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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2013-0028
Tatra Mt. Math. Publ. **56** (2013), 79–86

SIMULTANEOUS DIOPHANTINE APPROXIMATION IN $\mathbb{R}^2 \times \mathbb{C} \times \mathbb{Q}_p$

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ABSTRACT. An analogue of the convergence part of Khintchine's theorem (1924) for simultaneous approximation of integral polynomials at the points

$$(x_1, x_2, z, w) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{Q}_p$$

is proved. It is a solution of the more general problem than Sprindžuk's problem (1980) in the ring of adeles. We use a new form of the essential and nonessential domain methods in metric theory of Diophantine approximation.

1. Introduction

From the beginning of a current century a new direction in the metric theory of Diophantine approximation is developed [1]–[7]. This is simultaneous approximation of zero by values of integral polynomials P, $\deg P \leq n$, with respect to different valuations.

Primary a problem of simultaneous approximation in $\mathbb{R}^k \times \mathbb{C}^l \times \prod_{p \in S} \mathbb{Q}_p$, where $k \geq 0$, $l \geq 0$ are integers and S is a finite set of prime numbers, $n \geq k + 2l$, was formulated by V. Sprindžuk (1980). According to contemporary therminology it is *Diophantine approximation in the ring of adeles*.

Let $P = P(t) = a_n t^n + \cdots + a_1 t + a_0 \in \mathbb{Z}[t]$, $a_n \neq 0$, $H = H(P) = \max(|a_n|, \ldots, |a_0|)$. Let $p \geq 2$ be a prime number, $\mathbb{Q}_p$ be the field of p-adic numbers, $|\cdot|_p$ be the p-adic valuation. Suppose that $\mathcal{O} = \mathbb{R}^2 \times \mathbb{C} \times \mathbb{Q}_p$. We define a measure $\overline{\mu}$ in $\mathcal{O}$ as a product of the Lebesque measure μ_1 in $\mathbb{R}^2$, the Lebesque measure μ_2 in $\mathbb{C}$ and the Haar measure μ_p in $\mathbb{Q}_p$, that is, $\overline{\mu} = \mu_1 \mu_2 \mu_p$. Let $\Psi \colon \mathbb{N} \to \mathbb{R}^+$, Ψ be a monotonically decreasing function, $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $V = (v_1, v_2, v_3, v_4)$, where $\lambda_i \geq 0$ and $v_i \geq 0$ are vectors in $\mathbb{R}^4$. We consider the system of inequalities

© 2013 Mathematical Institute, Slovak Academy of Sciences.
2010 Mathematics Subject Classification: 11J61, 11J83, 11K60.
Keywords: metric theory of Diophantine approximation, ring of adeles.

$$|P(x_1)| < H^{-v_1}\Psi(H)^{\lambda_1}, \qquad |P(x_2)| < H^{-v_2}\Psi(H)^{\lambda_2},$$
$$|P(z)| < H^{-v_3}\Psi(H)^{\lambda_3}, \qquad |P(\omega)|_p < H^{-v_4}\Psi(H)^{\lambda_4}, \tag{1}$$

where $(x_1, x_2, z, \omega) \in \mathcal{O}$ and $v_1 + v_2 + 2v_3 + v_4 = n - 4$, $\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 = 1$. Let $M_n(V, \Psi, \Lambda)$ be a set of the points $(x_1, x_2, z, \omega) \in \mathcal{O}$ for which the system (1) has infinitely many solutions in polynomials $P \in \mathbb{Z}[t]$, $\deg P = n$. We prove

Theorem. *If $n \geq 4$ and $\sum_{H=1}^{\infty} \Psi(H) < \infty$ then $\overline{\mu}\big(M_n(V, \Psi, \Lambda)\big) = 0$.*

Notice that for proving of the theorem we apply *the essential and nonessential domains method* of Sprindžuk developed and improved by V. Bernik, M. Dodson, V. Beresnevich, D. Dickinson and S. Velani and the other representatives of the Number Theory schools in the Byelorussian Academy of Sciences (Minsk, Belarus) and the York University (York, UK) (1980–2013).

2. Sketch of proof

Our investigation is based on the method [8], the argumentations from [2]–[4], [7]–[10] and their development.

