VERSITA

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2013-0030
Tatra Mt. Math. Publ. **56** (2013), 87–92

NOTE ON SUMS OF UNITS IN REAL CUBIC FIELDS

Juraj Kostra — David Krčmarský

ABSTRACT. Let $K/\mathbb{Q}$ be a cyclic cubic field with an prime conductor l. In the paper there is given method for verification that K is ω-good and it is applied for conductors up to $l = 349$.

Introduction

In 1954 D. Zelinsky [4] investigated the additive unit structure of rings. Zelinsky's work gave rise to many investigations of rings that are generated by their units.

Definition 1 (see [1], [2]). An integral domain R is called n-good, if every element of R can be written as sum of n units, i.e., invertible elements of R, and it is called ω-good, if it is not n-good for any n, but each of its elements is a sum of units.

It has been proved by N. Ashrafi and P. Vámos [1] that the ring of integers of a quadratic number field is not n-good for any n, and the same holds in the case of cubic number fields having a negative discriminant and cyclotomic fields $\mathbb{Q}(\zeta_{2^N})$ for $N \geq 1$.

In [2] M. Jarden and W. Narkiewicz have shown that no finite extensions of rationals is n-good.

After the above result the interesting question remains:
In which algebraic number fields every algebraic integer can be expressed as sum of units, in other words, which algebraic number fields are ω-good?

2010 Mathematics Subject Classification: 11R04, 11R27.
Keywords: cyclotomic units, class number, sum of units, order.

In [3] R. Tichy and V. Ziegler characterized all complex cubic fields with maximal orders generated by their units.

In the present paper we are interested in special class of real algebraic number fields. Let K a cyclic extension of rationals of prime degree $p > 2$ such that $K \subseteq \mathbb{Q}(\zeta_m)$ and let $S_K = \{\alpha = \sum_{i=1}^{k} \epsilon_i \colon \epsilon_i \in U_K, n \in \mathbb{N}\}$. In the paper we show that the S_K is an order of K and give computations to decide whether the cubic field $K \subset \mathbb{Q}(\zeta_l)$ is or is not ω-good.

Results

Lemma 1. *Let K be a cyclic extension of rationals of prime degree $p > 2$ such that $K \subseteq \mathbb{Q}(\zeta_m)$, and let $S_K = \{\alpha = \sum_{i=1}^{k} \epsilon_i \colon \epsilon_i \in U_K, n \in \mathbb{N}\}$. Then S_K is an order of K.*

Proof. Clearly S_K is a subring and a submodule of Z_K with a unit because S_K is closed under addition and multiplication. Since p is a prime greater or equal to 3, there exists a nontrivial unit ϵ such that $1, \epsilon, \dots, \epsilon^{l-1}$ forms a basis of the field K over $\mathbb{Q}$. Hence S_K is submodule of Z_K of dimension p and thus an order of the field K. $\square$

Let $L = \mathbb{Q}(\zeta_l)$ be the lth cyclotomic extension of rationals such that l is a prime $l \equiv 1 \mod 3$. Let $K \subseteq L$ be the cyclic real cubic extension of rationals.

We will consider norms of cyclotomic units. We denote by ϵ_a cyclotomic units which generate the group of all cyclotomic units, namely

$$\epsilon_a = \zeta^{\frac{1-a}{2}} \frac{1-\zeta^a}{1-\zeta}, \quad \text{where } 1 < a < \frac{l}{2}.$$

As for every $\sigma \in \mathrm{Gal}(L/K)$, one has $\epsilon_a^{\sigma} = \epsilon_a^{-\sigma}$, it follows that

$$\gamma_a^2 = \mathrm{N_{L/K}}(\epsilon_a), \qquad \gamma_a \in K.$$

In the following computations we list just two such gammas, say γ_i, γ_j, for which it holds that for every γ_a we have either:

$$\gamma_a = \pm 1, \quad \text{or} \quad \gamma_a = \pm\gamma_i, \quad \text{or} \quad \gamma_a = \pm\gamma_j.$$

Moreover, if $h^+ = 1$, then

$$[U_K : \mathrm{N_{L/K}}(U_L)] = 2^{[K:\mathbb{Q}]-1}$$

and it follows that γ_i, γ_j are fundamental units of K.

Let c_j be the coefficients of a γ_i relative to the integral normal basis of K/Q

$$\{\mathrm{Tr_{L/K}}(\zeta)^{\sigma} : \sigma \in \mathrm{Gal}(K/\mathbb{Q})\},$$

in the following results we write

$$\gamma_i = (c_1, c_2, c_3).$$

Let D_K be discriminant of the field K. Denote

$$D_i = \operatorname{disc}\bigl(1, \gamma_i, \gamma_i^2\bigr), \quad \text{and} \quad D_{i,j} = \operatorname{disc}\bigl(1, \gamma_i, \gamma_j^2\bigr).$$

Then if the greatest common divisor of $(D_i, D_j, D_{i,j}) = D_K$, then K is ω-good. If $h^+ = 1$ and $(D_i, D_j, D_{i,j}) \neq D_K$, then K is not ω-good. If $h^+ > 1$ and $(D_i, D_j, D_{i,j}) \neq D_K$, then we do not know.

