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THE STURM SEPARATION THEOREM FOR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Wronskian is one of the classical objects in the theory of ordinary differential equations. Properties of Wronskian lead to important conclusions on behaviour of solutions of delay equations. For instance, non-vanishing Wronskian ensures validity of the Sturm separation theorem (between two adjacent zeros of any solution there is one and only one zero of every other nontrivial linearly independent solution) for delay equations. We propose the Sturm separation theorem in the case of impulsive delay differential equations and obtain assertions about its validity.

1. Introduction

Our paper is devoted to the Sturm separation theorem for impulsive equations. Let us describe this topic in more detail. We start with the following second order equation

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0, \qquad t \in [0, +\infty), \tag{1.1}$$

with locally summable coefficients p, q. This equation presents one of the classical objects in the theory of ordinary differential equations. For distribution of zeros of its solutions, the following assertion, known as the Sturm separation theorem, is true.

Theorem 1.1. *Between two adjacent zeros of any solution there is one and only one zero of each other nontrivial linearly independent solution.*

There were some attempts to formulate the Sturm separation theorem for delay differential equation

$$x''(t) + p(t)x'\big(t - \theta(t)\big) + q(t)x\big(t - \tau(t)\big) = 0, \qquad t \in [0, +\infty), \tag{1.2}$$

$$x(\xi) = 0, \quad x'(\xi) = 0 \qquad \text{for} \quad \xi < 0. \tag{1.3}$$

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Y. Domshlak in [3] used the idea of "extent big semi-circle". If t_1, t_2 are two adjacent zeros of solution x of equation (1.2), (1.3), then the interval $[t_1 - m, t_2]$, where $m = \operatorname{esssup}_{t\geq 0}\{\theta(t), \tau(t)\}$, is called an extent big semi-circle. He formulated the Sturm separation theorem for (1.2), (1.3) in the form: "between two adjacent zeros of a solution x there is no an extent big semi-circle of any other nontrivial linearly independent solution".

N. V. Azbelev in [1] introduced such a definition of homogeneous equation which excluded infinite number of zeros on every finite interval and preserved a finite fundamental system. He obtained the first analog of the classical Sturm separation theorem for the non-impulsive delay equation

$$x''(t) + \sum_{j=1}^{m} b_j(t)x\big(t - \tau_j(t)\big) = 0, \qquad t \in [0, +\infty), \;\; b_j(t) \geq 0, \tag{1.4}$$

$$x(\xi) = 0 \qquad \text{for} \quad \xi < 0, \tag{1.5}$$

under the conditions on a "smallness" of the delays $\tau_j(t)$, $j = 1, \ldots, m$. In the case of two-terms non-impulsive equation

$$x''(t) + b_1(t)x\big(t - \tau_1(t)\big) = 0, \qquad t \in [0, +\infty), \;\; b_1(t) \geq 0, \tag{1.6}$$

the following assertion was proven by Labovskii in [10], [11].

Theorem 1.2. *If the function $h_1(t) \equiv t - \tau_1(t)$ is nondecreasing, then the Sturm separation theorem is true for equation* (1.6) *with the initial function* (1.5).

The Sturm separation theorem for non-impulsive delay equation (1.4), (1.5)) was obtained in the paper [5] under the conditions on a "smallness" of the differences $\tau_i - \tau_j$, where $i, j = 1, \ldots, m$. In this paper, we obtain assertions about validity of the Sturm separation theorem for impulsive delay differential equation

$$x''(t) + \sum_{j=1}^{m} a_j(t)x'\big(t - \theta_j(t)\big) + \sum_{j=1}^{m} b_j(t)x\big(t - \tau_j(t)\big) = 0, \qquad t \in [0, +\infty), \tag{1.7}$$

$$x(\xi) = 0, \qquad x'(\xi) = 0 \qquad \text{for} \quad \xi < 0, \tag{1.8}$$

$$x(t_i) = \gamma_i x(t_i - 0), \quad x'(t_i) = \delta_i x'(t_i - 0), \qquad i = 1, 2, \ldots \tag{1.9}$$

where a_j and b_j are summable functions, θ_j and τ_j are nonnegative measurable functions, γ_i and δ_i are positive constants. Let D be a space of functions $x\colon [0, +\infty] \to \mathbb{R}$ such that their derivative $x'(t)$ is absolutely continuous on every interval $t \in [t_i, t_{i+1})$, $i = 1, 2, \ldots$, there exist the finite limits $x(t_i - 0) = \lim_{t\to t_i^-} x(t)$ and $x'(t_i - 0) = \lim_{t\to t_i^-} x'(t)$ and condition (1.9) is satisfied at points t_i ($i = 1, 2, \ldots$). Solution x is a function $x \in D$ satisfying (1.7)–(1.9).