Let $\mathbf{T} = I_1 \times I_2 \times K \times D_p \subset \mathcal{O}$, where I_1, I_2 are the intervals in $\mathbb{R}$, K is a circle in $\mathbb{C}$ and D_p is a disc in $\mathbb{Q}_p$. According to a metric character of the theorem we shall prove it for the points in $\mathbf{T}$. We shall call $\mathbf{T}$ a *parallelepiped.* Fix $\delta > 0$ and exclude from $\mathbf{T}$ a set of the points (x_1, x_2, z, ω) which satisfy $|x_i| < \delta$ $(i = 1, 2)$, $|\mathrm{Im} z| < \delta$ and $|\omega|_p < \delta$. Thus, from now on we shall assume that the points $(x_1, x_2, z, \omega) \in \mathbf{T}$ satisfy $|x_i| \geq \delta$ $(i = 1, 2)$, $|\mathrm{Im} z| \geq \delta$ and $|\omega|_p \geq \delta$. It will be without loss of generality if δ is an arbitrary small number.

Introduce a class of polynomials $\mathcal{P}_n(Q) = \big\{P \in \mathbb{Z}[t] : H(P) \leq Q\big\}$, where $Q > Q_0 > 0$. The *important* moment of the proof is a reduction to *irreducible* and *leading* polynomials $P \in \mathcal{P}_n(Q)$, i.e., $H(P) < c(n)|a_n|$, $c(n) \geq 1$ and $|a_n|_p > p^{-n}$ (as [8, Ch. 1, §§5, 8 and Ch. 2, §2] or [3]). We denote a set of such polynomials P as $\mathfrak{P}_n$.

Let $\mathfrak{P}_n(H)$ denote a set of polynomials $P \in \mathfrak{P}_n$ satisfying (1) for which $H(P) = H$ where H is a fix number, $0 < Q_0 < H \leq Q$. The set $\mathfrak{P}_n(H)$ is divided into *ε-classes* $\mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$ according to the values of a differences between their roots (§3, formulas (2), (3) and the text above and below these formulas). Next, we prove the theorem for each *ε-class.* For this, we introduce the notion of (i_1, i_2, i_3, i_4)-*linear* polynomial, where $i_j \in \{0, 1\}$ $(j = 1, 2, 3, 4)$. For example, $(0, 0, 0, 0)$-*linear* polynomial, $(1, 1, 1, 1)$-*linear* one, $(0, 1, 1, 0)$-*linear* one and so on). We have 16 cases of *linearity.* This notion is necessary to obtain the lower bounds of the derivatives $|P'(x_i)|$ $(i = 1, 2)$, $|P'(z)|$ and $|P'(\omega)|_p$ for $P \in \mathfrak{P}_n(H)$. Lemma 2 §3 gives the upper bounds of them. Fix a admissible vector (i_1, i_2, i_3, i_4) and denote by $\mathfrak{P}_n^{(i_1,i_2,i_3,i_4)}$ the class of (i_1, i_2, i_3, i_4)-*linear* polynomials $P \in \mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$.

Now, we take a fix $P \in \mathfrak{P}_n^{(i_1,i_2,i_3,i_4)}$ and consider a system of the small parallelepipeds $\Pi_j(P) \subset \mathbf{T}$, $(j = 1, 2, \ldots)$ at which P satisfies (1). These parallelepipeds $\Pi_j(P) \subset \mathbf{T}$ are divided into two classes: *the essential* and *the inessential* (analogously to [8, §§10, 11]). The parallelepiped $\Pi_j(P)$ is called *essential* if for all polynomials $P_j \neq P$, $P_j \in \mathfrak{P}_n^{(i_1,i_2,i_3,i_4)}$, we have $\overline{\mu}\big(\Pi_j(P) \bigcap \Pi_j(P_j)\big) < \frac{1}{2}\mu\Pi_j(P)$. If there exists $P_j \in \mathfrak{P}_n^{(i_1,i_2,i_3,i_4)}$, $P_j \neq P$, such that $\overline{\mu}\big(\Pi_j(P) \bigcap \Pi_j(P_j)\big) \geq \frac{1}{2}\mu\Pi_j(P)$ then the parallelepiped $\Pi_j(P)$ is called *inessential*.

Next, using Lemmas 1–4 §3 and the classic metric *Borel–Cantelli theorem* [8, Ch. 1, §3, Lemma 12] we show that the measure of the set of points lying in infinitely many *the essential* parallelepipeds $\Pi_j(P)$ equals *zero*, and that the measure of the set of points lying in infinitely many *the inessential* parallelepipeds $\Pi_j(P)$ also equals *zero*.