Computational results for selected primes

l=7, $\quad \mathrm{Z_K} = \mathrm{S_K}$ **:**

$$\gamma_4 = (0, 1, 0),$$
$$\gamma_5 = (0, 1, 1),$$

$$\operatorname{disc}\bigl(1, \gamma_4, \gamma_4^2\bigr) = \bigl(1\bigr)^2 \cdot 7^2,$$
$$\operatorname{disc}\bigl(1, \gamma_5, \gamma_5^2\bigr) = \bigl(1\bigr)^2 \cdot 7^2,$$
$$\operatorname{disc}\bigl(1, \gamma_4, \gamma_5\bigr) = \bigl(1\bigr)^2 \cdot 7^2.$$

l=13, $\quad \mathrm{Z_K} = \mathrm{S_K}$ **:**

$$\gamma_4 = (0, 1, 0),$$
$$\gamma_6 = (1, 1, 2),$$

$$\operatorname{disc}\bigl(1, \gamma_4, \gamma_4^2\bigr) = \bigl(1\bigr)^2 \cdot 13^2,$$
$$\operatorname{disc}\bigl(1, \gamma_6, \gamma_6^2\bigr) = \bigl(1\bigr)^2 \cdot 13^2,$$
$$\operatorname{disc}\bigl(1, \gamma_4, \gamma_6\bigr) = \bigl(1\bigr)^2 \cdot 13^2.$$

l=19, $\quad \mathrm{Z_K} = \mathrm{S_K}$ **:**

$$\gamma_4 = (1, 2, 2),$$
$$\gamma_6 = (1, 0, 1),$$

$$\operatorname{disc}\bigl(1, \gamma_4, \gamma_4^2\bigr) = \bigl(1\bigr)^2 \cdot 19^2,$$
$$\operatorname{disc}\bigl(1, \gamma_6, \gamma_6^2\bigr) = \bigl(1\bigr)^2 \cdot 19^2,$$
$$\operatorname{disc}\bigl(1, \gamma_4, \gamma_6\bigr) = \bigl(1\bigr)^2 \cdot 19^2.$$

The calculations for $l = 7, 13, 19$ give us the following small remark.

Remark 1. If the class number of L is equal to 1, then $S_K = Z_K$.

l=31, $\mathrm{Z_K} \neq \mathrm{S_K}$ **:**

$$\gamma_5 = (3, 5, 7) \equiv 1 \mod 2,$$
$$\gamma_7 = (11, 17, 21) \equiv 1 \mod 2,$$

$$\operatorname{disc}\big(1, \gamma_5, \gamma_5^2\big) = \big(2^5\big)^2 \cdot 31^2,$$
$$\operatorname{disc}\big(1, \gamma_7, \gamma_7^2\big) = \big(2^5\big)^2 \cdot 31^2,$$
$$\operatorname{disc}\big(1, \gamma_5, \gamma_7\big) = \big(2^2\big)^2 \cdot 31^2.$$

The units γ_5, γ_7 are fundamental units of the form $2\alpha + a$, where a is a rational integer. Then every unit in U_K is of that form and, thus, is every finite sum of units. But $\mathrm{Tr}_{\mathrm{L/K}}\big(\zeta_{31}\big)$ cannot be written in the above form.
Consequently,

$$Z_K \neq S_K.$$

l=67, $\mathrm{Z_K} = \mathrm{S_K}$ **:**

$$\gamma_4 = (5, 6, 9),$$
$$\gamma_6 = (42, 51, 64),$$

$$\operatorname{disc}\big(1, \gamma_4, \gamma_4^2\big) = \big(3 \cdot 5^2\big)^2 \cdot 67^2,$$
$$\operatorname{disc}\big(1, \gamma_6, \gamma_6^2\big) = \big(3 \cdot 5^2\big)^2 \cdot 67^2,$$
$$\operatorname{disc}\big(1, \gamma_4, \gamma_6\big) = \big(2 \cdot 7\big)^2 \cdot 67^2.$$

l=229, $h^+ = 3 \Rightarrow \mathrm{Z_K} \overset{?}{\neq} \mathrm{S_K}$ **:**

$$\gamma_5 = (129785, 162609, 154103) \equiv 1 \mod 2,$$
$$\gamma_7 = (5673, 7111, 6737) \equiv 1 \mod 2,$$

$$\operatorname{disc}\big(1, \gamma_5, \gamma_5^2,\big) = \big(2^7 \cdot 13 \cdot 101 \cdot 227\big)^2 \cdot 229^2,$$
$$\operatorname{disc}\big(1, \gamma_7, \gamma_7^2,\big) = \big(2^7 \cdot 13 \cdot 101 \cdot 227\big)^2 \cdot 229^2,$$
$$\operatorname{disc}\ \big(1, \gamma_5, \gamma_7\big) = \big(2^2 \cdot 7 \cdot 37 \cdot 43\big)^2 \cdot 229^2.$$

Results for all prime conductors $l \equiv 1 \mod 3$ up to $l = 349$ are listed in the following tables.