We propose the generalization of Theorem 1.1 and Theorem 1.2 in case of impulsive equations and obtain the validity of the Sturm separation theorem under assumptions on smallness of delays $\theta_j(t)$ and $\tau_j(t)$ for $j = 1, \dots, m$.

2. Preliminaries

Our methodology is based on the general theory of functional differential equations [2]. Basing on its concepts, we can write the general solution of equation (2.1)

$$x''(t) + \sum_{j=1}^{m} a_j(t)x'\big(t - \theta_j(t)\big) + \sum_{j=1}^{m} b_j(t)x\big(t - \tau_j(t)\big) = f(t), \quad t \in [0, +\infty), \tag{2.1}$$

with initial and impulsive conditions (1.8), (1.9) in the form [8]

$$x(t) = \int_0^t C(t,s)f(s)\,ds + x_1(t)x(0) + x_2(t)x'(0). \tag{2.2}$$

Here the functions x_1 and x_2 are solutions of the homogeneous equation satisfying the conditions $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 0$, $x_2'(0) = 1$, and $C(t,s)$ as a function of t for every fixed s is the solution of the following impulsive equation

$$x''(t) + \sum_{j=1}^{m} a_j(t)x'\big(t - \theta_j(t)\big) + \sum_{j=1}^{m} b_j(t)x\big(t - \tau_j(t)\big) = 0, \quad t \in [s, +\infty), \tag{2.3}$$

with zero initial functions $x(\xi) = 0$, $x'(\xi) = 0$, $\xi \in [s, +\infty)$, $x(t_i) = \gamma_i x(t_i - 0)$, $x'(t_i) = \delta_i x'(t_i - 0)$, $i = 1, 2, \dots$, satisfying the condition $C(s,s) = 0$, $C_t'(s,s) = 1$ and $C(t,s) = 0$, $C_t'(t,s) = 0$ for $t < s$.

The structure of $C(t,s)$ and its positivity are studied in [9].

The fundamental system of impulsive differential equation (1.7)–(1.9) consists of two linearly independent solutions x_1 and x_2. Like in the case of ordinary differential equations its Wronskian

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}$$

could be one of the classical objects in this theory. Various tests, where the Wroskian does not vanish for the non-impulsive equation were obtained in [1], [4], [5], [7], [10], [11]. It can be also noted that a corresponding growth of the Wronsian leads to existence of unbounded oscillating solutions. Results on unboundedness of solutions of (1.5), (1.6) based on this idea were obtained in [6], [7]. It will be proven in the next section that the classical Sturm separation theorem is valid for the impulsive equation (1.7)–(1.9) if the Wroskian does not vanish, i.e., $W(t) \neq 0$ for $t \in [0, +\infty)$.

3. The Sturm separation theorem in the case of $b_j \geq 0$

Let us start with the following generalization of the Sturm separation theorem.

Theorem 3.1. *Let $\gamma_i > 0$ and $\delta_i > 0$ for every i. If the Wronskian $W(t)$ of the fundamental system of* (1.7)–(1.9) *does not have zeros, then the Sturm separation theorem is valid.*

Proof. Let t_1, t_2 be two adjacent zeros of solution x_1, and let us suppose on the contrary the existence of two zeros $\bar{t_1}$, $\bar{t_2}$ ($t_1 < \bar{t_1} < \bar{t_2} < t_2$) of other nontrivial solution x_2 (see Fig. 1). Consider the following function $y(t) = \frac{x_2(t)}{x_1(t)}$. Its derivative is $y'(t) = \frac{W(t)}{[x_1(t)]^2}$. It is clear that under the conditions $\gamma_i > 0$ and $\delta_i > 0$ for every i, the functions $y(t)$ and $y'(t)$ preserve their signs for $t \in [\bar{t_1}, \bar{t_2}]$. From the form of $y(t)$ we have $y(\bar{t_1}) = y(\bar{t_2}) = 0$. This contradicts the fact that $y(t)$ is either monotonically increasing or monotonically decreasing function for $t \in [\bar{t_1}, \bar{t_2}]$ (depending on the sign of W(t)). □

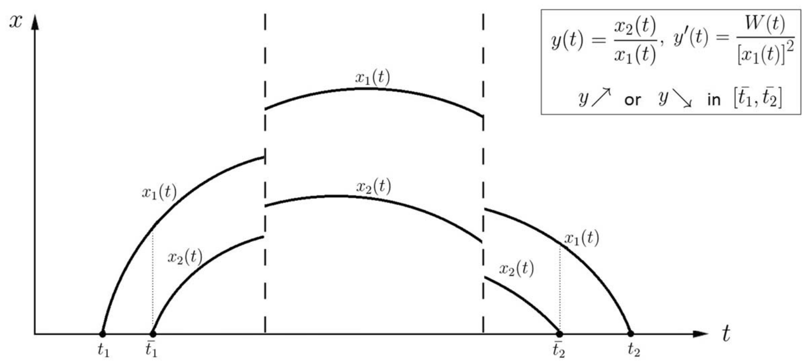

Figure 1. Sketch of the proof of Theorem 3.1

Let us continue now with the following generalization of Theorem 1.2.