3. Lemmas on polynomials

Let $P \in \mathfrak{P}_n(H)$ have roots $\alpha_1, \alpha_1, \ldots, \alpha_n$ in $\mathbb{C}$ and roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the smallest field containing $\mathbb{Q}_p$ and all algebraic numbers. According to Lemma 1 [8, Ch. 1, §2] and Lemma 4 [8, Ch. 2, §2] we have $|\alpha_j| \ll 1$, $|\gamma_j|_p \ll 1$ $(j = 1, \ldots, n)$. Let $\alpha_1, \ldots, \alpha_k$ be the *real* roots of P and $\beta_1, \ldots, \beta_{(n-k)/2}$ be the *complex* roots of it. Since P is irreducible then all of its roots are different. Choose two real roots $\alpha_{j_1} \in I_1$, $\alpha_{j_2} \in I_2$, a complex root $\beta_1 = \alpha_{j_3} \in K$, and a p-adic root $\gamma_1 \in D_p$. Remember that the parallelepiped $\mathbf{T} = I_1 \times I_2 \times K \times D_p$ was introduced at the beginning of §2. Define the sets

$$S_i(\alpha_{j_i}) = \left\{ u \in \mathbb{U} : |u - \alpha_{j_i}| = \min_{1 \leq k \leq n} |u - \alpha_k| \right\}, \qquad i = 1, 2, 3,$$

where u represents x_1 or x_2 or z, and α_{j_i} is a real or a complex root of P, and $\mathbb{U}$ is $I_1 \subset \mathbb{R}$ or $I_2 \subset \mathbb{R}$, or $K \subset \mathbb{C}$ as required, and

$$S_p(\gamma_s) = \left\{ \omega \in D_p \subset \mathbb{Q}_p : |\omega - \gamma_s|_p = \min_{1 \leq k \leq n} |\omega - \gamma_k|_p \right\}.$$

We shall consider these sets for a fixed vector (j_1, j_2, j_3, s) and for simplicity we shall assume that $j_1 = 1$, $j_2 = 2$, $\alpha_{j_3} = \beta_1$ and $s = 1$. Reorder the other roots of P in the following way:

(1) $|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_3| \leq \cdots \leq |\alpha_1 - \alpha_k|$,

(2) $|\alpha_2 - \alpha_1| \leq |\alpha_2 - \alpha_3| \leq \cdots \leq |\alpha_2 - \alpha_k|$,

(3) $|\beta_1 - \beta_2| \leq \cdots \leq |\beta_1 - \beta_{(n-k)/2}|$, and

(4) $|\gamma_1 - \gamma_2|_p \leq \cdots \leq |\gamma_1 - \gamma_n|_p$.

Also, for the polynomial $P \in \mathfrak{P}_n(H)$ define numbers $\rho_{ij} \in \mathbb{R}$ by $|\alpha_{i1} - \alpha_{ij}| = H^{-\rho_{ij}}$, $2 \leq j \leq n$, $\rho_{in} \leq \rho_{i2} \leq \cdots \leq \rho_{i2}$ $(i = 1, 2, 3, 4)$, where $\alpha_{11} = \alpha_1$, $\alpha_{21} = \alpha_2$, $\alpha_{31} = \beta_1$ and $\alpha_{41} = \gamma_1$. Since the roots $|\alpha_j|$, $|\beta_j|$, $|\gamma_j|_p$ are bounded (see the beginning §2), then there exists $\varepsilon_1 > 1$ such that $\rho_{ij} \geq -\varepsilon_1/2$ for $i = 1, 2, 3, 4$ and $j = 2, 3, \ldots, n$. Choose $\varepsilon > 0$ such that $\varepsilon_1 = \varepsilon/T_1$ for some sufficiently large $T_1 > T_0 > 0$. Let $T = [n/\varepsilon_1] + 1$. Define the integers $(k_{1j}, k_{2j}, l_j, m_j) = (t_{1j}, t_{2j}, t_{3j}, t_{4j})$ $(j = 2, 3, \ldots, n)$ by the relations

$$(t_{ij} - 1)/T \leq \rho_{ij} < t_{ij}/T, \quad t_{i2} \geq t_{i3} \geq \cdots \geq t_{in} \geq 0, \qquad i = 1, 2, 3, 4. \tag{2}$$

