asymptotic regular matrices in [12]. Pobyvancts introduced the concept of asymptotically regular matrices which preserve the asymptotic equivalence of two nonnegative numbers sequences in [16]. The frequent occurrence of terms having zero value makes a term-by-term ratio $\frac{x_k}{y_k}$ inapplicable in many cases. Fridy introduces new ways of comparing rates of convergence in [9]. If x is in ℓ^1, he uses the remainder sum whose nth term is $R_n x := \sum_{k=n}^{\infty} |x_k|$, and examines the ratio $\frac{R_n x}{R_n y}$ as $n \to \infty$. If x is a bounded sequence, he uses the supremum of the remaining terms which is given by $\mu_n x := \sup_{k\geq n} |x_k|$. Patterson extended these concepts by presenting an asymptotically statistical equivalent analogy of these definitions and natural regularity conditions for nonnegative summability matrices in [15].

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. One of such extensions considered in this paper is the concept of Wijsman convergence. The concept

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of Wijsman statistical convergence is implementation of the concept of statistical convergence to sequences of the sets presented by Nuray and Rhoades in [14]. Similarly, there is the concept of Wijsman lacunary statistical convergence presented by Ulusu and Nuray in [20]. For more works on convergence of sequences of sets, we refer to [1]–[5], [7], [10], [21]–[23].

The idea of statistical convergence was formerly given under the name "*almost convergence*" by Zygmund in the first edition of his celebrated monography published in Warsaw in 1935 [24]. The concept was formally introduced by Steinhaus [19] and Fast [8], later by Schoenberg [18], and also independently by Buck [6]. A lot of developments have been made in this area after the works of Šalát [17] and Fridy [9]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. In the recent years, generalization of statistical convergence has appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers.

In this paper we define asymptotically λ-statistical equivalent sequences of sets in sense of Wijsman and establish some basic results regarding the notions asymptotically λ-statistical equivalent sequences of sets in sense of Wijsman and asymptotically Wijsman statistical equivalent sequences of sets.

Now we recall the definitions statistical convergence, λ-statistical convergence and Wijsman convergence.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1.$$

Definition 1.1. A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

The generalized de la Vallée-Poussin idea is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V,λ)-summable to the number L [11] if $t_n(x) \to L$ for $n \to \infty$. If $\lambda_n = n$, then (V, λ)-summability reduces to (C,1)-summability.

Mursaleen [13] defined λ-statistically convergent sequence as follows

Definition 1.2. A sequence $x = (x_k)$ is said to be λ-statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n : |x_k - L| \geq \varepsilon\}\right| = 0.$$

Let S_λ denote the set of all λ-statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y\in A} \rho(x, y).$$

Definition 1.3 ([2])**.** Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman convergent to A if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case we write

$$W - \lim A_k = A.$$

2. Definitions and notations

Definition 2.1 ([12])**.** Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1,$$

denoted by $x \sim y$.

Definition 2.2 ([15])**.** Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L, provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| = 0,$$

denoted by $x \sim^{S^L} y$ and simply asymptotically statistical equivalent if $L = 1$.

Now we define asymptotically λ-statistical equivalent and strongly asymptotically λ-statistical equivalent sequences as follows

Definition 2.3. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically λ-statistical equivalent of multiple L, provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left|\left\{k \in I_n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| = 0,$$

denoted by $x \sim^{S_\lambda^L} y$ and simply asymptotically λ-statistical equivalent if $L = 1$. If we take $\lambda_n = n$, then we get the Definition 2.2.

Definition 2.4. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically λ-statistical equivalent of multiple L, provided that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{x_k}{y_k} - L \right| = 0,$$

denoted by $x \sim^{[V,\lambda]^L} y$ and simply strongly asymptotically λ-statistical equivalent if $L = 1$. If we take $\lambda_n = n$, then we get the following definition.

Definition 2.5. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically Cesàro statistical equivalent of multiple L, provided that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| = 0,$$

denoted by $x \sim^{[C,1]^L} y$ and simply strongly asymptotically Cesàro statistical equivalent if $L = 1$.

The concepts of Wijsman statistical convergence and boundedness for the sequence (A_k) were given by Nuray and Rhoades [14] as follows

Definition 2.6 ([14])**.** Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman statistical convergent to A if the sequence $\big(d(x, A_k)\big)$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \left| \{ k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon \} \right| = 0.$$

In this case, we write

$$\mathrm{st} - \lim_k A_k = A \quad \text{or} \quad A_k \to A \, (WS) \, .$$

The sequence (A_k) is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets is denoted by L_∞.

In [21], Ulusu and Nuray define asymptotically equivalent and asymptotically statistical equivalent sequences of sets as follows

Definition 2.7. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically equivalent (Wijsman sense) if for each $x \in X$,

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1,$$

denoted by $(A_k) \sim (B_k)$.

Definition 2.8. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically statistical equivalent of multiple L (Wijsman sense) if every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n}\left|\left\{k \le n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right| = 0,$$

denoted by $(A_k) \sim^{WS^L} (B_k)$ and simply asymptotically statistical equivalent (Wijsman sense) if $L = 1$.