Tables

0.1. $\mathbb{Z}_K = S_K$

The cyclotomic units are fundamental units as $h^+ = 1$ is the index of the cyclotomic units in group of all units. The equality $\mathbb{Z}_K = S_K$ follows since the index $d = 1$ of S_K in $\mathbb{Z}_K$.

TABLE 1.

l	h^+	γ_i	γ_j	$d = \frac{(D_i, D_j, D_{i,j})}{D_K}$
7	1	(0,1,0)	(0,1,1)	1
13	1	(0,1,0)	(1,1,2)	1
19	1	(1,2,2)	(1,0,1)	1
37	1	(2,3,2)	(1,1,2)	1
67	1	(5,6,9)	(42,51,64)	1
79	1	(2,3,2)	(3,3,4)	1
97	1	(3,4,4)	(3,2,3)	1
103	1	(66,85,93)	(9,12,14)	1
139	1	(4,5,4)	(3,3,4)	1
151	1	(184,229,243)	(17,18,20)	1
163	4	(49,62,63)	(42,53,54)	1
181	1	(199,258,233)	(10,15,12)	1
193	1	(135,171,142)	(1086,1030,1299)	1
211	1	(233,273,296)	(12,16,17)	1
313	7	(105,124,103)	(1660,1677,1968)	1
331	1	(28,34,31)	(32,29,35)	1
337	1	(116,127,144)	(5189,5685,6266)	1
349	16	(109,128,129)	(98,115,116)	1

0.2. $\mathbb{Z}_K \neq S_K$

As in the previous table, h^+ is 1, so the cyclotomic units are fundamental units. One has $\mathbb{Z}_K \neq S_K$ as the index $d > 1$ of S_K in $\mathbb{Z}_K$.

TABLE 2.

l	h^+	γ_i	γ_j	$d = \frac{(D_i, D_j, D_{i,j})}{D_K}$
31	1	(3,5,7)	(11,17,21)	16
43	1	(7,9,9)	(39,47,65)	16
61	1	(10,16,13)	(14,11,17)	729
73	1	(13,19,17)	(31,47,41)	16
109	1	(449,627,533)	(443,373,521)	16
127	1	(99,131,81)	(14565,15367,19277)	16
157	1	(2683,3509,2953)	(8883,8071,10557)	16
199	1	(95,105,121)	(35,39,45)	16
223	1	(49591,60217,61785)	(96959,117735,120801)	38416
241	1	(87,108,94)	(93,86,107)	117649
271	1	(28,37,19)	(6029,6254,7316)	59049
283	1	(39631,46569,38735)	(14607,14945,17561)	16
307	1	(3500815,3988957,4265923)	(1233880499,1405925741,1503532433)	104976
331	1	(28,34,31)	(32,29,35)	729

0.3. $\mathbb{Z}_K \overset{?}{\neq} S_K$

In the following table are cases which are the most complicated. We do not know the answer to our question. Sums of the cyclotomic units are not $\mathbb{Z}_K$ as the index $d > 1$ in $\mathbb{Z}_K$. But since $h^+ \neq 1$ the cyclotomic units are not fundamental, thus using our method one cannot say if $\mathbb{Z}_K = S_K$.

Table 3.

l	h^+	γ_i	γ_j	$d = \frac{(D_i, D_j, D_{i,j})}{D_K}$
229	3	(129785,162609,154103)	(5673,7111,6737)	16
277	4	(250641,263305,304937)	(2201,2545,2105)	4096

REFERENCES

[1] ASHRAFI, N.—VÁMOS, P.: *On unit sum number of some rings,* Q. J. Math. **56** (2005), 1-12.

[2] JARDEN, M.—NARKIEWICZ, W.: *On sum of units,* Monatsh. Math. **150** (2007), 327–332.

[3] TICHY, R. F.—ZIEGLER, V.: *Units generating the ring of integer of complex cubic fields,* Colloq. Math. **109** (2007), 71–83.

[4] ZELINSKY, D.: *Every linear transformation is a sum of nonsingular ones,* Proc. Am. Math. Soc. **5** (1954), 627–630.

Received September 23, 2013

Juraj Kostra
Institute of Applied Physics and Mathematics
Faculty of Chemical Technology
University of Pardubice
Studentská 95
CZ–532-10 Pardubice
CZECH REPUBLIC
E-mail: Juraj.Kostra@upce.cz

David Krčmarský
Mitrovická 426
CZ–724-00 Ostrava
CZECH REPUBLIC
E-mail: d.krcmarsky@aplex.cz