Finally, define the numbers q_{1i}, q_{2i}, r_i and s_i $(i = 1, 2, \ldots, n-1)$ by

$$q_{1i} = T^{-1} \sum_{t=i+1}^{n} k_{1t},\ q_{2i} = T^{-1} \sum_{t=i+1}^{n} k_{2t},\ r_i = T^{-1} \sum_{t=i+1}^{n} l_t,\ s_i = T^{-1} \sum_{t=i+1}^{n} m_t. \tag{3}$$

Each polynomial $P \in \mathfrak{P}_n(H)$ is now associated with four vectors $\mathbf{q}_1 = (q_{11}, q_{12}, \ldots$ $\ldots, q_{1(n-1)})$, $\mathbf{q}_2 = (q_{21}, q_{22}, \ldots, q_{2(n-1)})$, $\mathbf{r} = (r_1, r_2, \ldots, r_{n-1})$, $\mathbf{s} = (s_1, s_2, \ldots$ $\ldots, s_{n-1})$. The number of these vectors is finite and depends only on n, ρ and T (see [8, Ch. 1, Lemma 24 and Ch. 2, Lemma 12]). Let $\mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$ denote the set of polynomials $P \in \mathfrak{P}_n(H)$ having the *same* four vectors $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$. Thus, we divide the set $\mathfrak{P}_n(H)$ on *ε-classes* $\mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$.

From now on it will be assumed without loss generality that $x_1 \in S_1(\alpha_1) \subset I_1$, $x_2 \in S_2(\alpha_2) \subset I_2$, $z \in S_3(\beta_1) \subset K$ and $\omega \in S_p(\gamma_1) \subset D_p$. At many points of our proof the values of the polynomials $P \in \mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$ will be estimated by means of a Taylor series. To obtain an upper bounds on the terms in the Taylor series and the other purposes the following two lemmas will be used.

Lemma 1. *If $P \in \mathfrak{P}_n(H)$ then*

$$|\widetilde{u} - \widetilde{\alpha}| \leq 2^n |P_n(\widetilde{u})| |P_n'(\widetilde{\alpha})|^{-1}, \qquad |\omega - \gamma_1|_p \leq |P_n(\omega)|_p |P_n'(\gamma_1|_p^{-1},$$

$$\begin{cases} |\widetilde{u} - \widetilde{\alpha}| \leq \min_{2 \leq j \leq n} \left(2^{n-j} \ (|P_n(\widetilde{u})| |P_n'(\widetilde{\alpha}|^{-1} \prod_{k=2}^{j} |\widetilde{\alpha} - \alpha_k|)^{1/j}, \right. \\ |\omega - \gamma_1|_p \leq \min_{2 \leq j \leq n} \ (|P_n(\omega)|_p |P_n'(\gamma_1|_p^{-1} \prod_{k=2}^{j} |\gamma_1 - \gamma_k|_p)^{1/j} \end{cases}$$

where $\widetilde{u}$ represents x_1 or x_2 or z and $\widetilde{\alpha}$ is α_1, α_2 or β_1 as required.

P r o o f. See [2] and [10, pp. 36, 131]. □

Lemma 2. *Let $P \in \mathfrak{P}_n(H, \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}, \boldsymbol{s})$. Then*

$$|P^{(l)}(\alpha_{i1})| < c(n) H^{1 - q_{il} + (n-l)\varepsilon_1} \quad (i = 1, 2), \quad |P^{(l)}(\beta_1)| < c(n) H^{1 - r_l + (n-l)\varepsilon_1},$$

$$|P^{(l)}(\gamma_1)|_p < c(n) H^{-s_l + (n-l)\varepsilon_1} \quad (1 \leq l \leq n-1),$$

where the constant $c(n) > 0$ depends only on n.

P r o o f. The first, second and third inequalities are proved in [2] or [10, pp. 36–37]. The fourth inequality is proved in [7, Lemma 2]. □

At several points of the proof there are various cases for $P \in \mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$ to consider. Usually the existence of one case is disproved by finding a contradiction to the final inequality in the next lemma.