In [7], Esi and Hazarika introduced the notion Wijsman λ-statistical convergence of sequences of sets as follows

Definition 2.9. Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ $(k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman λ-statistical convergent to A if the sequence $\big(d(x,A_k)\big)$ is λ-statistically convergent to $d(x,A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{\lambda_n}\big|\big\{k \in I_n : |d(x,A_k) - d(x,A)| \ge \varepsilon\big\}\big| = 0.$$

In this case, we write

$$S_\lambda^W - \lim_k A_k = A \quad \text{or} \quad A_k \to A\left(S_\lambda^W\right).$$

3. Main results

In this section we define asymptotically Wijsman λ-statistical equivalent sequences of sets and prove some interesting results.

Definition 3.1. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman λ-equivalent of multiple L if for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\frac{d(x,A_k)}{d(x,B_k)} = L,$$

denoted by $(A_k) \sim^{W(V,\lambda)^L} (B_k)$ and simply asymptotically Wijsman λ-equivalent if $L = 1$.

Definition 3.2. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are strongly asymptotically Wijsman λ-statistical equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| = 0,$$

denoted by $(A_k) \sim^{W[V,\lambda]^L} (B_k)$ and simply strongly asymptotically Wijsman λ-equivalent if $L = 1$.

Example 3.1. We consider the following sequences:

$$A_k = \begin{cases} \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{(x-\sqrt{k})^2}{k} + \frac{(y-\sqrt{2k})^2}{2k} + \frac{z^2}{3k} = 1 \right\} & \text{if } n - [|\lambda_n|] + 1 \le k \le n, \\ \{(1,1,1)\}, & \text{otherwise} \end{cases}$$

and

$$B_k = \begin{cases} \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{(x+\sqrt{k})^2}{k} + \frac{(y+\sqrt{2k})^2}{2k} + \frac{z^2}{3k} = 1 \right\} & \text{if } n - [|\lambda_n|] + 1 \le k \le n, \\ \{(1,1,1)\}, & \text{otherwise.} \end{cases}$$

Since

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{d(x,A_k)}{d(x,B_k)} - 1 \right| = 0,$$

therefore the sequences (A_k) and (B_k) are strongly asymptotically Wijsman λ-equivalent, i.e.,

$$(A_k) \sim^{W[V,\lambda]^1} (B_k).$$

If we take $\lambda_n = n$, then we get the definitions.

Definition 3.3. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman Cesàro-equivalent of multiple L if for each $x \in X$,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \frac{d(x,A_k)}{d(x,B_k)} = L,$$

denoted by $(A_k) \sim^{W(C,1)^L} (B_k)$ and simply asymptotically Wijsman Cesáro-equivalent if $L = 1$.

Definition 3.4. Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$.

We say that the sequences (A_k) and (B_k) are strongly asymptotically statistical equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| = 0,$$

denoted by $(A_k) \sim^{W[C,1]^L} (B_k)$ and simply strongly asymptotically Wijsman Cesàro-equivalent if $L = 1$.

Definition 3.5. Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences (A_k) and (B_k) are asymptotically Wijsman λ-statistical multiple L equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $(A_k) \sim^{WS_\lambda^L} (B_k)$ and simply asymptotically Wijsman λ-statistical equivalent if $L = 1$.

Example 3.2. We consider the following sequences:

$$A_k = \begin{cases} \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{k^2} \right\} & \text{if } n - [|\lambda_n|] + 1 \leq k \leq n, \\ & \text{and } k = i^2, \quad i = 1, 2, 3, \ldots, \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

and

$$B_k = \begin{cases} \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (y+1)^2 = \frac{1}{k^2} \right\} & \text{if } n - [|\lambda_n|] + 1 \leq k \leq n, \\ & \text{and } k = i^2, \quad i = 1, 2, 3, \ldots, \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$

Since

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| \geq \varepsilon \right\} \right| = 0,$$

therefore the sequences (A_k) and (B_k) are asymptotically Wijsman λ-equivalent, i.e., $(A_k) \sim^{WS_\lambda^1} (B_k)$.

Theorem 3.1. *Let (X, ρ) be a metric space and A_k, B_k be non-empty closed subsets X $(k \in \mathbb{N})$. Then*

(a) $(A_k) \sim^{W[V,\lambda]^L} (B_k) \Rightarrow (A_k) \sim^{WS_\lambda^L} (B_k)$,

(b) $W[V, \lambda]^L$ *is a proper subset of* WS_λ^L,

(c) *Let* $(A_k) \in L_\infty$ *and* $(A_k) \sim^{WS_\lambda^L} (B_k)$, *then* $(A_k) \sim^{W[V,\lambda]^L} (B_k)$,

(d) $W[V, \lambda]^L \cap L_\infty = WS_\lambda^L \cap L_\infty$.

Proof.

(a) Let $\varepsilon > 0$ and $(A_k) \sim^{W[V,\lambda]^L} (B_k)$. Then we can write

$$\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| \geq \sum_{\substack{k\in I_n\\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|$$
$$\geq \varepsilon\left|\left\{k\in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|$$

which gives the result.