Theorem 3.2. *Let $\gamma_i > 0$ and $\delta_i > 0$ for every i. If the function $h_1(t) \equiv t - \tau_1(t)$ in two-terms impulsive equation* (1.6), (1.5), (1.9) *is nondecreasing, then the Sturm separation theorem is true.*

Proof. Let us first show that a solution x of (1.6), (1.5), (1.9) cannot have multiple zeros. $b_1(t) \geq 0$, thus according to (1.6) $x(t)$ and $x''(t)$ must have the opposite signs. Let us assume, without loss of generality, that $x(0) > 0$, then $x(t) > 0$ and $x''(t) < 0$ on $[0, t_1]$ (where t_1 is the first zero of $x(t)$). It means that the function x is concave on interval $[0, t_1]$, so t_1 cannot be multiple. If t_2 ($t_2 > t_1$) is the second zero of x, then at some $H(t_1)$, such that

$t_1 < h_1(t)$ for $t > H(t_1)$, the function x has to change its shape from concave to convex, so on the interval $[H(t_1), t_2)$ x is convex, thus t_2 cannot be multiple. In the same way one can continue and conclude that a solution x cannot have multiple zeros on $[0, +\infty)$ (see Fig. 2).

$$\mathbf{x''(t) + b_1(t)x(h_1(t)) = 0, \quad t \in [0, +\infty),\ b_1(t) \geq 0,\ h_1(t) \nearrow}$$

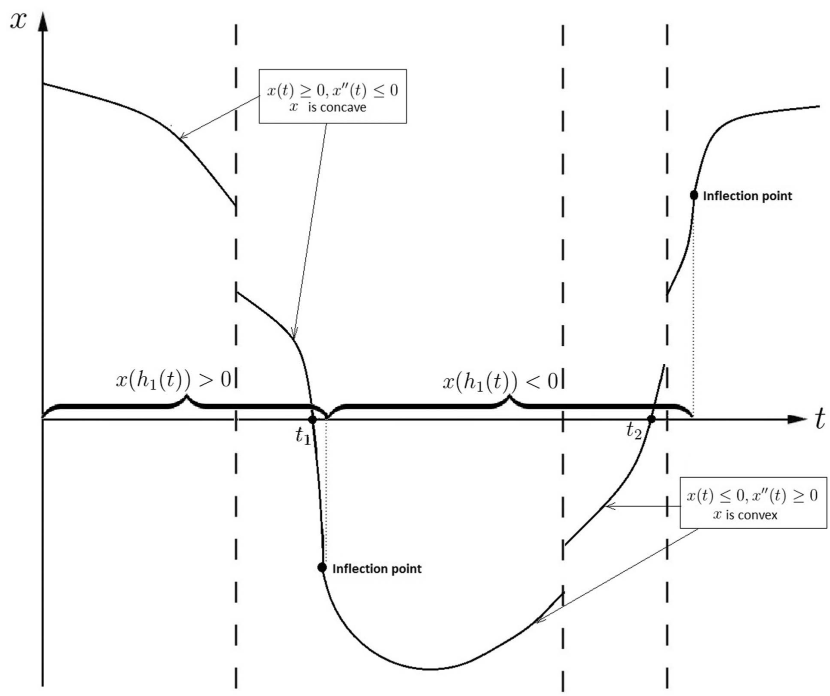

FIGURE 2. Sketch of the proof of Theorem 3.2

It is known from the general theory of functional differential equations [2] that zeros of the Wronskian do not depend on choosing of fundamental system. The Wronskian of the fundamental system of equation (1.7)–(1.9) is equal to zero at the point η (i.e., $W(\eta) = 0$) if and only if the corresponding boundary value problem where $x(\eta) = 0, x'(\eta) = 0$ has a nontrivial solution, i.e., there exists a nontrivial solution of (1.7)–(1.9) which has a multiple zero at the point η.

Thus, the fact that each solution x of equation (1.6), (1.5), (1.9) cannot have multiple zeros on $[0, +\infty)$ is equivalent to the fact that $W(t) \neq 0$ for $t \in [0, +\infty)$, and this means (see Theorem 3.1), that the Sturm separation theorem holds for this equation. □

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