Lemma 3. *Let $P_1, P_2 \in \mathbb{Z}[t]$ of degree at most n with no common roots and $\max\big(H(P_1), H(P_2)\big) \leq H$ $(H > H_0)$. Let $\delta > 0$ and $\eta_i > 0$ $(i = 1,2,3,4)$. Let $J_i \subset \mathbb{R}$ be the intervals, $\mu_i J_i = H^{-\eta_i}$ $(i = 1,2)$, $K \subset \mathbb{C}$ be a circle,* $\operatorname{diam} K = H^{-\eta_3}$ *and $D \subset \mathbb{Q}_p$ be a disk, $\mu_p|D|_p = H^{-\eta_4}$. If there exist $\tau_i > -1$ $(i = 1,2,3)$ and $\tau_4 > 0$ such that for all $(x_1, x_2, z, \omega) \in J_1 \times J_2 \times K \times D$ we have*

$$\max\big(|P_1(x_i)|, |P_2(x_i)|\big) < H^{-\tau_i} \qquad (i = 1,2),$$

$$\max\big(|P_1(z)|, |P_2(z)|\big) < H^{-\tau_3}, \quad \max\big(|P_1(\omega)|_p, |P_2(\omega)|_p\big) < H^{-\tau_4},$$

then

$$\tau_1 + \tau_2 + 2\tau_3 + \tau_4 + 4 + 2\max(\tau_1 + 1 - \eta_1, 0) + 2\max(\tau_2 + 1 - \eta_2, 0) \\ + 4\max(\tau_3 + 1 - \eta_3, 0) + 2\max(\tau_4 + 1 - \eta_4, 0) < 2n + \delta.$$

P r o o f. It is analogous to [4]. Distinctions consist only in the sets of $\overline{X} = (X_1, X_2, X_3, X_4)$ and in the metrics of the corresponding spaces. Namely, in [4] we have $\overline{X} = (x_1, z, \omega) \in \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$, in our case we have $\overline{X} = (x_1, x_2, z, \omega) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{Q}_p$.

The *sense* of the lemma is the following: if the values of two polynomials are small at a given $J_1 \times J_2 \times K \times D$, then the parameters τ_i and η_i are connected by the final inequality of lemma. □

Lemma 4. *Let $P \in \mathbb{Z}[t]$, $\deg P = n \geq 4$ and $v > 0$. Let $G(v)$ be the set of points $(x_1, x_2, z, \omega) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{Q}_p$ for which the inequality $|P(x_1)| \cdot |P(x_2)| \cdot |P(z)| \cdot |P(\omega)|_p < H^{-v}$, $H = H(P)$, has infinitely many solutions P. Then $\overline{\mu} G(v) = 0$ for $v > n - 3$.*

P r o o f. See [9]. □

4. Proof of Theorem

Remember that we consider the points $(x_1, x_2, z, \omega) \in \mathbf{T}$ and $P \in \mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$. We prove the theorem for $n \geq 5$. The case $n = 4$ follows from Lemma 1 and the Borel–Cantelli theorem.

Definition. Let $i_j \in \{0,1\}$ $(j = 1,2,3,4)$. A polynomial $P \in \mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$ is called (i_1, i_2, i_3, i_4)-*linear* if:

(1) for $(i_1, i_2, i_3, i_4) = (0, 0, 0, 0)$ the system of inequalities

$$r_{i1} + s_{i2}/T < v_i + 1 \qquad (i = 1, 2, 3, 4), \tag{4}$$

holds, where $(r_{11}, r_{21}, r_{31}, r_{41}) = (q_{11}, q_{21}, r_1, s_1)$ defined in (2), (3);

(2) for $(i_1, i_2, i_3, i_4) = (1, 1, 1, 1)$ the inequality sings in (4) are reversed;

(3) for $(0, 1, 1, 1)$ the first inequality in (4) has the sign $<$ and the other inequalities have sings $\geq$; and so on. There exist 16 kinds of *linear* polynomials.

Denote by $\mathfrak{P}_n^{(i_1,i_2,i_3,i_4)}$ the class of (i_1, i_2, i_3, i_4)-*linear* polynomials $P \in \mathfrak{P}_n(H, \mathbf{q}_1, \mathbf{q}_2, \mathbf{r}, \mathbf{s})$. If $(x_1, x_2, z, \omega) \in M_n(V, \Psi, \Lambda)$ (see §1), then there exist infinitely many polynomials satisfying at least one of these 16 kinds of linearity. Let $M_n^{(i_1,i_2,i_3,i_4)}(V, \Psi, \Lambda)$ denote the set of $(x_1, x_2, z, \omega) \in \mathbf{T}$ for which the system of inequalities (1) holds for infinitely many polynomials $P \in \mathfrak{P}_n^{(i_1,i_2,i_3,i_4)}$. It should be clear that $M_n(V, \Psi, \Lambda) = \bigcup_{i_j \in \{0,1\},\ (j=1,2,3,4)} M_n^{(i_1,i_2,i_3,i_4)}(V, \Psi, \Lambda)$.