(b) Suppose that $W[V,\lambda]^L \subset WS^L_\lambda$. Let (A_k) and (B_k) be two sequences defined as follows:

$$A_k = \begin{cases} \{k\} & \text{if } n-[|\lambda_n|]+1\leq k\leq n,\ k=1,2,3,\dots, \\ \{0\}, & \text{otherwise}\end{cases}$$

and

$$B_k = \{0\} \qquad \text{for all} \quad k\in\mathbb{N}.$$

It is clear that $(A_k)\notin L_\infty$ and $\varepsilon>0$ and for each $x\in X$,

$$\lim_n \frac{1}{\lambda_n}\left|\left\{k\in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)}-1\right|\geq\varepsilon\right\}\right| = \lim_n \frac{1}{\lambda_n}[|\lambda_n|] = 0.$$

So

$$(A_k) \sim^{WS^1_\lambda} (B_k), \quad \text{but} \quad \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| \neq 0.$$

Therefore $(A_k) \not\sim^{W[V,\lambda]^L} (B_k)$.

(c) Suppose that $(A_k) \sim^{WS^L_\lambda} (B_k)$ and $(A_k)\in L_\infty$. We assume that

$$\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\leq M$$

for each $x\in X$ and for all $k\in\mathbb{N}$. Given $\varepsilon>0$, we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| = \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|$$
$$+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|<\varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|$$
$$\leq \frac{M}{\lambda_n}\left|\left\{k\in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|+\varepsilon$$

from which the result follows.

(d) It follows from (a), (b) and (c). □

If we let $\lambda_n = n$ in Theorem 3.1, then we have the following corollary.

Corollary 3.2. *Let (X,ρ) be a metric space and A_k, B_k be non-empty closed subsets X $(k \in \mathbb{N})$. Then*

(a) $(A_k) \sim^{W[C,1]^L} (B_k) \Rightarrow (A_k) \sim^{WS^L} (B_k)$,

(b) $W[C,1]^L$ *is a proper subset of* WS^L,

(c) *Let* $(A_k) \in L_\infty$ *and* $(A_k) \sim^{WS^L} (B_k)$, *then* $(A_k) \sim^{W[C,1]^L} (B_k)$,

(d) $W[C,1]^L \cap L_\infty = WS^L \cap L_\infty$.

Theorem 3.3. *Let (X,ρ) be a metric space and A_k, B_k be non-empty closed subsets X $(k \in \mathbb{N})$. Then $(A_k) \sim^{WS^L} (B_k) \Rightarrow (A_k) \sim^{WS_\lambda^L} (B_k)$ if and only if $\liminf \frac{\lambda_n}{n} > 0$.*

Proof. Suppose that $\liminf \frac{\lambda_n}{n} > 0$. For given $\varepsilon > 0$, we have

$$\left\{k \le n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\} \supset \left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}.$$

Therefore,

$$\frac{1}{n}\left|\left\{k \le n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right| \ge \frac{1}{n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right|$$

$$\ge \frac{\lambda_n}{n}.\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right|.$$

Taking the limit as $n \to \infty$ and using $\liminf \frac{\lambda_n}{n} > 0$, we get the desired result.

Conversely, suppose that $\liminf_n \frac{\lambda_n}{n} = 0$. Then we can select a subsequence $\big(n(i)\big)_{i=1}^{\infty}$ such that

$$\frac{\lambda_{n(i)}}{n(i)} < \frac{1}{i}.$$

We define sequences (A_k) and (B_k) as follows

$$A_k = \begin{cases} \{1\} & \text{if } n(i) - \big[\big|\lambda_{n(i)}\big|\big] + 1 \le k \le n(i),],\ i = 1,2,3,\dots, \\ \{0\}, & \text{otherwise.} \end{cases}$$

and

$$B_k = \{0\} \qquad \text{for all} \quad k \in \mathbb{N}.$$

Then

$$(A_k) \sim^{WS^L} (B_k), \quad \text{but} \quad (A_k) \nsim^{WS_\lambda^L} (B_k)(A_k).$$

This completes the proof. $\square$

Theorem 3.4. *Let (X,ρ) be a metric space and A_k, B_k be non-empty closed subsets X $(k \in \mathbb{N})$. Then $(A_k) \sim^{WS^L_\lambda} (B_k) \Rightarrow (A_k) \sim^{WS^L} (B_k)$ if* $\liminf \frac{\lambda_n}{n} = 1$.

Proof. Since $\lim_n \frac{\lambda_n}{n} = 1$, then for $\varepsilon > 0$, we observe that

$$\frac{1}{n}\left|\left\{k \le n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right|$$

$$\le \frac{1}{n}\left|\left\{k \le n-\lambda_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right| + \frac{1}{n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right|$$

$$\le \frac{n-\lambda_n}{n} + \frac{1}{n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right|$$

$$= \frac{n-\lambda_n}{n} + \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \ge \varepsilon\right\}\right|.$$

This implies that

$$(A_k) \sim^{WS^L_\lambda} (B_k) \Longrightarrow (A_k) \sim^{WS^L} (B_k).$$

□

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