Two constants

$$d_1 = q_{11} + q_{21} + 2r_1 + s_1 \quad \text{and} \quad d_2 = (k_{12} + k_{22} + 2l_2 + m_2)/T \tag{5}$$

which are connected with (2), (3), will be used further in our proof. The proof consists of a series of propositions with different *linearity* conditions and different *ranges* of $d_1 + d_2$ considered separately.

Besides, we have $|P'(\alpha_{i1})| = H|\alpha_{i1} - \alpha_{i2}| \cdots |\alpha_{i1} - \alpha_{in}| = H^{1-r_{ij}}$ $(i = 1, 2, 3)$, where $(r_{1j}, r_{2j}, r_{3j}) = (q_{11}, q_{21}, r_3)$ and $|P'(\gamma_i)|_p = H^{-s_1}$. These relations follow directly from (3).

Proposition 1. *Let* $P \in \mathfrak{P}_n^{(0,0,0,0,)}$. *Then* $\overline{\mu} M_n^{(0,0,0,0)}(V, \Psi, \Lambda) = 0$.

Proof. According to (4) and (5) we have $d_1 + d_2 < n + 1$. The proof includes four cases:

(1) $n + \varepsilon \leq d_1 + d_2 < n + 1$;

(2) $5 - \varepsilon \leq d_1 + d_2 < n + \varepsilon$;

(3) $\varepsilon \leq d_1 + d_2 < 5 - \varepsilon$;

(4) $d_1 + d_2 < \varepsilon$.

We use scheme of the proofs of propositions 1, 2, 3 4 of [3], correspondingly but there exist some *distinctions*. The distinctions appear in the sets of $\overline{X} = (X_1, X_2, X_3, X_4)$ of the corresponding spaces. Namely, in [3] we have $\overline{X} = (x_1, z, \omega) \in \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$, in our case we have $\overline{X} = (x_1, x_2, z, \omega) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{Q}_p$. Note that in

(1) we use Lemmas 1–3 §3; in

(2) we use Lemmas 1–4 §3 and make a reduction to polynomials of the *third* degree (in [3] the reduced polynomials have the *second* degree); in

(3) we use Lemmas 1–4 §3 and make a reduction to polynomials of the *forth* degree (in [3] the ones have the *third* degree); in

(4) we use Lemmas 1–3 §3 and make a reduction to polynomials of the *third* degree (in [3] the ones have the *second* degree).

□

Proposition 2. *Let* $P \in \mathfrak{P}_n^{(1,1,1,1)}$*. Then* $\overline{\mu} M_n^{(1,1,1,1)}(V, \Psi, \Lambda) = 0$.

Proof. According to (4) and (5) we have $d_1 + d_2 \geq n+1$. The proof is similarly to Proposition 5 [3]. □

Propositions 1, 2 are *basic* in proving of theorem. The other cases of *linearity* are the combinations of previous two cases on the corresponding coordinates. Namely, the cases $(1,0,0,0)$-, $(0,1,0,0)$-, $(0,0,1,0)$-, $(0,0,0,1)$-*linearity* are considered in the same manner since they are the permutations of the coordinates. Thus, only the $(1,0,0,0)$-*linearity* case will be investigated. It is proved analogues to Proposition 6 [3], where for the second coordinate i_2 ($i_2 = 0$) we add the inequality $q_{21} + k_{22}/T < 1 + v_2 + \lambda_2$.

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The cases $(1,1,1,0)$-, $(1,1,0,1)$-, $(1,0,1,1)$-, $(0,1,1,1)$-*linearity* are considered in the same manner since they are also the permutations of the coordinates. Thus, only the $(1,1,1,0)$-*linearity* case will be investigated. It is a combination of Propositions 6, 7 [3], where for the second coordinate i_2 ($i_2 = 1$) we add the inequality $q_{21} + k_{22}/T \geq 1 + v_2 + \lambda_2$, and for the third coordinate i_3 ($i_3 = 1$) we take $r_1 + l_2/T \geq 1 + v_3 + \lambda_3$.

The theorem is proved. Note that the similar method was used earlier in [7].

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Received August 19, 2013

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